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The Anomalous Magnetic Moments of the Electron and the Muon – Improved QED Predictions using Padé Approximants*

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Abstract

We use Padé Approximants to obtain improved predictions for the anomalous magnetic moments of the electron and the muon. These are needed because of the very precise experimental values presently obtained for the electron, and soon to be obtained at BNL for the muon. The Padé prediction for the QED contribution to the anomalous magnetic moment of the muon differs significantly from the naive perturbative prediction.

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Two of the most important tests of quantum electrodynamics (QED) are the comparisons between theory and experiment of the anomalous magnetic moments of the electron and the muon, a_e and a_μ respectively, where $a = (g-2)/2$. The latest Penning trap measurements of the electron and positron anomalies obtained by the University of Washington group[1] are:

$$a_{e^-}^{\rm expt} = 1159652188.4(4.3) \times 10^{-12} \qquad (1)$$

and

$$a_{e^+}^{\rm expt} = 1159652187.9(4.3) \times 10^{-12} \qquad (2)$$

The figures in brackets represent the error in the last 2 figures, a convention we will follow throughout this paper. Taking the average of eqs (1) and (2), one finds

$$a_{e}^{\rm expt} = 1159652188.2(3.0) \times 10^{-12} \qquad (3)$$

The most accurate measurement for the muon anomaly comes from the CERN $g-2$ experiment[2] in which it was found that

$$a_{\mu^-}^{\rm expt} = 1165936(12) \times 10^{-9} \qquad (4)$$

and

$$a_{\mu^+}^{\rm expt} = 1165910(11) \times 10^{-9} \qquad (5)$$

and the combined result is

$$a_{\mu}^{\rm expt} = 1165923(9) \times 10^{-9} \qquad (6)$$

where correlations are taken into account in combining the errors. A new $g-2$ muon experiment is being done at Brookhaven National Laboratory (BNL)[3], and an improvement in the accuracy by a factor of about 20 is expected. In order to compare properly theory and experiment, one must improve correspondingly the accuracy of the theoretical predictions.

In an heroic feat, Kinoshita[4] has calculated a_e in eighth order and Kinoshita, Nizic, Okamoto[5] and Marciano[6] have calculated a_μ in eighth order. Moreover, there have been some recent improvements in the analytic calculations[7,8] of a_e and a_μ.

There have recently been several papers estimating coefficients in Perturbative Quantum Field Theory (PQFT) using Padé Approximants[9,10].

This procedure is known to give significant improvements on naive perturbative calculations in many condensed-matter applications[11], removes a large part of the discrepancy between experiment and QED calculations of the ortho-positronium decay rate[10,12] and agrees well with other estimates of higher-order perturbative coefficients in QCD[9,13].

In this paper we will use Padé Approximants (PA's) to estimate, not just the next term in the perturbation series, but the entire sum of the series (as is frequently done in condensed-matter applications), for both a_e and a_μ. We obtain in this way a more accurate theoretical prediction of the QED contribution to a_μ, in particular, which lies outside the errors quoted previously.

The first step is to obtain an accurate value for the fine-structure constant α. The two most precise measurements of α are[14]

$$\alpha^{-1} = 137.0359979(32) \tag{7}$$

and[15]

$$\alpha^{-1} = 137.0359840(50) \tag{8}$$

We note that these two values differ by more than 2 standard deviations, but nevertheless take the average of eqs (7) and (8) to obtain

$$\alpha_{\rm exp}^{-1} = 137.0359939(27) \tag{9}$$

The accuracy of this result limits the precision of tests of QED in the case of a_e, where both theory and experiment are extremely precise. The perturbation series for a_e is[4]

$$a_e = \frac{1}{2}(\frac{\alpha}{\pi}) - 0.328478965(\frac{\alpha}{\pi})^2 + 1.17611(42)(\frac{\alpha}{\pi})^3 - 1.434(138)(\frac{\alpha}{\pi})^4 \tag{10}$$

and the error in the theoretical prediction is dominated by the error in $\alpha_{\rm exp}$.

The [N/M] Padé Approximant to a series

$$S = S_0 + S_1 x + S_2 x^2 + ... + S_{N+M} x^{N+M} \tag{11}$$

is given by

$$[N/M] = \frac{a_0 + a_1 x + ... + a_N x^N}{1 + b_1 x + ... + b_M x^M} \tag{12}$$

where one chooses the coefficients a_i, b_j so that

$$[N/M] = S + O(x^{N+M+1}) \tag{13}$$

One can use such a PA either to predict the next coefficient S_{N+M+1} or to evaluate $[N/M]$ for the relevant value of x (in our case $x = \frac{\alpha}{\pi}$), and obtain an estimate for the sum of the series. Here we do the latter. The PA's are known to accelerate the convergence of many series by including the effects of higher (unknown) terms, thus providing a more accurate estimate of the series[11]. The PA's also provide reliable estimates of many asymptotic series, as is the case in QED[10] and QCD[9].

For our application, we first construct PA's to a_e after removing an overall multiplicative factor of $(\frac{\alpha}{\pi})$. Our result for the [1/2] PA is

$$[1/2] = 1159652169.1(24.0) \times 10^{-12} \tag{14}$$

and the [2/1] PA agrees very well with the [1/2]:

$$[2/1] = 1159652169.0(24.0) \times 10^{-12} \tag{15}$$

The errors consist of 22.8 from α and 7.4 from the theoretical uncertainty. To obtain a_e one must add the contribution due to muon diagrams

$$\Delta a_e(\text{muon}) = 2.8 \times 10^{-12} \tag{16}$$

the contribution due to τ diagrams

$$\Delta a_e(\text{tau}) = 0.01 \times 10^{-12} \tag{17}$$

the hadronic contribution

$$\Delta a_e(\text{hadron}) = 1.6(2) \times 10^{-12} \tag{18}$$

and the purely weak contribution

$$\Delta a_e(\text{weak}) = 0.05 \times 10^{-12} \tag{19}$$

for a total of

$$\Delta a_e = 4.5(2) \times 10^{-12} \tag{20}$$

Thus the theoretical prediction for a_e is

$$a_e = 1159652173.5(24.0) \times 10^{-12} \tag{21}$$

Comparing eq (21) with eq (3), we see that there is beautiful agreement between theory and experiment:

$$a_e^{\mathrm{expt}} - a_e = 14.7(24.0) \times 10^{-12}\ (0.61\sigma) \tag{22}$$

As noted before, the error in eq (21) is dominated by the error in α. If one now assumes that QED is correct, and hence that theory and experiment agree, one obtains a new and more accurate value of α: α_{th}, where the [1/2] PA gives

$$\alpha_{\mathrm{th}} = 137.03599228(86) \tag{23}$$

and the [2/1] PA leads to

$$\alpha_{\mathrm{th}} = 137.03599227(86) \tag{24}$$

Comparing eq (23) with eq (9), one sees that there is beautiful agreement with the less-precise experimental value.

$$\alpha_{\mathrm{expt}}^{-1} - \alpha_{\mathrm{th}}^{-1} = 16(28) \times 10^{-7}\ (0.57\sigma) \tag{25}$$

corresponding to the good agreement in eq (22). We note in passing that this provides an *a posteriori* justification for averaging naively the two most accurate measurements[14,15] of α, and that the difference between the values of $\alpha_{\mathrm{th}}^{-1}$ extracted using the perturbative series and the PA's is just 3×10^{-8}.

We now turn to the anomalous magnetic moment of the muon, a_μ. As is usual, we first consider the difference[6,8]

$$a_\mu - a_e = 1.09433583(7)(\frac{\alpha}{\pi})^2 + 22.869265(4)(\frac{\alpha}{\pi})^3 + 127.00(41)(\frac{\alpha}{\pi})^4 \tag{26}$$

In constructing a PA to this series, we must first remove an overall factor $(\frac{\alpha}{\pi})^2$ from the perturbative series. In this way, we obtain the [1/1] PA value

$$(a_\mu - a_e)[1/1] = 6194839(12) \times 10^{-12} \tag{27}$$

whereas the value from the series in eq (26) is

$$a_\mu - a_e = 6194791(12) \times 10^{-12} \tag{28}$$

Adding a_e from eq (21), after subtracting Δa_e from eq (20), we obtain

$$a_\mu^{\rm QED} = 1165847008(12)(27) \times 10^{-12} \tag{29}$$

where the first error is due to numerical integrations used in evaluating the perturbative series, and the second error is due to α. This should be compared the value of Kinoshita and Marciano[6]

$$a_\mu^{\rm QED} = 1165846955(44)(27) \times 10^{-12} \tag{30}$$

We note that the difference between these two estimates of $a_\mu^{\rm QED}$ is considerably larger than the error propagated from α. The reason for our smaller error is that we have used the new more precise values in ref. [8].

If one now adds the hadronic[16] and the weak[6] contributions

$$\Delta a_\mu({\rm had}) = 7011(76) \times 10^{-11} \tag{31}$$

and

$$\Delta a_\mu({\rm weak}) = 195(10) \times 10^{-11} \tag{32}$$

one obtains the theoretical value

$$a_\mu = 116591907(77) \times 10^{-11} \tag{33}$$

The error is dominated by the error in Δa_μ(had), and new, more precise experiments are underway in Novosibirsk and Frascati[17] to reduce this error. Comparing eq (33) with eq (6), we obtain

$$a_\mu^{\rm expt} - a_\mu = 4(9) \times 10^{-9}\ (0.4\sigma) \tag{34}$$

The error in the difference between theory and experiment is dominated by the experimental error in eq (6), which should be reduced by a factor of 20 in the forthcoming BNL experiment[3].

In summary, we have used PA to obtain new more precise values for the QED values of a_e and a_μ . These PA values, in effect, estimate the unknown higher-order contributions, and should be more precise than the naive perturbative values used previously. It would be interesting to compare our estimates with values obtained in a different way, for example using the effective charge approach[13] which agrees very well with PA's in QCD applications[9]. Although smaller than some of the other uncertainties, the shift we find in a_μ, in particular, is significantly larger than other theoretical uncertainties and the error due to α.

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