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ONE-LOOP GAUGE THEORY AMPLITUDES WITH AN ARBITRARY NUMBER OF EXTERNAL LEGS*

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ABSTRACT

We review recent progress in calculations of one-loop QCD amplitudes. By imposing the consistency requirements of unitarity and correct behavior as the momenta of two legs become collinear, we construct ansätze for one-loop amplitudes with an arbitrary number of external legs. For supersymmetric amplitudes, which can be thought of as components of QCD amplitudes, the cuts uniquely specify the amplitude.

1. Introduction.

Multi-jet processes at colliders require knowledge of matrix elements with multiple final state partons. The discovery of new physics relies to a large extent on the subtraction of known QCD physics from the data. Unfortunately, perturbative QCD amplitudes are notoriously difficult to calculate even at tree level [1]. It has nevertheless been possible to derive a set of extremely simple formulae at tree level for "maximally helicity-violating" (MHV) amplitudes with an arbitrary number of external gluons [2,3,4].

Three main ideas contributed to progress at tree-level: the use of a spinor

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helicity basis [5] for gluon polarization vectors; the color decomposition of the amplitudes [6,7]; and a recursive technique for calculating the kinematical coefficients of the different color factors [3,4]. Supersymmetry Ward identities [8] have also proven useful [9].

In 1986, R. K. Ellis and J. Sexton defined the state of the art for one-loop computations by their calculation of the four-parton matrix elements [10]. The next step beyond this, the calculation of five-point amplitudes, is significantly more complicated and seemed unobtainable with standard techniques. As an example of the complexity, consider the pentagon diagram one would encounter in a brute-force five-gluon computation. A naive count of the number of terms gives approximately 6^5 terms since each non-abelian vertex contains six terms. (This count is reduced by the use of on-shell conditions but increased since each internal momentum is a sum of external momenta and the loop momentum.) Each term is associated with an integral which evaluates to an expression on the order of a page in length. This means that one is faced with the order of 10^4 pages of algebra for this diagram alone.

Last year, all one-loop five-gluon helicity amplitudes [11] were obtained by use of string-based techniques [12] coupled with improvements in integration [13] and spinor helicity methods. This method helps minimize the algebra by reducing the size of the initial expressions, as well as organizing the results into smaller gauge invariant pieces, called partial amplitudes. This leads to much smaller intermediate expressions. Much of the simplification can be understood in terms of field theory [14,15]. Some of the ingredients which improve the computational efficiency beyond the tree-level techniques, are better gauge choices [16,17,14], a supersymmetric decomposition of amplitudes [11,18,19,20] and a relation which allows one to obtain all subleading-color partial amplitudes [21] in terms of leading-color amplitudes [20]. These three ideas were motivated by string theory but may also be derived in field theory. Supersymmetry Ward identities provide an additional helpful tool at one loop [22]. These types of simplifications can also be used for external fermions [23]. An explicit example of one such amplitude will be provided here.

Here we will discuss exact one-loop results in QCD for particular helicity configurations and an arbitrary number of external legs [24,25,26,20]. The three additional ingredients which enter are:

1) Consistency of amplitudes as two legs become collinear [2,27]. At tree level, the collinear (and soft) behavior of amplitudes has been widely used as a consistency check; the same technique can be applied at one-loop [25,20]. Using the collinear limits as a consistency condition, one can construct guesses for higher-point amplitudes based on previously calculated amplitudes.
2) Unitarity, in the form of the Cutkosky rules [28] which fix the cuts of any loop amplitude in terms of tree amplitudes. It is often easier to reconstruct the loop amplitudes from the cuts than to calculate the full amplitude directly; unitarity therefore provides a powerful tool for obtaining parts of amplitudes containing cuts.
3) A one-loop unitarity result which states that an amplitude is uniquely determined by its cuts if the m-point loop integrals contributing to it have at most $m-2$ powers of the loop momentum in the numerator; i.e., if the diagrams lead to integrands with two powers less than the maximum possible in gauge theory. Examples of such amplitudes are provided by supersymmetric gauge theories.

Mahlon has also introduced recursive techniques for constructing one-loop amplitudes [29,26], which are complementary to these.

Why do we need compact analytic results when the matrix elements are ultimately inserted into numerical programs? Without compact results, numerical instabilities generically arise from the vanishing of spurious Gram determinant denominators in the expression; such spuriously singular denominators have been removed from the results which we present. Compact analytic results also make it easier to compare independent calculations. Finally, compact analytic results are crucial for extensions to an arbitrary number of external legs.

For the identical-helicity case, we used the collinear constraints to construct an ansatz [24,25] for an arbitrary number of external legs which was later proven correct by a recursive procedure [26]. Mahlon has also constructed an all-n formula for the configuration with one leg of opposite helicity from the rest [26]. Since tree-level amplitudes vanish for these helicity configurations, these loop amplitudes do not contribute to multi-jet QCD cross-sections at next-to-leading order in α_s.

Amplitudes with two opposite-helicity gluons are more useful in this regard. For these helicity configurations, we have obtained results for an arbitrary number of external legs for both $N=4$ [20] and $N=1$ supersymmetric amplitudes [30]. When using a supersymmetric decomposition of amplitudes discussed in refs. [11,18,19,20] (as described in the next section), these may be thought of as two of the three terms in an n-gluon QCD amplitude. The third component is the contribution of a scalar in the loop, for which unitarity will not give the complete result and which is more difficult to obtain.

Other related examples of loop amplitudes that are known for all n include the n-photon massless QED amplitudes where all photon helicities are identical, or all but one are identical, which have recently been shown to vanish for five or more legs by Mahlon [29]. The QED results can be generalized to amplitudes with external photons and gluons, interacting via a massless quark loop. Amplitudes with five or more legs, where three or more legs are photons instead of gluons, also vanish when all the helicities are identical [25], and when one of the photon helicities is reversed [31].

2. Review of Previous Results.

Tree-level amplitudes for $U(N_c)$ or $SU(N_c)$ gauge theory with n external gluons can be decomposed into color-ordered partial amplitudes, multiplied by an associated color trace [32,6,7]. Summing over all non-cyclic permutations reconstructs the full amplitude $\mathcal{A}_n^{\rm tree}$ from the partial amplitudes $A_n^{\rm tree}(\sigma)$,

$$\mathcal{A}_n^{\rm tree}(\{k_i,\lambda_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\ A_n^{\rm tree}(k_{\sigma(1)}^{\lambda_{\sigma(1)}},\ldots,k_{\sigma(n)}^{\lambda_{\sigma(n)}})\ , \tag{1}$$

where k_i, λ_i, and a_i are respectively the momentum, helicity ($\pm$), and color index of the i-th external gluon, g is the coupling constant, and S_n/Z_n is the set of non-cyclic permutations of $\{1,\ldots,n\}$. The $U(N_c)$ ($SU(N_c)$) generators T^a are the set of hermitian (traceless hermitian) $N_c\times N_c$ matrices, normalized so that ${\rm Tr}\left(T^aT^b\right)=\delta^{ab}$. The color decomposition (1) can be derived in conventional field theory simply by using $f^{abc} = -i\,{\rm Tr}\left(\left[T^a,T^b\right]T^c\right)/\sqrt{2}$, where the T^a may by either $SU(N_c)$ matrices or $U(N_c)$ matrices.

In a supersymmetric theory, amplitudes with all helicities identical, or all but one identical, vanish due to supersymmetry Ward identities [8]. Tree-level gluon amplitudes in super-Yang-Mills and in purely gluonic Yang-Mills are identical (fermions do not appear at this order), so that

$$A_n^{\rm tree}(1^\pm, 2^+, \ldots, n^+) = 0. \tag{2}$$

Parity may of course be used to simultaneously reverse all helicities in a partial amplitude. The non-vanishing Parke-Taylor formulæ [2] are for maximally helicity-violating (MHV) partial amplitudes, those with two negative helicities and the rest positive,

$$A_{jk}^{\rm tree\ MHV}(1,2,\ldots,n) \;=\; i\, \frac{\langle j\, k\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n\,1\rangle}\,, \tag{3}$$

where we have introduced the notation

$$A_{jk}^{\rm MHV}(1,2,\ldots,n) \;\equiv\; A_n(1^+,\ldots,j^-,\ldots,k^-,\ldots,n^+), \tag{4}$$

for a partial amplitude where j and k are the only legs with negative helicity. Our convention is that all legs are outgoing. The result (3) is written in terms of spinor inner-products, $\langle j\, l\rangle = \langle j^-|l^+\rangle = \bar u_-(k_j)u_+(k_l)$ and $[j\, l] = \langle j^+|l^-\rangle = \bar u_+(k_j)u_-(k_l)$, where $u_\pm(k)$ is a massless Weyl spinor with momentum k and chirality $\pm$ [5,1].

For one-loop amplitudes, one may perform a similar color decomposition to the tree-level decomposition (1); in this case, there are up to two traces over color matrices [21], and one must also sum over the different spins J of the internal particles circulating in the loop. When all internal particles transform as color adjoints, the result takes the form

$$\mathcal{A}_n\left(\{k_i,\lambda_i,a_i\}\right) = \sum_J n_J \sum_{c=1}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_n/S_{n;c}} {\rm Gr}_{n;c}\left(\sigma\right)\, A_{n;c}^{[J]}(\sigma), \tag{5}$$

where $\lfloor x\rfloor$ is the largest integer less than or equal to x and n_J is the number of particles of spin J. The leading color-structure factor,

$${\rm Gr}_{n;1}(1) = N_c\ \ {\rm Tr}\left(T^{a_1}\cdots T^{a_n}\right)\,, \tag{6}$$

is just N_c times the tree color factor, and the subleading color structures ($c > 1$) are given by

$${\rm Gr}_{n;c}(1) = {\rm Tr}\left(T^{a_1}\cdots T^{a_{c-1}}\right)\, {\rm Tr}\left(T^{a_c}\cdots T^{a_n}\right)\,. \tag{7}$$

S_n is the set of all permutations of n objects, and $S_{n;c}$ is the subset leaving $\mathrm{Gr}_{n;c}$ invariant. Once again it is convenient to use $U(N_c)$ matrices; the extra $U(1)$ decouples from all final results [21]. (For internal particles in the fundamental $(N_c+\bar{N}_c)$ representation, only the single-trace color structure ($c=1$) would be present, and the corresponding color factor would be smaller by a factor of N_c. In this case the $U(1)$ gauge boson will *not* decouple from the partial amplitude, so one should only sum over $SU(N_c)$ indices when color-summing the cross-section.) In each case the massless spin-J particle is taken to have two helicity states: gauge bosons, Weyl fermions, and complex scalars.

It is very convenient to organize the spin-J partial amplitudes in a supersymmetry-inspired fashion,

$$
\begin{aligned}
A_n^{[0]} &= A_n^{\rm scalar}\,, \\
A_n^{[1/2]} &= A_n^{\rm fermion} \;=\; -A_n^{\rm scalar} + A_n^{N=1\ \rm susy}\,, \\
A_n^{[1]} &= A_n^{\rm gluon} = A_n^{\rm scalar} - 4A_n^{N=1\ \rm susy} + A_n^{N=4\ \rm susy}\,,
\end{aligned}
\tag{8}
$$

where the $N=1$ supersymmetry label refers to the contribution of a chiral multiplet consisting of a complex scalar and a Weyl fermion, while the $N=4$ supersymmetry label refers to a vector multiplet consisting of three complex scalars, four Weyl fermions and a single gluon. We have assumed the use of a supersymmetry preserving regulator [33,12,18,22].

The utility of the decomposition (8) stems from the simplicity of $N=4$ and $N=1$ supersymmetric calculations, in any approach where supersymmetric cancellations are made manifest in each diagram, such as string-based [12,14] or superspace approaches [34]. The string-based approach is related to use of the background field gauge effective action [17] together with a second-order form for the fermion loop, summarized by the determinants [14],

$$
\begin{aligned}
\Gamma_{\rm scalar}[A] &= \ln\det{}^{-1}[D^2]\,, \\
\Gamma_{\rm fermion}[A] &= \frac{1}{2}\ln\det[\not{D}] = \frac{1}{2}\ln\det{}^{1/2}[D^2 - \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}]\,, \\
\Gamma_{\rm gluon}[A] &= \ln\det{}^{-1/2}[D^2\eta_{\alpha\beta} - g(\Sigma_{\mu\nu})_{\alpha\beta}F^{\mu\nu}] + \ln\det[D^2]\,,
\end{aligned}
\tag{9}
$$

where D is the covariant derivative, the particle labels refer to the states circulating in the loop, $\frac{1}{2}\sigma_{\mu\nu}$ ($\Sigma_{\mu\nu}$) are the spin-$\frac{1}{2}$ (spin-1) Lorentz generators, and we have used the fact that the contribution of a Weyl fermion in a non-chiral theory is half that of a Dirac fermion. (Note that the gluon determinant contains Lorentz indices and the fermion determinant spinor indices.) The similarity of these determinants to each other suggests that there may be significant overlap between the calculation of a gluon loop and the calculation of a fermion loop. Indeed from (9) one can see that for the $N=1$ chiral matter multiplet contribution (scalar plus fermion) the pure D^2 contributions cancel, which means that the m-point loop integrands will contain at most $m-2$ powers of the loop momentum in the numerator, or two fewer than that for an individual scalar, fermion or gluon contribution. For the $N=4$ contribution, as a result of further cancellations (which are manifest in superspace or string-based approaches), m-point loop integrands contain at most $m-4$ powers of the loop momentum. A reduction of two powers of the loop momentum in each diagram permits one to apply a result, described in section 5, which dictates that the cuts completely specify the amplitude [30]; thus, loop diagrams can be bypassed in favor of simpler cut calculations for the $N=1$ and $N=4$ supersymmetric contributions. In a conventional formalism using ordinary gauges and fermion Feynman rules, one would not find the supersymmetric simplifications until all diagrams are summed, and it would not be clear how to apply this result.

The relative simplicity of $N=4$ loop amplitudes was first observed by Green, Schwarz and Brink in their calculation of the four-gluon amplitude as the low-energy limit of a superstring [35]; the same simplicity was found to extend to five-gluon amplitudes (for both $N=4$ and $N=1$ components) in ref. [11]. The supersymmetric decomposition can also reveal structure in electroweak amplitudes that would otherwise remain hidden [36,19].

3. A Subleading from Leading Color Formula.

It turns out that the subleading-color partial amplitudes associated with adjoint representation states appearing in eq. (5) can be obtained from the leading color partial amplitudes by summing over an appropriate set of permutations given by

$$A_{n;c}(1,2,\ldots,c-1;c,c+1,\ldots,n) \;=\; (-1)^{c-1} \sum_{\sigma\in COP\{\alpha\}\{\beta\}} A_{n;1}(\sigma) \qquad (10)$$

where $\alpha_i \in \{\alpha\} \equiv \{c-1,c-2,\ldots,2,1\}$, $\beta_i \in \{\beta\} \equiv \{c,c+1,\ldots,n-1,n\}$, and

$COP\{\alpha\}\{\beta\}$ is the set of all permutations of $\{1,2,\ldots,n\}$ with n held fixed that preserve the cyclic ordering of the α_i within $\{\alpha\}$ and of the β_i within $\{\beta\}$, while allowing for all possible relative orderings of the α_i with respect to the β_i. Note that the ordering of the first sets of indices is reversed with respect to the second. This formula may be easily derived using string-based rules [20], but can also be derived in field theory using the color-ordered Feynman rules described in refs. [1,18]. This formula eliminates the need to do a separate calculation for the subleading-color parts of an n-gluon amplitude; we therefore need only the leading-color partial amplitudes.

4. Collinear Limits Constraint.

Consider first the n-point tree-level partial amplitude $A_n(1,2,\ldots,n)$ with an arbitrary helicity configuration. The external legs may be fermions or gluons. There is an implicit color ordering of the vertices $1,2,\ldots,n$, so that collinear singularities arise only from neighboring legs a and b becoming collinear [27,1]. These singularities have the form

$$A_n^{\rm tree} \stackrel{a\parallel b}{\longrightarrow} \sum_{\lambda=\pm} \mathrm{Split}^{\rm tree}_{-\lambda}(a^{\lambda_a},b^{\lambda_b})\, A_{n-1}^{\rm tree}(\ldots(a+b)^{\lambda}\ldots)\,, \tag{11}$$

where the non-vanishing splitting amplitudes diverge as $1/\sqrt{s_{ab}}$ in the collinear limit $s_{ab}=(k_a+k_b)^2\to 0$. In the collinear limit $k_a = z\,P$, $k_b=(1-z)\,P$, where P is the sum of the collinear momenta; λ is the helicity of the intermediate state with momentum P. The tree splitting amplitudes $\mathrm{Split}^{\rm tree}_{-\lambda}(a^{\lambda_a},b^{\lambda_b})$ may be found in refs. [2,7,3,1].

The collinear limits of the (color-ordered) one-loop partial amplitudes have the form

$$A_{n;1}^{\rm loop} \stackrel{a\parallel b}{\longrightarrow} \sum_{\lambda=\pm}\biggl(\mathrm{Split}^{\rm tree}_{-\lambda}(a^{\lambda_a},b^{\lambda_b})\, A_{n-1;1}^{\rm loop}(\ldots(a+b)^{\lambda}\ldots) + \mathrm{Split}^{\rm loop}_{-\lambda}(a^{\lambda_a},b^{\lambda_b})\, A_{n-1}^{\rm tree}(\ldots(a+b)^{\lambda}\ldots)\biggr), \tag{12}$$

which is schematically depicted in fig. 1. The splitting amplitudes $\mathrm{Split}^{\rm tree}_{-\lambda}(a^{\lambda_a},b^{\lambda_b})$ and $\mathrm{Split}^{\rm loop}_{-\lambda}(a^{\lambda_a},b^{\lambda_b})$ are universal: they depend only on the two legs becoming collinear, and not upon the specific amplitude under consideration. This universal behavior is expected to hold for all one-loop amplitudes, with external (massless) fermions as well as gluons; all one-loop amplitudes that we have inspected do indeed obey eq. (12). (A similar equation is expected to govern the limit of one-loop partial amplitudes as one external gluon momentum becomes soft.) The explicit $\mathrm{Split}^{\mathrm{loop}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})$ have been determined from the known four- and five-point one-loop amplitudes [11,23], and are collected in appendix II of ref. [20]. An outline of a direct proof of the universality of the splitting amplitudes for the scalar-loop contributions to amplitudes with external gluons was presented in ref. [25]; a more general discussion will be presented elsewhere [37].

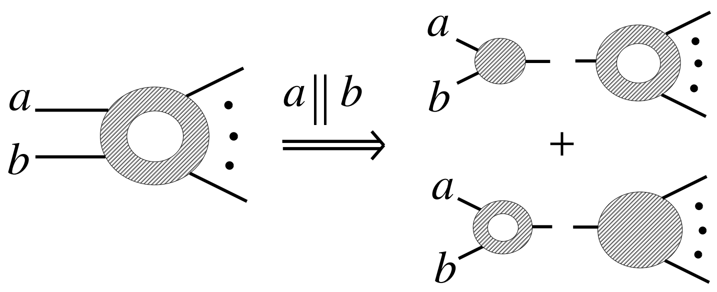

Fig. 1: A schematic representation of the behavior of one-loop amplitudes as the momenta of two legs become collinear.

The collinear behavior places tight constraints on the possible form of one-loop amplitudes and allows one to construct ansätze for higher-point amplitudes. To do this one writes down a general form for a higher-point amplitude containing arbitrary coefficients which may then be fixed by demanding that the expression have the correct collinear behavior. In this way a 'collinear bootstrap' can be constructed in certain cases to an arbitrary number of external legs [25].

However, functions lacking singular behavior in any collinear limit may appear in amplitudes, and would thus be omitted in such a construction. The simplest non-trivial example of such a function is the five-point function

$$\frac{\varepsilon(1,2,3,4)}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\,, \tag{13}$$

since the contracted antisymmetric tensor $\varepsilon(1,2,3,4) \equiv 4i\varepsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho k_4^\sigma$ vanishes when any two of the five vectors k_i become collinear ($\sum_{i=1}^5 k_i = 0$). Another example is the six-point function

$$\sum_{P(1,\ldots,5)} \frac{\ln(-s_{12}) + \ln(-s_{23}) + \cdots + \ln(-s_{61})}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle}\,, \tag{14}$$

where the summation is over all 120 permutations of legs 1 through 5 and $s_{i,i+1} = (k_i + k_{i+1})^2$. Without additional information, functions such as (13) and (14) represent additive ambiguities in the collinear bootstrap.

5. Unitarity Constraint.

The Cutkosky cutting rules [28] provide a relatively simple way to fix the absorptive (cut) parts of amplitudes. These cut amplitudes are generally much simpler to evaluate than the full amplitude. We calculate cuts in terms of the imaginary parts of one-loop integrals that would have been encountered in a direct calculation. This makes it straightforward to write down an analytic expression with the correct cuts in all channels, thus avoiding the need to do a dispersion integral to reconstruct the full amplitudes.

In order to evaluate the cuts, consider the amplitude, not in a physical kinematic configuration, but in a region where exactly one of the momentum invariants is taken to be positive (time-like), and the rest are negative (space-like). In this way cuts are isolated in a single momentum channel. The Cutkosky rules are applied at the amplitude level, rather than at the diagram level. That is, write the sum of all cut diagrams as the sum of all tree diagrams on one side of the cut, multiplied by the sum of all tree diagrams on the other side of the cut. Thus the cut in the one-loop amplitude is given by the integral over a two-body phase-space of the product of two tree amplitudes, which is then summed over each intermediate helicity configuration that contributes. By summing over the various cuts one can construct an expression which has all the correct cuts in all channels.

Unitarity determines the cuts uniquely — and hence the dilogarithms and logarithms — but does not directly provide any information about polynomial terms in the amplitude. (By 'polynomial terms' we actually mean any cut-free function of the kinematic invariants and spinor products, that is any rational function of these variables.)

The supersymmetric case is however special: a knowledge of the cuts completely determines the amplitude. The key to this result is the property discussed in section 2, that in a supersymmetric theory the loop-momentum polynomials encountered in every string-based or superspace diagram have a degree that is at least two less than the purely gluonic case, namely $m-2$ for an m-point integral. This means that only a restricted set of integrals (after reduction [38] to boxes, triangles, and bubbles [39,13]) can appear in an explicit calculation of supersymmetric amplitudes. One can show that there is no linear combination of integrals (with coefficients rational functions of the momentum invariants) in this restricted set which is free of cuts, that is which yields a 'polynomial term' in the sense used above. Further details of this procedure are given in refs. [20,30].

In general, for theories other than supersymmetric ones, the cuts may not uniquely determine the full amplitude. As a simple example, the five-point helicity amplitudes $A_{5;1}(1^-,2^+,\ldots,5^+)$ and $A_{5;1}(1^+,2^+,\ldots,5^+)$ each have no nontrivial cuts but are not equal. In such cases the collinear limits provide restrictions on the form of rational functions that may appear in the amplitudes [25].

6. Obtaining New Amplitudes from Known Amplitudes.

Using string-based methods all one-loop five-gluon helicity amplitudes [11] have been computed. Besides the intrinsic value of these amplitudes to the computation of next-to-leading order corrections to the three-jet cross-section at hadron colliders, these amplitudes are also useful as a starting point to generate further amplitudes. One can use the collinear bootstrap discussed in section 4 to obtain amplitudes with a larger number of external legs. Supersymmetry identities can also be used to generate contributions to amplitudes with external fermions [8,9,22]. We now present examples of amplitudes and point out how they are used to generate further amplitudes.

One amplitude that we present is for the all-plus helicity configuration

$$A_{5;1}(1^+,2^+,3^+,4^+,5^+) = \frac{iN_p}{192\pi^2}\,\frac{s_{12}s_{23}+s_{23}s_{34}+s_{34}s_{45}+s_{45}s_{51}+s_{51}s_{12} \;+\; \varepsilon(1,2,3,4)}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\,, \tag{15}$$

where $\varepsilon(i,j,m,n) = 4i\varepsilon_{\mu\nu\rho\sigma}k_i^\mu k_j^\nu k_m^\rho k_n^\sigma \;=\; [i\,j]\,\langle j\,m\rangle\,[m\,n]\,\langle n\,i\rangle - \langle i\,j\rangle\,[j\,m]\,\langle m\,n\rangle\,[n\,i]$, and N_p is the number of color-weighted bosonic states minus fermionic states circulating in the loop; for QCD with n_f quarks, $N_p = 2(1 - n_f/N_c)$ with the number of colors $N_c = 3$. This amplitude forms the basis for a collinear bootstrap to larger numbers of external legs, discussed in the next section.

Another example which can be extended to an arbitrary number of external legs is the set of $N = 4$ supersymmetric five-gluon amplitudes

$$A_{5;1}^{N=4\ \mathrm{susy}} = c_\Gamma A_5^{\mathrm{tree}} \left[\sum_{i=1}^{5} -\frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + \sum_{i=1}^{5} \ln\left(\frac{-s_{i,i+1}}{-s_{i+1,i+2}} \right) \ln\left(\frac{-s_{i+2,i+3}}{-s_{i-2,i-1}} \right) + \frac{5}{6}\pi^2 \right] \tag{16}$$

where A_5^{tree} is the tree amplitude for the same (MHV) helicity configuration, given by eq. (3) for $n = 5$. The overall constant is

$$c_\Gamma = \frac{(4\pi)^\epsilon}{16\pi^2} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \ . \tag{17}$$

In this case, the cuts turn out to provide a more powerful tool for extending (16) to an arbitrary number of legs, than do the collinear limits, because in the $N = 4$ case the cuts uniquely determine the amplitude [20].

As an example of how supersymmetry can be used to aid in the calculation of amplitudes with external fermions, consider $A_{5;1}(1_{\bar q}^-, 2_q^+, 3^+, 4^+, 5^-)$, where the first two legs are fermionic and the remaining ones gluonic. In QCD this partial amplitude is associated with the leading color factor $(T^{a_3}T^{a_4}T^{a_5})_{i_2}{}^{\bar\imath_1}$. For the calculation of five-point leading-color amplitudes with external fermions [23], we used a field theory approach containing a number of improvements motivated by superstring theory, including better gauge choices, supersymmetry decompositions, and an improved decomposition into gauge invariant pieces. A more recent development has been to make use of the observation that the cuts completely determine large parts of the amplitudes.

To illustrate the use of supersymmetry identities, first consider the contribution of an $N = 1$ chiral multiplet (one complex scalar and one Weyl fermion) to the known five-gluon amplitude with the same helicity configuration given by [11]

$$
\begin{aligned}
& A^{N=1\ \mathrm{susy}}(1^-,2^+,3^+,4^+,5^-) = \\
& \qquad c_\Gamma A_5^{\mathrm{tree}}(1^-,2^+,3^+,4^+,5^-)\Biggl\{ \frac{1}{2\epsilon(1-2\epsilon)}\left(\left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon}+\left(\frac{\mu^2}{-s_{45}}\right)^{\epsilon}\right) \\
& \qquad\qquad - \frac{1}{2s_{51}}\Bigl(\mathrm{tr}[5123]+\mathrm{tr}[5134]\Bigr)\frac{\mathrm{L}_0\left(\frac{-s_{12}}{-s_{45}}\right)}{s_{45}}\Biggr\}\,,
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
\mathrm{tr}[a_1 a_2\cdots a_{2m}] &\equiv \mathrm{tr}_+[\not k_{a_1}\not k_{a_2}\cdots \not k_{2m}] \equiv \frac{1}{2}\,\mathrm{tr}[(1+\gamma_5)\not k_{a_1}\not k_{a_2}\cdots\not k_{2m}] \\
&= [a_1\ a_2]\,\langle a_2\ a_3\rangle\cdots[a_{2m-1}\ a_{2m}]\,\langle a_{2m}\ a_1\rangle\ ,
\end{aligned}
\tag{19}
$$

and

$$
A_5^{\mathrm{tree}}(1^-,2^+,3^+,4^+,5^-) = i\frac{\langle 1\,5\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\,. \tag{20}
$$

The poles in s_{ij} are fictitious and cancel against factors in the numerators and in A_5^{tree}. We have defined an auxiliary set of functions,

$$
\begin{aligned}
& \mathrm{L}_0(r) = \frac{\ln(r)}{1-r}\,, \qquad \mathrm{L}_1(r) = \frac{\ln(r)+1-r}{(1-r)^2}\,, \qquad \mathrm{L}_2(r) = \frac{\ln(r)-\frac{1}{2}(r-1/r)}{(1-r)^3}\,, \\
& \mathrm{Ls}_1(r_1,r_2) = \frac{1}{(1-r_1-r_2)^2}\Biggl[\mathrm{Li}_2(1-r_1)+\mathrm{Li}_2(1-r_2)+\ln r_1\,\ln r_2-\frac{\pi^2}{6} \\
& \qquad\qquad + (1-r_1-r_2)(\mathrm{L}_0(r_1)+\mathrm{L}_0(r_2))\Biggr]\ ,
\end{aligned}
\tag{21}
$$

and $\mathrm{Li}_2(x) = -\int_0^x dz \ln(1-z)/z$ is the dilogarithm [40]. In eq. (18), as for all subsequent amplitudes, no ultraviolet subtraction has been performed; in modified minimal subtraction the quantity $-3A^{\mathrm{tree}}c_\Gamma/2\epsilon$ should be subtracted from $A^{N=1\ \mathrm{susy}}$. Note that there are no polynomial terms in (18) that are independent of the terms with cuts, in accordance with the result discussed in section 5, that for a supersymmetric amplitude the cuts uniquely specify the amplitude.

Now consider the corresponding amplitude with two external quarks and three gluons. In order to make the supersymmetry relations more apparent decompose the loop amplitude in terms of supersymmetric and non-supersymmetric pieces

$$
\begin{aligned}
A_{n;1}(1_{\bar q},2_q;3,\ldots,n) = A^{N=4\ \mathrm{susy}} - 3A^{N=1\ \mathrm{susy}} &- \left(1+\frac{1}{N^2}\right)A^{\mathrm{R}} \\
&+ \left(1+\frac{n_s}{N}-\frac{n_f}{N}\right)A^s + \left(1-\frac{n_f}{N}\right)A^f\ ,
\end{aligned}
\tag{22}
$$

where $A^{N=4\ \mathrm{susy}}$ and $A^{N=1\ \mathrm{susy}}$ are the contributions of an $N=4$ vector supermultiplet and an $N=1$ chiral matter multiplet. This decomposition, although more complicated, is analogous to the one given in eq. (8) for purely external gluons. The $N=4$ and $N=1$ contributions are given from the corresponding five-gluon amplitudes in eqs. (16) and (18) by the supersymmetry Ward identity [8]

$$A_{5;1}^{\mathrm{susy}}(1_{\bar q}^-,2_q^+,3^+,4^+,5^-) = -\frac{\langle 2\,5\rangle}{\langle 1\,5\rangle} A_{5;1}^{\mathrm{susy}}(1^-,2^+,3^+,4^+,5^-)\,, \tag{23}$$

so there is actually no need to recalculate the supersymmetric parts of the amplitudes; alternatively one can use the supersymmetry identity as a non-trivial check on results, which we have performed for this amplitude. The other components of this helicity amplitude are

$$\begin{aligned}
A^{\mathrm{R}} \;&=\; c_\Gamma A^{\mathrm{tree}}\Bigg\{-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} - \frac{3}{2\epsilon}\left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} - \frac{7}{2} + \frac{\mathrm{tr}[1543]\,\mathrm{tr}[153452]}{2s_{51}^2 s_{25}}\frac{\mathrm{L}_1\left(\frac{-s_{12}}{-s_{45}}\right)}{s_{45}^2} \\
&\quad - \frac{s_{14}\,\mathrm{tr}[1452]}{s_{25}}\frac{\mathrm{Ls}_1\left(\frac{-s_{45}}{-s_{23}},\frac{-s_{51}}{-s_{23}}\right)}{s_{23}^2} - \frac{s_{13}\,\mathrm{tr}[1352]}{s_{25}}\frac{\mathrm{Ls}_1\left(\frac{-s_{12}}{-s_{45}},\frac{-s_{23}}{-s_{45}}\right)}{s_{45}^2} + \frac{\mathrm{tr}[1452]}{s_{25}}\frac{\mathrm{L}_0\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}} \\
&\quad - s_{12}\frac{\mathrm{L}_0\left(\frac{-s_{23}}{-s_{45}}\right)}{s_{45}} + \frac{\mathrm{tr}[1532]}{s_{51}}\frac{\mathrm{L}_0\left(\frac{-s_{12}}{-s_{45}}\right)}{s_{45}} + \frac{\mathrm{tr}[1542]\,\mathrm{tr}[3452]}{2s_{51}^2 s_{25} s_{45}}\Bigg\}\,, \\
A^{s} \;&=\; -\frac{c_\Gamma}{3}A^{\mathrm{tree}}\Bigg\{-2\frac{\mathrm{tr}^2[1543]\,\mathrm{tr}[3452]}{s_{51}^2 s_{25}}\frac{\mathrm{L}_2\left(\frac{-s_{45}}{-s_{12}}\right)}{s_{12}^3} + 3\frac{\mathrm{tr}[1543]\,\mathrm{tr}[3452]}{s_{51}s_{25}}\frac{\mathrm{L}_1\left(\frac{-s_{45}}{-s_{12}}\right)}{s_{12}^2} \\
&\qquad - 2\frac{\mathrm{tr}[3452]}{s_{12}s_{25}} + \frac{\mathrm{tr}[132452]}{s_{12}s_{51}s_{25}} + \frac{s_{14}\,\mathrm{tr}[1542]\,\mathrm{tr}[3452]}{s_{12}s_{51}^2 s_{25} s_{45}}\Bigg\}\,, \\
A^{f} \;&=\; c_\Gamma A^{\mathrm{tree}}\frac{\mathrm{tr}[3452]}{s_{25}}\frac{\mathrm{L}_0\left(\frac{-s_{12}}{-s_{45}}\right)}{s_{45}}\,.
\end{aligned} \tag{24}$$

(Again no ultraviolet subtraction has been performed and we have used a supersymmetry preserving regulator [33,12,18,22].) The tree amplitude for this helicity configuration is

$$A_5^{\mathrm{tree}}(1_{\bar q}^-,2_q^+,3^+,4^+,5^-) = -i\,\frac{\langle 1\,5\rangle^3\,\langle 2\,5\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\,. \tag{25}$$

Terms containing logarithms and dilogorithms may be computed most efficiently via the Cutkosky rules. The polynomial terms, however, are significantly more difficult to compute. Even though expression (24) is complicated, it should be possible to extend this result to at least one more leg via the 'collinear-unitarity bootstrap' discussed in sections 4 and 5.

7. The All-Plus Helicity Amplitudes.

The structure of $A_{n;1}(1^+, 2^+, \cdots n^+)$ is particularly simple, making it an ideal first candidate for finding an all-n expression. The all-plus helicity structure is cyclicly symmetric, and no logarithms or other functions containing branch cuts can appear. This can be seen by considering the cutting rules: the cut in a given channel is given by a phase space integral of the product of the two tree amplitudes obtained from cutting. One of these tree amplitudes will vanish for all assignments of helicities on the cut internal legs since $A_n^{\rm tree}(1^\pm, 2^+, 3^+, \ldots, n^+) = 0$, so that all cuts vanish. For the same reason, consideration of factorization on poles in the sum of three or more momenta shows that the all-plus helicity loop amplitude does not contain such multi-particle poles.

The starting point in constructing an n-point expression is the previously calculated [11] five-point one-loop helicity amplitude (15). Using eq. (12), the explicit form of the tree splitting amplitudes [27,1], $A_n^{\rm tree}(1^\pm, 2^+, \cdots, n^+) = 0$, and experimenting at small n, higher point amplitudes can be constructed by writing down general forms with only two particle-poles, and requiring that they have the correct collinear limits. Doing so, one may obtain the all n ansatz [24,25],

$$A_{n;1}(1^+, 2^+, \ldots, n^+) \;=\; -\frac{iN_p}{96\pi^2} \sum_{1\leq i_1<i_2<i_3<i_4\leq n} \frac{{\rm tr}[i_1 i_2 i_3 i_4]}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n\, 1\rangle}\,, \qquad (26)$$

which has been confirmed by a recursive calculation [26].

In massless QED, through use of recursion relations, Mahlon has demonstrated that the one-loop n-photon helicity amplitudes $A_n(\gamma_1^\pm, \gamma_2^+, \cdots, \gamma_n^+)$ vanish for $n > 4$ [29]. It is easy to argue that the collinear limits are consistent with this result, and that many more "mixed" photon-gluon amplitudes should also vanish. Charge conjugation invariance implies that photon amplitudes with an odd number of legs vanish. This also implies that the amplitude with three photons and two gluons $A_{5;1}(\gamma_1,\gamma_2,\gamma_3,g_4,g_5) = 0$, since this amplitude is proportional to the corresponding five-photon amplitude. Using the collinear behavior (12) leads one to suspect that the six-point helicity amplitudes $A_{6;1}(\gamma_1^\pm,\gamma_2^+,\gamma_3^+,g_4^+,g_5^+,g_6^+)$ and $A_{6;1}(\gamma_1^+,\gamma_2^+,\gamma_3^+,g_4^+,g_5^+,g_6^\pm)$ may vanish since both terms on the right-hand-side of the collinear limits formula (12) vanish. Continuing recursively in this way, one may surmise that all other one-loop cut-free amplitudes with three photons and additional gluons vanish. Photon-gluon amplitudes which contain cuts — those with two or more helicities of opposite sign from the rest — need not vanish by the same argument, due to the existence of functions such as (14), which have cuts but are nonsingular in every collinear limit.

To verify that the all-plus helicity amplitude with three photons and $(n-3)$ gluons vanishes, one can convert three of the gluons in expression (26) into photons. Amplitudes with r external photons and $(n-r)$ gluons have a color decomposition similar to that of the pure-gluon amplitudes, except that charge matrices are set to unity for the photon legs. This implies that partial amplitudes with photons are given by sums over the permutations of the n-gluon partial amplitude that leave the color trace factor for the $(n-r)$ "true" gluons invariant. Explicitly performing the permutation sum for three external photons, one finds that the amplitude vanishes,

$$A_{n>4}^{\rm loop}(\gamma_1^+,\gamma_2^+,\gamma_3^+,g_4^+,\ldots,g_n^+) = 0\,, \tag{27}$$

in agreement with the collinear limits argument. Since amplitudes with even more photon legs are obtained by further sums over permutations of legs, the all-plus helicity amplitudes with three or more photon legs vanish (for $n>4$) in agreement with the expectation from the collinear limits. Equation (27) has been shown to hold also in the case where one of the photon legs is of opposite helicity [31]. There is numerical evidence that it is true as well in the remaining cut-free case, when one of the gluon legs is of opposite helicity [41].

8. Amplitudes with Two Negative Helicities.

In this section amplitudes with two negative helicities (MHV amplitudes) are considered, as these contribute to next-to-leading order multi-jet cross-sections. As discussed in section 2, it is useful to decompose a QCD amplitude into supersymmetry-based pieces. In particular, the n-gluon amplitude may be split up as in eq. (8).

As the $N=4$ supersymmetric amplitude is the simplest piece of an n-gluon QCD amplitude we discuss it first. One can again use the collinear limits to construct an all-n $N=4$ MHV ansatz starting from the five-point amplitude (16). Such a construction is similar to the previous all-plus example. It turns out, however, that in the supersymmetric case the unitarity cuts provide a more powerful tool for obtaining an all-n formula for the amplitude, because the cuts fix the amplitude uniquely [20].

The cuts can be evaluated directly from the Cutkosky rules [28], which turn out to require only the Parke-Taylor (MHV) tree amplitudes (3) for their evaluation. For example for the cut where the two negative helicity gluons are on the same side of the cut is

$$\int d\mathrm{LIPS}(-\ell_1,\ell_2)\; A_{jk}^{\mathrm{tree\ MHV}}(-\ell_1,m_1,\ldots,m_2,\ell_2)\; A_{(-\ell_2)\ell_1}^{\mathrm{tree\ MHV}}(-\ell_2,m_2+1,\ldots,m_1-1,\ell_1), \tag{28}$$

where $d\mathrm{LIPS}(-\ell_1,\ell_2)$ denotes the Lorentz-invariant phase space measure. After replacing the cut propagators with ordinary propagators we obtain, for any cut and any value of n, a single cut hexagon integral to evaluate, which can be further reduced to a sum of four cut scalar box integrals. Combining the expressions for the various cuts, one obtains a function whose cuts all match the cuts of the amplitude. The result for the one loop $N=4$ MHV amplitudes with an arbitrary number of external legs are schematically depicted in fig. 2 in terms of box integral functions. The explicit expression, together with a more detailed derivation, may be found in ref. [20].

$$-\frac{A_{\mathrm{tree}}}{2}\sum\Big[(P_1+P_2)^2(P_4+P_1)^2-P_1^2P_3^2\Big]\bullet$$

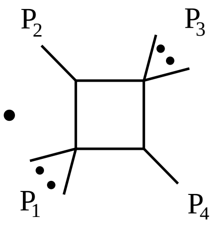

Fig. 2: A schematic representation of the $N=4$ supersymmetry MHV amplitudes.

The contributions of an $N=1$ chiral multiplet to MHV amplitudes have also been constructed via their cuts [30]. The explicit form where the two negative helicity legs are nearest neighbors is given by

$$
\begin{aligned}
A_{n;1}^{N=1\ \mathrm{susy}}(1^-,2^-,3^+,\cdots,n^+) = \\
& c_\Gamma A_n^{\mathrm{tree}} \left\{ \frac{1}{2\epsilon(1-2\epsilon)} \left[\left(\frac{\mu^2}{-s_{23}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{n1}} \right)^\epsilon \right] \right. \\
& \left. + \frac{1}{2 s_{12}} \sum_{m=2}^{n-3} \frac{\mathrm{L}_0\left(\frac{-t_2^{[m]}}{-t_2^{[m+1]}} \right)}{t_2^{[m+1]}} (\mathrm{tr}_+[\not k_1 \not k_2 \not k_{m+2} \not q_m] - \mathrm{tr}_+[\not k_1 \not k_2 \not q_m \not k_{m+2}]) \right\} ,
\end{aligned}
\tag{29}
$$

where $q_m = \sum_{j=2}^{m+1} k_j$, and $t_2^{[m]} = q_m^2$. As a check, we have confirmed that this expression has all the correct collinear limits.

The $N=1$ and $N=4$ amplitudes are two parts of the supersymmetry-based decomposition of the QCD gluon amplitude; the third piece is the contribution of a scalar in the loop, which tends to be the most complicated of the three. The terms in the scalar contribution containing logarithms and dilogarithms can of course be computed using the unitarity cuts, but there are additional 'polynomial' terms which cannot be determined in this way; however, they should be amenable either to the 'collinear bootstrap' technique [25,20] or to a recursive technique [29,26].

For supersymmetric theories, supersymmetry Ward identities can be used to generate amplitudes with external fermions from the n-gluon amplitudes presented above, in the same way as discussed in section 6 for five-gluon amplitudes.

9. Prospects for Two Loops.

As an exercise, it is not difficult to calculate the cuts in the two-loop leading-color amplitude $A_4^{\mathrm{two-loop}}(1^+,2^+,3^+,4^+)$. The four non-vanishing cuts are depicted in fig. 3. The three-particle cuts vanish since one side of the cut will always contain a tree which vanishes by eq. (2), making the absorptive parts particularly easy to evaluate. Another simplifying feature is the absence of logarithms in the one-loop amplitude

$$
A_{4;1}^{\mathrm{one-loop}}(1^+,2^+,3^+,4^+) = \frac{i}{48\pi^2} \frac{st}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle} , \tag{30}
$$

where $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$ are the usual Mandelstam variables. This means that the two-loop cuts can be evaluated entirely in terms of one-loop integrals. The result for the pure glue contributions to the four-gluon leading-color partial amplitude is

$$A_4^{\rm two-loop}(1^+,2^+,3^+,4^+) \sim -c_\Gamma(\mu^2)^\epsilon A_{4;1}^{\rm one-loop} \frac{2}{\epsilon^2}\Big((-s)^{-\epsilon}+(-t)^{-\epsilon}\Big) + \text{polynomial}\,. \tag{31}$$

The form of the $1/\epsilon^2$ terms can also be determined from the requirement that the singularities cancel against singularities encountered in phase-space integrals of physical cross-sections. The cuts show that no further logarithms are present in the amplitude. The polynomial terms however cannot be determined by unitarity, nor can they be determined from collinear limits since there is no lower-point amplitude from which to extrapolate.

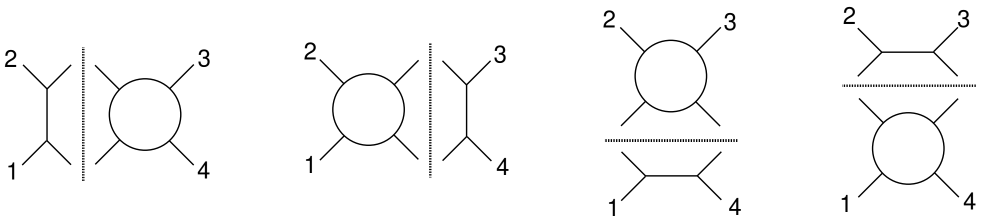

Fig. 3: The non-vanishing cuts in the two loop amplitude.

The polynomial terms, and more generally other two-loop amplitudes, await further advances in calculational technology. One must construct a two-loop integral table; and further improvements in the formalism will also be important. Two possible avenues of approach are either string-based [42] or recursive techniques [29,26]. We are hopeful that two-loop calculations will be made possible through extensions of one-loop techniques.

10. Summary and Conclusions.

In this talk we presented techniques for obtaining new one-loop amplitudes from known ones. From the collinear (or soft) limits [27] one constructs higher amplitudes by demanding the correct collinear behavior [2,25]. From the Cutkosky rules [28] one can use known tree amplitudes to generate terms in one-loop amplitudes containing cuts. In certain cases, such as supersymmetric amplitudes, the cuts uniquely determine the amplitude [20,30]. Finally, supersymmetry Ward identities [8,9] can be used to obtain supersymmetric parts of one-loop amplitudes [22] with external fermions, using known gluon amplitudes.

Using these techniques, we have obtained a variety of exact one-loop results for amplitudes. In particular, we have constructed, using unitarity constraints, one-loop supersymmetric n-gluon amplitudes with maximal helicity violation. For these amplitudes the supersymmetry Ward identities can be immediately used to obtain amplitudes with two external fermions and additional gluons. We also presented a five-point QCD example which illustrates the use of supersymmetry to obtain parts of amplitudes with external fermions from gluon amplitudes. The supersymmetric amplitudes may be interpreted as two of three terms which make up a QCD gluonic amplitude, following the organization suggested by the string-based method [11,18,19]. The third piece is in general more complicated and cannot be obtained by unitarity alone. For example, in the all-plus helicity amplitudes the cuts are trivial (they all vanish) and it is the use of collinear limits that allowed us to give an ansatz for the amplitude [24,25], which was later proven correct [26]. All five-gluon amplitudes have been computed [11,23], and thus provide a starting point for ansätze for the polynomial terms in higher-point amplitudes, relying on the constraints imposed by the collinear limits. (Ambiguities in the collinear bootstrap make it problematic to start from four-point amplitudes.) In general, the restrictions imposed by collinear behavior and unitarity complement each other.

In summary, we have constructed certain classes of one-loop amplitudes with an arbitrary number of external gluons through the reliance on the dual constraints of unitarity and collinear behavior. We expect that this method will generate further fixed-n and all-n amplitudes.

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