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Two-Dimensional Quantum Cosmology*

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ABSTRACT

Two-dimensional quantum gravity coupled to conformally invariant matter with central charge $c > 25$ has been proposed as a toy model for quantum gravity in higher dimensions. The associated "Wheeler-DeWitt equation" is non-linear and unstable to forming a condensate of baby universes. This will occur even in the classical $c \to \infty$ limit. Small fluctuations about this background describe the propagation of single universes and satisfy a more conventional linear Wheeler-DeWitt equation. The resulting two-dimensional cosmology depends on details of the non-linear dynamics. In particular the existence of a large scale cosmological constant is determined by the behavior of a string theoretic tachyon potential near its minimum.

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1. Introduction

This is a revised and expanded version of our previous preprint entitled "The Semi-Classical Limit of Quantum Gravity Isn't" (SLAC-PUB-5413). The new material in Sections 7 and 8 provides an alternative possibility to our previous conclusion that a non-zero conventional cosmological constant in two-dimensional gravity is inconsistent with a large scale continuum geometry.

According to the traditional view of quantum cosmology, the universe is governed by a linear Wheeler-DeWitt equation [1]

$$H_{WD}\,|\Psi\rangle = 0. \tag{1.1}$$

The exact form of H_{WD} is determined by the field-theoretic Lagrangian which includes both gravitational and matter degrees of freedom. The form of this Lagrangian is constrained only by the principle of general covariance. In recent years this conventional view has been challenged and a far richer picture proposed, in which the Wheeler-DeWitt equation is itself quantized and non-linear [2-12]. The number of universes is indefinite, being subject to the rules of "third quantization", and the laws of nature are given by dynamically determined condensates of universes. In particular, it has been argued that the emission and absorption of baby universes shifts coupling constants, and even forces the cosmological constant to zero [13-17].

In this paper we shall analyze these issues in the relatively simple context of two-dimensional gravity coupled to conformally invariant matter. This theory has received a lot of attention in recent years, both due to its intrinsic interest, and because of its application to non-critical strings. Most of the effort has been focused on coupling gravity to matter theories with central charge $D \leq 1$, using either continuum field theory or a discrete description in terms of matrix models of random surfaces. However, the dynamics more closely resembles higher dimensional gravity when one studies non-critical string theory with $D > 25$ [18,19]. In that case, the kinetic energy associated with the scale factor of the metric is negative, just as it is in higher dimensions, and familiar cosmological solutions (such as de Sitter space with positive cosmological constant) are obtained. Two-dimensional quantum cosmology, with $D > 25$, is identical to string theory in a Lorentzian target-space with background fields. The non-linear Wheeler-DeWitt equation is the condition that the string theory beta-functions vanish, and the behavior of the cosmological constant is replaced by the dynamics of the "tachyon field" in target-space.† Our main result is that the non-linear terms of the Wheeler-DeWitt equation inevitably become important, even in the classical limit. This will correspond to the universe interacting with a background of "baby universes". We then address the question of whether the condensate shifts the cosmological constant to zero as suggested by Coleman [13].‡ This appears to be the case if the tachyon potential has a smooth minimum. On the other hand, a very mild singularity at the potential minimum leads to a large scale behavior with arbitrary cosmological constant. Recent matrix model results of Moore *et al.* [21] vindicate the latter possibility.

The outline of the paper is as follows. In Section 2 we briefly review how the matrix model description of two-dimensional gravity leads to an essentially non-linear Wheeler-DeWitt equation. In Section 3 we describe the continuum formulation of two-dimensional gravity in the conformal gauge, and show how it leads to an effective action for the target-space fields. In Section 4 we consider the cosmologically interesting case of $D > 25$, and derive the non-linear Wheeler-DeWitt equation from the effective action. In Section 5 we consider the evolution of couplings with scale, and examine the relation between the target-space equations of motion and the renormalization group. In Sections 6 and 7 we discuss whether these ideas can lead to the vanishing of the cosmological constant at large scale. In Section 8 we give an explicit calculation of the tachyon beta-function using two

† This approach to two-dimensional quantum cosmology was discussed by Banks in [20].
‡ The Euclidean saddle point, of Baum [15], Hawking [16] and Coleman [13], has an analog in the two-dimensional theory for $D > 25$ [19].

different renormalization procedures. We conclude with speculations about how these ideas may carry over into four-dimensional quantum gravity.

2. Matrix Models and the Non-linear Wheeler-DeWitt Equation

In this section we briefly review the derivation of the non-linear Wheeler-DeWitt equation from a matrix model of discrete quantum gravity [22]. In particular, we consider the single matrix model described by the path integral

$$Z = \frac{1}{N^2} \int dM^{N^2} e^{-N \operatorname{tr} V(M)} , \tag{2.1}$$

where M is a Hermitian $N \times N$ matrix. Perturbation theory generates a set of Feynman diagrams with vertices depending on V. Diagrams of genus h are weighted by a factor of N^{-2h}, so in the $N \to \infty$ limit the surviving graphs have the topology of a sphere. By considering the dual graphs, (2.1) can be identified with a sum over discrete approximations to Euclidean signature two-dimensional geometries, and in the continuum limit this can be thought of as a quantization of Euclidean gravity.

Consider next the discretization of surfaces with a boundary of length l. The sum over such geometries defines a Wheeler-DeWitt amplitude $Z(l)$ which is given by

$$Z(l) = \frac{g^{1+\frac{l}{2}}}{N^2} \int dM^{N^2}\, N \operatorname{tr} M^l\, e^{-N \operatorname{tr} V(M)} , \tag{2.2}$$

where we have chosen for illustration $V(M) = \frac{1}{2}M^2 + gM^4$. Similarly, the amplitudes for geometries with m boundaries of lengths $l_1, l_2, \ldots, l_m$ are given by

$$Z(l_1, l_2, \ldots, l_m) = \frac{g^{m+\frac{1}{2}\sum l_i}}{N^2} \int dM^{N^2}\, N \operatorname{tr} M^{l_1} \ldots N \operatorname{tr} M^{l_m}\, e^{-N \operatorname{tr} V(M)} . \tag{2.3}$$

It is a simple matter to derive the Schwinger-Dyson equations corresponding to a shift of M [22,23]. The simplest such equation is

$$Z(l+1) + 4\,Z(l+3) = \frac{1}{N^2}\sum_{p=0}^{l-1} Z(p,l-1-p)\,. \tag{2.4}$$

In the large N limit, $Z(l_1,l_2) \sim N^2 Z(l_1)Z(l_2)$ and (2.4) reduces to the non-linear loop equation

$$Z(l+1) + 4\,Z(l+3) = \sum_{p=0}^{l-1} Z(p)Z(l-1-p)\,, \tag{2.5}$$

with boundary condition $Z(0) = g$. The non-linearity of this equation has a natural interpretation in terms of one-dimensional universes, or strings, splitting and joining. Notice that the string coupling constant $1/N$ does not appear in (2.5), so there is no limit in which the non-linear terms can be ignored. Furthermore, the parameter g which controls the two-dimensional cosmological constant appears as an initial condition in (2.5). A linear equation for fluctuations in Z, or equivalently for variations of Z with respect to g, can easily be derived from (2.5). It is these fluctuations which should be identified with the conventional Wheeler-DeWitt amplitudes of the single universe theory. Although the linearized version of (2.5) does not look like a conventional Wheeler-DeWitt equation, Moore *et al.* [21] have in fact shown that the continuum limit describes the same physics.

In the remainder of the paper we will be considering the continuum theory. The above calculations were presented in order to emphasize from another viewpoint the importance of non-linear effects in the Wheeler-DeWitt equation.

3. Two-dimensional Quantum Gravity and String Theory

We shall now review the continuum formulation of non-critical string theory. This will serve to fix our notation and make contact with previous work. For the moment, assume that both the world-sheet (σ^0, σ^1) and the space of matter fields

X^i, $i = 1, \ldots, D$ are Euclidean, and that $D < 25$. The action is

$$S = \frac{1}{8\pi} \int d^2\sigma \sqrt{\gamma} \left\{ \gamma^{ab} \partial_a X \cdot \partial_b X + \lambda_0 \right\} , \tag{3.1}$$

where γ_{ab} is the two-dimensional metric and λ_0 is the bare cosmological constant.

To carry out the path integral over metrics γ_{ab} and matter fields X^i, the following steps are taken:

- Gauge fixing: The over-counting of metrics due to general coordinate invariance is removed by introducing an arbitrary background, or fiducial, metric $\hat{\gamma}_{ab}(\sigma)$ and defining a *conformal gauge*,

$$\gamma_{ab} = e^{\phi}\, \hat{\gamma}_{ab} . \tag{3.2}$$

  The remaining degree of freedom, ϕ, is called the Liouville field or conformal mode. The path integral over metrics reduces to an integral over ϕ, with a ϕ-dependent Faddeev-Popov determinant, which has the familiar Liouville form [24],

$$S_L = \frac{26-D}{96\pi} \int d^2\sigma \sqrt{\hat{\gamma}} \left\{ \hat{\gamma}^{ab} \partial_a \phi \partial_b \phi + 2\hat{R}\phi + \lambda e^{\phi} \right\} . \tag{3.3}$$

  Note that the couplings of the conformal mode in the Liouville action are exactly those of a string matter field in a non-trivial background. This is the first indication of the close parallel between ϕ and the X^i that we will elaborate on in due course.

- Regularization: The theory has ultra-violet divergences which need to be regularized by introducing a cutoff, or shortest distance. The cutoff scale is defined with reference to the fiducial metric, $\hat{\gamma}_{ab}$, rather than the original metric, γ_{ab}. Therefore the regularized theory is *not* manifestly covariant.

- Renormalization: In order to define the theory at some size scale, we have to integrate out both gravitational and matter field fluctuations on smaller scales. The result is an effective Lagrangian, which does not necessarily look covariant, since the renormalization procedure is to be carried out with reference to the fiducial metric. In particular one might have quite large geometries, as measured in the original metric, γ_{ab}, which nevertheless appear as short distance fluctuations on a scale set by $\hat{\gamma}_{ab}$. An example of this is illustrated in Figure 1.

The requirement that the original theory be covariant can be stated as a set of conditions, that the path integral does not depend on our choice of fiducial metric, $\hat{\gamma}_{ab}$. These turn out to place quite strong restrictions on the allowed couplings of the fields, X^i and ϕ. In particular, the theory must be reparametrization invariant with respect to $\hat{\gamma}_{ab}$, and if we consider variations of the conformal part of $\hat{\gamma}_{ab}$, *i.e.* of $\det \hat{\gamma}$, we find that the path integral has to be conformally invariant, which in turn implies that beta-functions of all couplings must vanish.

All this can be summarized as follows. We start with a generally covariant theory of gravity coupled to scalar fields, X^i. In order to define the path integral we fix a gauge and regularize in a non-covariant manner. The resulting theory involves a scalar field, ϕ, in addition to the matter fields, and is in general quite complicated. The original covariance appears as a set of restrictions on the couplings, which include the requirement that all the beta-functions vanish. Notice that in this way of stating things ϕ and X^i are placed on equal footing. The Liouville field has been promoted to an additional target-space dimension. This approach to the quantization of two-dimensional gravity has been advocated by a number of authors [19,20,25,26,27].

The object of interest is thus some reparametrization invariant scalar field theory in two dimensions, with *a priori* quite general couplings,

$$S = \frac{1}{8\pi} \int d^2\sigma \, \sqrt{\hat{\gamma}} \left\{ T(X) + \hat{\gamma}^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + 2\hat{R}\, \Phi(X) + \cdots \right\}. \quad (3.4)$$

There are $D+1$ scalar fields X^μ, including both the matter fields, X^i, and the Liouville mode. In order to have a standard kinetic term, the Liouville field has been rescaled to $X^0 = \frac{Q}{2}\phi$, with $Q^2 = \frac{25-D}{3}$. We have only written down the terms of scaling dimensions zero and two,* but there is an infinite sequence of possible couplings involving more derivatives on the X^μ and higher powers of the two-dimensional curvature $\hat{R}$.

This class of theories has been extensively investigated in string theory, where the action (3.4) describes strings in background fields in $D+1$ spacetime dimensions. The beta-function equations, implementing the conformal invariance of the two-dimensional theory, have the form of field equations in target-space for $T(X)$, $\Phi(X)$ and $G_{\mu\nu}(X)$ (tachyon, dilaton and graviton fields respectively), along with additional fields representing higher order couplings. These field equations describe the propagation and creation and annihilation of the particle-like eigenmodes of strings in spacetime, or more to the point of this paper, one-dimensional universes containing matter fields.† The tachyon field, $T(X)$, is of primary interest because it controls the two-dimensional cosmological constant. To see this, note that the cosmological term in the Liouville action (3.3) corresponds to a particular tachyon background in target-space,

$$T(X) = \lambda\, e^{\frac{2}{Q}X^0}\ . \tag{3.5}$$

The string theory equations of motion, obtained by setting beta-functions to zero, are derivable from an action. For simplicity, we will work within a truncated theory, containing only the lowest order couplings, $T(X)$, $\Phi(X)$ and $G_{\mu\nu}(X)$. To leading order in derivatives, the target-space action for these fields is [28]

$$I = \frac{-1}{2g_0^2}\int d^{D+1}X\sqrt{G}\, e^{-2\Phi}\left\{\frac{25-D}{3} + R + 4(\nabla\Phi)^2 - (\nabla T)^2 - 2V(T) + \cdots\right\}, \tag{3.6}$$

where $V(T) = -T^2 + \frac{1}{24}T^3 + \ldots$ is the tachyon effective potential. The general

* For simplicity, we have not included the anti-symmetric tensor field. Its presence would not qualitatively alter our conclusions.

† We will use the string theory names for the target-space fields, but the reader should keep in mind their cosmological interpretation.

form of $V(T)$ is not known, but we show how to obtain the leading terms from two-dimensional renormalization theory in Section 8. Our answer is, of course, not universal because renormalization group beta-functions always depend on the regularization procedure used. This prescription dependence is believed to correspond to field redefinition ambiguities in target space. In fact all higher order terms in the beta-function can be arranged to involve target space derivatives, and therefore be removed from the potential leaving only $-T^2$ [33,34]. On the other hand, the object defined in this way will probably not correspond to the Wheeler-DeWitt amplitude of a one-dimensional universe. We will return to this point in Sections 7 and 8. The equations of motion which follow from the above action are

$$\begin{aligned} \nabla^2 T - 2\nabla\Phi\cdot\nabla T &= V'(T)\,, \\ \nabla^2\Phi - 2(\nabla\Phi)^2 &= -\frac{25-D}{6} + V(T)\,, \\ R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R &= -2\nabla_\mu\nabla_\nu\Phi + G_{\mu\nu}\nabla^2\Phi + \nabla_\mu T\nabla_\nu T - \frac{1}{2}G_{\mu\nu}(\nabla T)^2\,. \end{aligned} \tag{3.7}$$

For $D < 25$ these equations have a simple solution, which is known to be exact even when higher order terms in the beta-functions are included [29],

$$\begin{aligned} T &= 0\,, \\ G_{\mu\nu} &= \delta_{\mu\nu}\,, \\ \Phi &= \frac{Q}{2}X^0\,. \end{aligned} \tag{3.8}$$

An important feature of this linear dilaton background is that the strength of the string loop coupling constant is related to the two-dimensional scale,

$$g = g_0\, e^{\Phi} = g_0\, e^{\frac{Q}{2}X^0}\,. \tag{3.9}$$

We can only expect the effective field theory to be simple where this coupling is weak.

A background tachyon field can be added to the exact solution (3.8). Its beta-function equation depends on the shape of the effective potential, $V(T)$, and is non-linear in general. If we assume that the background field is weak, and only depends on X^0, the equation can be linearized as follows,

$$\partial_0^2 T - Q\partial_0 T + 2T = 0\,, \tag{3.10}$$

and has solutions

$$T(X^0) = \lambda\, e^{(\frac{Q}{2}\mp\sqrt{\frac{Q^2}{4}-2})X^0}\,. \tag{3.11}$$

Such a homogeneous background configuration corresponds precisely to the two-dimensional theory discussed by David [30] and by Distler and Kawai [31].* In the weak-coupling regime, $X^0 \to -\infty$, the solution with the negative sign in the exponent is more important to the physics, since the other one is more rapidly damped. Note also that in the $D \to -\infty$ semi-classical limit of the Liouville theory this choice of sign makes the cosmological term (3.11) reduce to $T = \lambda e^{\frac{2}{Q}X^0} = \lambda e^{\phi}$, as it should.

An important feature of the $D < 25$ theory is that the coupling $g_0 e^{\Phi}$ governing the strength of quantum corrections (string loops) tends to zero when $X^0 \to -\infty$. In this limit strings are metrically small. Furthermore, we can consider a background solution which tends to $T = 0$ as $X^0 \to -\infty$. In the vicinity of $T = 0$ the non-linear terms in the classical target-space equations disappear. On the other hand, as $X^0 \to \infty$, the limit of large scale factor, the interaction strength increases. It has been argued by Polchinski [32] that this leads to a reflecting wall which prevents the particle-like strings from penetrating to large sizes.

* Note that our X^0 differs, by a sign, from the rescaled Liouville field of Distler and Kawai.

4. Quantum Cosmology in Two Dimensions

We have so far been concentrating on theories with $D < 25$.† Let us now follow Polchinski [19] and consider the case $D > 25$, which is more relevant to cosmology. The derivation of the target-space equations (3.7) does not depend on the value of D. On the other hand, the nature of their solutions qualitatively changes when D becomes greater than 25. In particular, if $G_{\mu\nu}$ has Euclidean signature, a linear dilaton background will no longer be a solution for $D > 25$. However, there exists a solution analogous to (3.8) if the target-space is Lorentzian. It is given by

$$\begin{aligned} T &= 0\,, \\ G_{\mu\nu} &= \eta_{\mu\nu}\,, \\ \Phi &= -\frac{q}{2} X^0\,, \end{aligned} \tag{4.1}$$

where $q^2 = \frac{D-25}{3}$ and the conformal mode, X^0, is time-like.‡ Strictly speaking, it is not consistent to treat the world-sheet as Euclidean if the target-space is Lorentzian. The reason is that the two-dimensional action is then unbounded, since the X^0 kinetic term has opposite sign to that of X^i. We should therefore reformulate the two-dimensional field theory on a world-sheet of Lorentzian signature in order to discuss the $D > 25$ case. Unfortunately it is not at all clear how to perform the steps involved in the quantization of such a theory (regularization, renormalization, *etc.*). Our working assumption, which may be unwarranted, is that a consistent Lorentzian formulation will lead to the same covariant target-space equations as the formal Euclidean calculation. In interpreting the equations of motion as a Lorentzian field theory of strings, the two-dimensional path integral implicitly includes geometries which describe splitting and joining. Such configurations inevitably include well-defined events at which the universe bifurcates and the

† Actually $D \leq 1$, since the exponent in (3.11) is complex for $D > 1$, and the two-dimensional action is unbounded from below.

‡ We are free to choose the sign of the dilaton background. This corresponds, in fact, to the choice of the direction of time. The physics just tells us that small universes are weakly coupled for $D < 25$ and strongly coupled for $D > 25$. Our solutions (3.8) and (4.1) reflect the convention that the universe was small at *early* times.

two-dimensional metric is singular. These will be observable, or even catastrophic, for one-dimensional observers. In addition, the path integral receives contributions from universes being absorbed or emitted from the background, which also involve two-dimensional singularities (see Figure 2). By contrast, in Euclidean space the metric can be chosen with no singularities. It should be noted that we only use Euclidean methods to compute renormalization group beta-functions, but our subsequent discussion of the cosmology takes place with Lorentzian signature.

The gravitational coupling in two-dimensions is dimensionless, so there is no proper Planck-scale. However, in the case $D > 25$ the coupling strength $g_0 e^{\Phi}$ increases as $X^0 \to -\infty$ so that the theory is strongly coupled for sufficiently small strings. No longer can quantum mechanics (string loops) be ignored in the ultraviolet. One can say that a Planck-scale is spontaneously induced, and define it by the point at which $g_0 e^{\Phi} = 1$. The factor of g_0 can be absorbed by a constant shift of the dilaton. The effective Planck-scale is then set by q^{-1}, and depends on the number of scalar fields in the theory. In particular, $D \to \infty$ is a semi-classical limit for gravitational fluctuations. The question of initial conditions is complicated because the theory is strongly coupled at early times. The short distance physics is summarized by some unknown initial state at the Planck-time, which then evolves in the weakly coupled theory. In a quantum theory this means a wave-function in target-space and in a classical theory it corresponds to initial conditions on the target-space fields.

In the background (4.1) the linearized equation for a homogeneous tachyon field reads

$$-\partial_0^2 T - q\partial_0 T + 2T = 0\,, \tag{4.2}$$

and is solved by

$$T(X^0) = \lambda\, e^{(-\frac{q}{2} \pm \sqrt{\frac{q^2}{4}+2})X^0}\,. \tag{4.3}$$

One of the solutions decays with time, but the other one grows exponentially. The system is unstable and is likely to form a condensate of background tachyons.

Now consider fluctuations of some field in the exponentially growing tachyon background, $T_B(X^0) = \lambda\, e^{(-\frac{q}{2}+\sqrt{\frac{q^2}{4}+2})X^0}$. Take, for example, a tachyon with some non-zero space-like momentum k. The target-space action (3.6) is not time-translation invariant. In order to describe physical fluctuations it is convenient to absorb the $e^{-2\Phi}$ pre-factor by a field redefinition, which has the form

$$U(X) = e^{-\Phi(X)}\, T(X) \tag{4.4}$$

for tachyons. A fluctuation $U_k(X) = U_k(X_0)\, e^{ik_iX^i}$ satisfies a linear equation,

$$\partial_0^2 U_k + (k^2 + V''(T_B) - \frac{q^2}{4})U_k = 0\,. \tag{4.5}$$

Near the top of the potential the tachyon background is well approximated by the exponential form (4.3) and we can drop the contribution of all but the leading terms of the potential $V(T) = -T^2 + \frac{1}{24}T^3 + \ldots$ in the fluctuation equation, whereupon (4.5) becomes

$$\partial_0^2 U_k + (k^2{-}2{-}\frac{q^2}{4})U_k + \frac{1}{4}\lambda e^{(-\frac{q}{2}+\sqrt{\frac{q^2}{4}+2})X^0}\, U_k \,=\, 0. \tag{4.6}$$

This has the form of a Wheeler-DeWitt equation for a universe with some matter excitation. To see that, define a scale factor $a = e^{\frac{\gamma}{2}X^0}$, where $\gamma = -\frac{q}{2} + \sqrt{\frac{q^2}{4}+2}^{\,\star}$, and reexpress (4.6) in terms of a,

$$\Big\{(\frac{\gamma}{2}a\frac{\partial}{\partial a})^2 + (k^2{-}2{-}\frac{q^2}{4}) + \frac{1}{4}\lambda a^2\Big\}U_k \,=\, 0. \tag{4.7}$$

Up to factor-ordering ambiguities, this is the Wheeler-DeWitt equation derived from the mini-superspace Lagrangian of two-dimensional gravity,

$$L = -(\frac{\dot a}{a})^2 - \frac{1}{\gamma^2}\big[k^2 - (2{+}\frac{q^2}{4}) + \frac{1}{4}\lambda a^2\big]\,. \tag{4.8}$$

The three terms in square brackets are the matter, curvature and cosmological constant energy densities.

$\star$ The scale factor reduces to its classical value $a = e^{\frac{1}{q}X^0} = e^{\phi/2}$ in the $q \to \infty$ limit.

It seems that we have recovered a more or less conventional Wheeler-DeWitt description of large scale cosmology. In particular, the problem of the cosmological constant is the usual one. In order to obtain vanishing cosmological constant, the exponentially increasing solution for $T(X^0)$ must be fine-tuned to zero. In other words, the tachyon must be delicately balanced at the top of the potential. We are ignorant about the short distance physics, which is supposed to determine the initial state, so we have no way of gauging how likely it is to find the system balanced at the top of this potential. At any rate, such a fine-tuned initial state is not allowed in a quantum theory, because of the uncertainty principle.

However, this is not the whole story. Even if the tachyon background starts out near the top of the potential it will eventually roll into the region where the higher-order non-linear terms in $V(T)$ cannot be ignored. At that point it is entirely possible that the linear Wheeler-DeWitt equation for fluctuations will deviate from (4.7). In particular, the cosmological term could be modified in such a way that it grows less rapidly than a^2, causing one-dimensional observers to measure a decreasing "cosmological constant" as their universe expands. One such cosmological model will be presented in the following section. On the other hand, as we shall see in Section 6, an entirely conventional cosmological constant can appear in the linearized Wheeler-DeWitt equation at all scales if the tachyon potential has a particular form.

As we have already mentioned, different renormalization prescriptions in the two-dimensional theory will lead to different evolutions for the tachyon background. Since the $-T^2$ term in $V(T)$ is universal the different schemes will all agree near $T = 0$, but away from the origin they can present very different pictures. For example, the question of whether $V(T)$ has a minimum is scheme-dependent. The key issue here is to identify the definition of the tachyon field most closely corresponding to Wheeler-DeWitt amplitudes in the two-dimensional cosmology. This is a non-trivial task which has not been carried out, but we will outline a promising candidate scheme at the end of Section 8.

It should be emphasized that the non-linear effects that we are talking about do not disappear in the semi-classical limit $D \to \infty$. In particular, the splitting and joining events described by the non-linear terms of the target-space equations are unsuppressed even at late times. This may seem surprising because the string coupling is becoming weak, with $e^{\Phi} = e^{-\frac{q}{2}X^0}$. Indeed, the canonical tachyon field $U(X^0)$ defined in (4.4) satisfies

$$[\nabla^2 + (2 + \frac{q^2}{4})]U = \frac{1}{8} e^{-\frac{q}{2}X^0} U^2 \, . \tag{4.9}$$

As we move toward the semi-classical limit $q \to \infty$, though, the tachyon mass squared increases as $\frac{q^2}{4}$, so that the unstable exponential growth of U compensates the decreasing coupling strength.

Since very little is known about higher order non-linearities in string theory, we can only speculate about their detailed effect on the physics. However, the very existence of string interactions, along with the tachyon instability, shows that the usual Liouville model described by an exponentially growing tachyon background is not the complete theory.

5. The Running of Coupling Constants

Before delving further into the two-dimensional cosmology, we shall clarify the connection between the target space equations of motion and the renormalization group flow of couplings in the two-dimensional field theory.

The equations of motion for the target-space fields are that the beta-functions of all two-dimensional couplings vanish. From this one might conclude that the couplings seen by a two-dimensional observer would not run. This, however, is not the correct interpretation. We can think of the equations for the target-space fields as renormalization group equations with $\frac{\gamma}{2}X^0$ identified with the logarithm of the renormalization scale. The X^0 dependence of the coupling functions $T, \Phi, G_{\mu\nu} \ldots$ hence determines their evolution with scale.

This connection may appear unfamiliar because the equations of motion are second-order in X^0 derivatives, whereas the usual renormalization flows are controlled by first-order equations. The second-order nature of the flows is a special feature of theories containing gravity where the scale itself is a dynamical variable.

The situation is similar to the issue of time evolution in the Wheeler-DeWitt formulation of quantum gravity. We begin with an equation $H_{WD}\,|\Psi\rangle = 0$ which seems to imply that no time evolution occurs. Reinterpreted, though, the equation tells us how the wave function of matter evolves with the expansion of the universe. The Wheeler-DeWitt equation, like the equations of motion for $T, \Phi, G_{\mu\nu}$, is second-order. The first-order Schrödinger equation is only recovered in a semi-classical limit in which gravitational fluctuations become unimportant [36]. In our two-dimensional theory, this semi-classical limit corresponds to taking $D \longrightarrow \infty$ or equivalently $q \rightarrow \infty$. In this limit we will see how the target-space field equations reduce to the familiar renormalization group equations.

We consider first the case of fluctuations about the linear dilaton background at the top of the tachyon potential with a *flat* target-space metric. A field A_n at the $n^{\rm th}$ mass level in string theory will contribute to the effective action a term

$$\frac{1}{2g_0^2}\int \sqrt{G}e^{-2\Phi}\left\{(\nabla A_n)^2 - 2(1-n)A_n^2 + \cdots\right\} . \tag{5.1}$$

Its equation of motion in a linear dilaton background is

$$\left[\left(\frac{\partial}{\partial X^0}\right)^2 + q\,\frac{\partial}{\partial X^0} + k^2 - 2(1-n)\right] A_n = 0 \,. \tag{5.2}$$

As we saw previously, this equation has unstable solutions for $n = 0$, which describe the tachyon rolling off the top of its potential. Note, however, that the solutions for $n \geq 1$ are stable for all values of q. In other words, the dilaton, graviton and higher couplings do not become "tachyonic" for large D.

Now, recalling that the scale factor is $a = e^{\frac{\gamma}{2}X^0}$, we can rewrite (5.2) as

$$\left[\frac{\gamma^2}{4}\left(a\frac{\partial}{\partial a}\right)^2 + \frac{\gamma q}{2} a\frac{\partial}{\partial a} + k^2 - 2(1-n)\right] A_n = 0\,. \tag{5.3}$$

Thus when $q \to \infty$ we find the first-order equation

$$a\frac{\partial}{\partial a} A_n = -\big(k^2 + 2(n-1)\big)\, A_n\,. \tag{5.4}$$

This is the usual lowest order Callan-Symanzik equation for a coupling of bare dimension $2n$. In particular the field $h_{\mu\nu}$ has anomalous dimension $-k^2$ as expected.

We have so far been considering a trivial matter sector, consisting of several free fields. A more stringent test of the above ideas should involve an interacting matter sector such as an asymptotically free sigma-model coupled to gravity. Experience in flat space indicates that such a model will generate a new mass-scale and renormalize the vacuum energy accordingly. To investigate this, we consider the example of a theory in which three of the target-space dimensions are compactified to a sphere of time-dependent radius $r(X^0)$. The remaining $D-3$ spatial coordinates are left flat. The sigma-model coupling constant is then $1/r$, and the X^0 dependence of $r(X^0)$ is just the running of the coupling with scale. We find that in the semi-classical limit the equations of motion (3.7) reproduce the standard renormalization group flow

$$\frac{d(\frac{1}{r})}{dX^0} \propto \left(\frac{1}{r}\right)^3\,. \tag{5.5}$$

To see this we insert into (3.7) the metric

$$ds^2 = -(dX^0)^2 + r(X^0)^2\, d\Omega_3^2 + \sum_4^D (dX^i)^2 \tag{5.6}$$

where $d\Omega_3^2$ is the line element on a unit three-sphere. The equations of motion then become

$$
\begin{aligned}
-\ddot{T} - 3\frac{\dot{r}}{r}\dot{T} + 2\dot{\Phi}\dot{T} =& V'(T)\,, \\
-\ddot{\Phi} - 3\frac{\dot{r}}{r}\dot{\Phi} + 2\dot{\Phi}^2 =& \frac{q^2}{2} + V(T)\,, \\
3\frac{\dot{r}^2}{r^2} + 3\frac{1}{r^2} =& -\ddot{\Phi} + 3\frac{\dot{r}}{r}\dot{\Phi} + \frac{1}{2}\dot{T}^2\,, \\
-2\frac{\ddot{r}}{r} - \frac{\dot{r}^2}{r^2} - \frac{1}{r^2} =& -\ddot{\Phi} - \frac{\dot{r}}{r}\dot{\Phi} + \frac{1}{2}\dot{T}^2\,.
\end{aligned}
\tag{5.7}
$$

Taking the semi-classical $q \to \infty$ limit, we find that to leading order these have solution

$$
\begin{aligned}
T =& 0\,, \\
\Phi =& -\frac{q}{2}X^0\,, \\
r =& \sqrt{\frac{4}{q}(c - X^0)}\,,
\end{aligned}
\tag{5.8}
$$

where c is an integration constant corresponding to the induced dynamical mass scale determined by initial conditions. We see that this solution indeed satisfies the renormalization group equation (5.5).

The equations of motion (5.7) governing the target-space gravitational field are valid only to first order in X-derivatives. This corresponds to the region of validity of the one-loop beta-function in the usual sigma-model perturbation theory. As $X^0 \sim c$, the sigma-model becomes strongly coupled, and higher order terms in the gravitational equations are important.

The solution (5.8) has a vanishing tachyon field. On the other hand, we expect the non-trivial sigma-model dynamics to generate a two-dimensional vacuum energy, which manifests itself as a source term in the tachyon equation of motion. To see this consider the effect of the non-trivial matter couplings on the cosmological constant. As explained before, it is the exponential growth of the tachyon field as it rolls off the top of the hill that gives rise to the cosmological constant term in the Wheeler-DeWitt equation (4.7). We might imagine that it would be possible to "fine tune" the initial conditions so that the tachyon stays balanced at the top, and the cosmological constant would thus vanish. In our simpler examples in which the target-space was flat, we saw that this could indeed be done. Now, however, the coupling of the sigma-model to the two-dimensional gravity will make it impossible. To see this, consider the higher order terms in the effective action coupling T and $G_{\mu\nu}$. One such term will be of the general form $T\nabla\nabla RR$. To determine it properly we should calculate the two-loop graviton beta-function, but for our argument it will suffice to note that there must be some term of this form because string theory has a non-zero graviton-graviton-tachyon vertex. There will thus be an extra source term $\nabla\nabla RR$ in the tachyon equation of motion. As the three-sphere contracts, this will knock the tachyon from the top of the potential. We would therefore have to search for new fine tuned initial conditions to make the tachyon end up balanced at the top of the potential at large scales. This need to account for the matter vacuum energy is just the familiar cosmological constant problem.

Note that we are not able to use these techniques to follow the system into the strongly coupled regime in which r becomes small. However, since we know that the flat space sigma-model contains only massive particles [37], we may speculate that well below the induced mass scale, the sigma-model degrees of freedom decouple. This would correspond to the effective central charge of the matter becoming smaller at some point in the evolution of the universe.

6. A Vanishing Cosmological Constant

In this section and the next we will follow the evolution of the cosmological constant as the two-dimensional universe expands. Recall that the spacetime equation of motion for the tachyon field is obtained from the tachyon beta-function

$$\beta^T = \nabla^2 T - 2\nabla\Phi\cdot\nabla T - V'(T) + \cdots . \tag{6.1}$$

In Section 8 we find that, in a particular regularization, the leading terms in the potential are given by $V(T) = -T^2 + \frac{1}{24}T^3 + \ldots$. There will also be higher-order contributions to the beta-function and couplings to other target-space fields.

As we discussed at the end of Section 4, the evolution of the tachyon background depends on the regularization procedure used to define the two-dimensional field theory. In what follows, we will assume that a prescription exists for which the tachyon field continues to be closely connected to the Wheeler-DeWitt amplitude as T rolls off the top of the potential. The discussion should be understood as speculations about the non-linear dynamics obtained using such a scheme.

The large scale two-dimensional cosmology is very sensitive to the form of the tachyon potential $V(T)$. It is particularly interesting to consider the case when $V(T)$ has a local minimum. This is consistent with the cubic behavior of $V(T)$, which we have found at leading order, but the minimum may or may not survive higher order corrections. First let us assume that the minimum is smooth (See Figure 3). There will then be another exact solution of the target-space equations given by

$$\begin{aligned} T &= T_0\,, \\ G_{\mu\nu} &= \eta_{\mu\nu}\,, \\ \Phi &= -\frac{\tilde{q}}{2}X^0 \equiv \Phi_B\,, \end{aligned} \tag{6.2}$$

where $T = T_0$ is the location of the potential minimum and $\tilde{q} = \sqrt{\frac{D-25}{3} + V(T_0)}$. Unlike the solution at $T = 0$, this solution is stable. We see this by considering the linearized equations for fluctuations about the background (6.2):

$$\begin{aligned} k^2\,\tau + \partial_0^2\tau + \tilde{q}\,\partial_0\tau + V''(T_0)\,\tau &= 0\,, \\ k^2\,\varphi + \partial_0^2\varphi + \tilde{q}\,\partial_0\varphi &= 0\,, \\ k^2\,h_{\mu\nu} + \partial_0^2 h_{\mu\nu} + \tilde{q}\,\partial_0 h_{\mu\nu} &= 0\,, \end{aligned} \tag{6.3}$$

where $\tau = T - T_0$, $\varphi = \Phi - \Phi_B$, $h_{\mu\nu} = G_{\mu\nu} - \eta_{\mu\nu}$, and k is the spatial momentum of the fluctuation. These equations have the solution

$$\begin{aligned}\tau(X^0) &= \mu\, e^{\left(-\frac{\tilde{q}}{2}\pm\sqrt{\frac{\tilde{q}^2}{4}-V''(T_0)-k^2}\right)X^0}\,,\\ \varphi\,,\, h_{\mu\nu} &\propto e^{\left(-\frac{\tilde{q}}{2}\pm\sqrt{\frac{\tilde{q}^2}{4}-k^2}\right)X^0}\,.\end{aligned} \tag{6.4}$$

For $k \neq 0$ all fluctuations are damped, whereas for $k = 0$ the dilaton and graviton have a constant mode, corresponding to a rescaling of the string coupling and the metric. The fluctuations of the higher modes are more strongly damped.

From a two-dimensional point of view, then, the theory at the bottom of the potential must be a field theory with no unstable fluctuations. Because the tachyon fluctuations have no exponentially growing mode, the corresponding Wheeler-DeWitt equation will not contain a cosmological constant term. Thus, while we do not know the exact form of the field theory seen by a one-dimensional observer, we can say that he measures a zero cosmological constant.

Having studied the solutions in the neighborhood of the stationary points of the potential $V(T)$, we can consider a solution that begins at or near $T = 0$ and rolls toward $T = T_0$. Because of the time-dependent dilaton background that starts off as

$$\Phi(X) = -\frac{q}{2}X^0\,, \tag{6.5}$$

and eventually becomes

$$\Phi(X) = -\frac{\tilde{q}}{2}X^0\,, \tag{6.6}$$

there will be a varying friction term in the equation of motion for $T(X^0)$. Initially T will behave as discussed before, with

$$T(X^0) \sim \lambda\, e^{\left(-\frac{q}{2}+\sqrt{\frac{q^2}{4}+2}\right)X^0}\,, \tag{6.7}$$

and eventually it will settle toward T_0 with a damped motion

$$T \sim T_0 + \mu\, e^{\left(-\frac{\tilde{q}}{2}+\sqrt{\frac{\tilde{q}^2}{4}-V''(T_0)}\right)X^0}\,, \tag{6.8}$$

In the semi-classical limit, the form of the Wheeler-DeWitt equation will thus evolve from

$$\left\{(\frac{\gamma}{2}a\frac{\partial}{\partial a})^2 + (k^2-2-\frac{q^2}{4}) + \frac{1}{4}\lambda a^2\right\}\Psi = 0 \tag{6.9}$$

for small a toward

$$\left\{(\frac{\gamma}{2}a\frac{\partial}{\partial a})^2 + (k^2+V''(T_0)-\frac{\tilde{q}^2}{4}) + V'''(T_0)\mu a^{-V''(T_0)}\right\}\Psi = 0 \tag{6.10}$$

for large a. This means that the universe will initially inflate, but will eventually settle down to a behavior with vanishing cosmological constant.

Although we have not yet thoroughly investigated the quantum theory of the system, it seems very probable that the quantum wave functional will become centered at T_0, with a dispersion which will shrink with X^0, since the quantum corrections are of order $e^{\Phi} \sim e^{-\frac{\tilde{q}}{2}X^0}$ at late times. This behavior is in marked contrast to that encountered in the $D \leq 1$ case in which the string coupling grows with the scale factor a. In those theories the effects of higher topologies become increasingly important as a increases, whereas for $D > 25$ their effect diminishes.

The picture that is suggested is that as the universe evolves, a condensate of baby universes interacts with it and forces the cosmological constant to zero. We are then faced with the question of the size of the corresponding wormholes. If they turn out to be macroscopic, then this scenario loses its appeal as a toy model for the real world ("The Giant Wormhole Problem"). Naively one might suppose that since the background tachyon field takes the value T_0 for arbitrarily large values of ϕ, the baby universes would form with arbitrarily large necks. In fact the correct relationship between the scale size and the metric is more subtle than this, since physical size depends on both ϕ and the fiducial size. It is not clear to us how to correctly derive the distribution of wormhole sizes from $T(\phi)$.

Whilst the above scenario, where the cosmological constant relaxes to zero at large scales, has some appeal, there are reasons to doubt its consistency. In particular one may ask what two-dimensional field theory can describe the stationary point at the potential minimum. Kutasov and Seiberg [35,38] have argued that a generic two-dimensional conformal field theory coupled to gravity contains physical tachyons. This seems to imply that the theory at the bottom of the potential cannot contain the standard gravitational degrees of freedom.

7. A Non-Vanishing Cosmological Constant

In the previous section we considered a two-dimensional cosmological model which showed very different behavior at large scales from what one would naively expect based upon semi-classical considerations in mini-superspace. It turns out that an apparently minor modification of the tachyon potential near its minimum can lead to drastic changes in the associated cosmology. In fact there exists a potential $V(T)$ such that the linearized fluctuations about the tachyon background satisfy a Wheeler-DeWitt equation with a conventional cosmological constant term throughout the non-linear evolution of the tachyon background. In other words, it is possible that the background dynamics is perfectly consistent with having a non-zero cosmological constant at all scales. This is most easily seen in the semi-classical $q \to \infty$ limit. In this case, we can ignore the effect of a non-vanishing tachyon field on a target-space linear dilaton background, *i.e.* $q \sim \tilde{q}$.

Consider a homogeneous tachyon background. Its equation of motion is

$$-\partial_0^2 T - q\partial_0 T = V'(T) \; . \tag{7.1}$$

In terms of the conformal mode $\phi = \frac{2}{q}X^0$ we have

$$\frac{4}{q^2}\partial_\phi^2 T + 2\partial_\phi T + V'(T) = 0 \; . \tag{7.2}$$

In the $q \to \infty$ limit the second-order term can be dropped and we are left with a first-order equation for the tachyon background. Fluctuations about this background will satisfy a linear equation,

$$2\partial_\phi \tau + V''(T)\tau = 0\,. \tag{7.3}$$

The idea is to look for a potential $V(T)$ such that $V''(T) = -2 + \frac{\lambda}{4}e^\phi$ throughout the evolution of $T(\phi)$. This is in fact satisfied by

$$V(T) = -T_0 T + \frac{1}{2}T^2 - T_0^2(1-\frac{T}{T_0})^2 \ln(1-\frac{T}{T_0})\,, \tag{7.4}$$

which has the form of an unstable tachyon potential near $T = 0$ and a stationary point at $T = T_0$, which is singular ($V'' \sim \infty$). The potential cannot be continued past that singularity but, as we shall see, the tachyon field never rolls beyond $T = T_0$. It is this singular behavior, due to the logarithm in (7.4), which allows a non-vanishing cosmological constant at large scales. To see how this works, insert (7.4) into the equation of motion and take the $q \to \infty$ limit,

$$2\partial_\phi T + 2T_0(1-\frac{T}{T_0})\ln(1-\frac{T}{T_0}) = 0\,. \tag{7.5}$$

This is easily solved by writing $(1-\frac{T}{T_0}) = e^S$, so that

$$2\partial_\phi S = S \tag{7.6}$$

from which we obtain

$$T = T_0(1-e^{-\frac{\lambda}{8}e^\phi})\,. \tag{7.7}$$

The cosmological constant of a two-dimensional universe interacting with this background is equal to the parameter λ, which is determined by the initial conditions on T. The equation for S is linear so we see that this example provides a realization of the fact that tachyon field redefinition can eliminate the non-linear terms in the equation of motion. However, the resulting field S is no longer proportional to the Wheeler-DeWitt amplitude in equation (4.4).

It should be noted once again that in spite of the apparently complicated evolution of T given by (7.7), the linear Wheeler-DeWitt equation obtained from (7.3) is precisely that of mini-superspace Liouville theory in the $q \to \infty$ limit [19, 21, 35]. It is interesting in this context to note that the tachyon background also satisfies a *linear* equation,

$$2\partial_\phi(T_0 - T) + \frac{\lambda}{4} e^{\phi}(T_0 - T) = 0 \,. \tag{7.8}$$

This suggests the alternate definition for the canonical tachyon field (4.4),

$$\tilde{U}(X) = e^{-\Phi(X)}(T_0 - T(X)) \,. \tag{7.9}$$

In the large q limit the following linear second order equation for $\tilde{U}$,

$$\partial_0^2 \tilde{U} - \frac{q^2}{4}\tilde{U} + \frac{\lambda}{4} e^{(-\frac{q}{2}+\sqrt{\frac{q^2}{4}+2})X^0}\tilde{U} = 0 \,, \tag{7.10}$$

is equivalent to (7.8). Equation (7.10) differs from the $k = 0$ Wheeler-DeWitt equation (4.6) by the term due to the bare dimension of the tachyon. This is the equation that the $SL(2,\mathrm{C})$ vacuum of string theory satisfies [19], and it is natural to identify that state with the most symmetric or Hartle-Hawking state of a one-dimensional universe [19,39].

The above considerations illustrate that having a tachyon potential with a minimum does not rule out a two-dimensional cosmological constant. Apparently a very special form of potential is required, with a mild singularity at the minimum. A deeper understanding of string theory is needed to determine the actual form of the tachyon beta-function, but it is certainly possible that the dynamics will eventually turn out to be as described in this section. We need to be able to identify the relation between the tachyon field and the Wheeler-DeWitt amplitudes for a given renormalization prescription in the two-dimensional theory. At the end of the following section we set up a renormalization scheme, in which this identification is particularly straightforward. We are able to obtain the beta-function for zero momentum tachyons to all orders in T and find a potential precisely of the form (7.4). For $D \leq 1$ there is further evidence to this effect coming from matrix models. Consider the non-linear loop equation (2.5). It can be solved explicitly to obtain the background amplitude $Z(l)$. Fluctuations about this background satisfy the linear equation

$$z(l{+}1) + 4z(l{+}3) = 2\sum_{p=0}^{l-1} z(p)Z(l{-}1{-}p)\ . \tag{7.11}$$

Moore *et al.* [21] have recently shown that in the continuum limit $z(l)$ is a solution of the mini-superspace Wheeler-DeWitt equation discussed in Section 4, with non-vanishing cosmological constant.

It is unclear what conformal field theory, if any, corresponds to a tachyon field sitting at rest at $T = T_0$, but we suspect it to be a rather trivial one. By the arguments of Kutasov and Seiberg [35,38] it cannot be a standard matter theory coupled to gravity. Apparently the $T = T_0$ fixed point describes the asymptotic behavior of an expanding universe long after all relevant scales (*e.g.* the cosmological constant scale) have been passed. The only remaining degrees of freedom are conformal matter fields from which the scale of the metric decouples. The situation is analogous to that in QCD at very large distance scales where the only degrees of freedom are massless pions. Another closer example is provided by the $D = 0$ one-matrix model. In this case a non-zero cosmological constant corresponds to a matrix potential slightly off criticality. The model flows to the trivial Gaussian matrix model at large scales and the random surface interpretation breaks down.

If the picture presented in this section is correct then there is nothing in the classical target space dynamics which favors vanishing cosmological constant at large scales. Higher worldsheet topologies are not likely to influence this. Recall that the string coupling rapidly decreases with increasing scale in the $D > 25$ theory. Target space quantization is therefore only important at small scales and its main effect will be to provide a distribution of initial conditions for the target space fields, which subsequently evolve classically. Since a small cosmological constant is dynamically insignificant at small scales it appears unlikely that it can force the initial conditions to be infinitely peaked at $\lambda = 0$.

We conclude this section by mentioning some consequences of a bottomless potential. In this case, $T(X^0)$ will continue to rapidly increase, and the theory is probably too sick to describe cosmology. To see why, we note that the increasing tachyon field will act as a source for all other fields, and the renormalization flows of the higher order couplings, which we discussed in the Section 5, will be disturbed even at late times. This would even be true in the semi-classical limit of large q.

8. The Tachyon Beta-Function

In this section, we present a calculation of the leading non-linear contribution to the tachyon beta-function. One might initially suspect that there are no non-linear terms, since individual diagrams in the loop expansion for the product of two normal ordered tachyon vertices are finite. On the other hand, we are interested in the *exact* renormalization of couplings, since we want the theory to be exactly independent of the fiducial metric. This means that we must keep track of the non-divergent cutoff dependence. When this is done, a non-vanishing beta-function will indeed emerge.

We shall not rigorously carry out this procedure, but will instead adopt the method originally used in [40] for open string theory. This approach is perturbative in the strength of the tachyon field, but sums all orders of the loop expansion. For a certain range of target-space momenta k_i, the sum over loops introduces short-distance divergences from which the beta-function can be read off. We then extend the results to the region of interest by assuming that the target space theory is analytic in the k_i.

Following reference [40] our starting point is a general renormalization group equation for a set of couplings g^i:

$$\beta^i \equiv \frac{dg^i}{dt} = \lambda_i g^i + \sum_{j,k} \alpha^i_{jk}\, g^j g^k + \sum_{j,k,l} \gamma^i_{jkl}\, g^j g^k g^l + \cdots . \tag{8.1}$$

Here t is the renormalization scale and λ_i is the anomalous dimension of the operator which carries the coupling g^i. The flow equation (8.1) can easily be integrated order by order in the couplings. Let $g^i(0)$ be the value of g^i at some infrared scale, $t = 0$. Then the renormalized coupling is given by

$$g^i(t) = e^{\lambda_i t}\, g^i(0) + \sum_{j,k} [e^{(\lambda_j+\lambda_k)t} - e^{\lambda_i t}]\, \frac{\alpha^i_{jk}}{\lambda_j+\lambda_k-\lambda_i}\, g^j(0) g^k(0) + \cdots . \tag{8.2}$$

For the time being, we are primarily interested in tachyon backgrounds, so we will assume a flat target-space with vanishing dilaton field. In due course we will consider the effect of a linear dilaton background. The tachyon term in (3.4) can then be written

$$\frac{1}{8\pi\epsilon^2} \int d^2\sigma\, \sqrt{\hat{\gamma}} \int d^{D+1}k\, T(k)\, e^{ik\cdot X(\sigma)} . \tag{8.3}$$

The target-space momentum k^μ plays the role of the index i and we have included an explicit dependence on the cutoff ϵ to make the tachyon couplings $T(k)$ dimensionless. We will calculate the two-dimensional effective action using standard background-field techniques. The fields X^μ are expanded around some classical background, which varies slowly on the cutoff scale: $X^\mu(\sigma) = X^\mu_b(\sigma) + \pi^\mu(\sigma)$. The path integral is performed over the quantum field $\pi^\mu(\sigma)$,

$$\begin{aligned} Z[X_b] = e^{-S_0[X_b]} \int \mathcal{D}\pi\, \exp -\frac{1}{8\pi} \int d^2\sigma\, \sqrt{\hat{\gamma}}\, \hat{\gamma}^{ab} \partial_a \pi^\mu \partial_b \pi_\mu \\ \times \exp\big(-\frac{1}{8\pi\epsilon^2} \int d^2\sigma\, \sqrt{\hat{\gamma}} \int d^{D+1}k\, T(k)\, e^{ik\cdot X_b}\, e^{ik\cdot\pi}\big) . \end{aligned} \tag{8.4}$$

The effective action is given by $S_{eff}[X_b] = -\log Z[X_b]$. The two-dimensional propagator is

$$\langle \pi^\mu(\sigma_1)\pi^\nu(\sigma_2)\rangle = -2\eta^{\mu\nu}\log\left(|\sigma_1 - \sigma_2|\right). \tag{8.5}$$

The path integral is evaluated by expanding the exponential in powers of $T(k)$, and evaluating diagrams with ever increasing numbers of interaction vertices. Because of the exponential form of the operator multiplied by $T(k)$, a given vertex can have any number of legs, and we have to sum over an infinite set of graphs to obtain the full answer at a given order in $T(k)$ (see Figure 4). There are divergences which do not show up at any finite order in the standard two-dimensional loop expansion, but only appear when the diagrams are added up. To regulate these divergences, we need a well-defined cutoff procedure. In general, the exact beta-function will depend on the choice of regulator. This is believed to correspond to the field redefinition ambiguities in the target space equations. Our assumption is that a regulator scheme can be found in which the target space fields are identified with Wheeler-DeWitt amplitudes. For reasons of simplicity and tractability, we will use a "hard sphere" regulator defined by cutting off position space integrals at finite separation. It is not clear that this prescription allows us to view the renormalized couplings as Wheeler-DeWitt amplitudes. Another approach that should satisfy this criterion is outlined at the end of this section. In general, explicit calculations are awkward in this scheme, but we're able to compute the beta-function of zero-momentum tachyons to all orders, and that allows us to determine the full form of the tachyon potential.

The first-order contribution to the effective action comes from the graphs on the left in Figure 4.

$$\begin{aligned} S_{eff}[X_b] =& S_0[X_b] + \frac{1}{8\pi\epsilon^2}\int d^2\sigma \int d^{D+1}k\, T(k)\, e^{ik\cdot X_b}\left\langle e^{ik\cdot\pi}\right\rangle \\ =& S_0[X_b] + \frac{1}{8\pi}\int d^2\sigma \int d^{D+1}k\, \epsilon^{k^2-2}\, T(k)\, e^{ik\cdot X_b}, \end{aligned} \tag{8.6}$$

where ϵ is the hard-sphere diameter. We want to compare this with the integrated renormalization group flow (8.2). It is natural to identify ϵ with e^{-t}, so that the cutoff is removed as $t \to \infty$. The renormalized coupling in (8.6) is simply

$$T_t(k) = \epsilon^{k^2-2} T_0(k) = e^{(2-k^2)t} T_0(k) \tag{8.7}$$

and we read off the familiar anomalous dimension $\lambda_k = 2-k^2$ for a tachyon of momentum k.

At second order we have to sum over the graphs on the right in Figure 4. Their contribution to $Z[X_b]$ is given by

$$\begin{aligned} \frac{1}{2}\Big(\frac{1}{8\pi\epsilon^2}\Big)^2 \int d^2\sigma_1\, d^2\sigma_2 \int d^{D+1}k_1\, d^{D+1}k_2\; T(k_1)\, T(k_2) \\ \times\, e^{ik_1\cdot X_b(\sigma_1)+ik_2\cdot X_b(\sigma_2)} \left\langle e^{ik_1\cdot\pi(\sigma_1)} e^{ik_2\cdot\pi(\sigma_2)} \right\rangle . \end{aligned} \tag{8.8}$$

We are assuming that the background field X_b varies slowly on the cutoff scale, so we can expand

$$X_b^\mu(\sigma_2) = X_b^\mu(\sigma_1) + (\sigma_2-\sigma_1)^a \partial_a X_b^\mu(\sigma_1) + \cdots \tag{8.9}$$

in (8.8). The sub-leading terms, involving derivatives of X_b^μ, contribute to terms of dimension higher than zero in the effective action, and do not affect the renormalization of tachyon couplings. The dimension zero piece of the effective action coming from (8.8) is therefore

$$\begin{aligned} -\pi \left(\frac{1}{8\pi}\right)^2 \int d^2\sigma \int d^{D+1}k_1\, d^{D+1}k_2\, \epsilon^{(k_1+k_2)^2-2} T(k_1) T(k_2) \\ \times\, e^{i(k_1+k_2)X_b(\sigma)} \int_1^\infty dy\; y^{2k_1\cdot k_2+1} . \end{aligned} \tag{8.10}$$

The y integral is convergent for $k_1 \cdot k_2 < -1$ and one can define its value outside that region by analytic continuation in the momenta. Using this prescription we find that the renormalized coupling to second order is

$$T_t(k) = e^{(2-k^2)t}\left[T_0(k) + \frac{1}{8}\int d^{D+1}k_1\, d^{D+1}k_2 \frac{\delta(k_1+k_2-k)}{2+2k_1\cdot k_2} T_0(k_1)T_0(k_2)\right] . \quad (8.11)$$

This is to be compared with the general solution (8.2). The denominator contains precisely the correct combination of anomalous dimensions, $2+2k_1\cdot k_2 = \lambda_{k_1}+\lambda_{k_2}-\lambda_{k_1+k_2}$, but we appear to be missing the $e^{(\lambda_{k_1}+\lambda_{k_2})t}$ term. This can be explained as follows. The integral in (8.10) is convergent for $k_1\cdot k_2+1<0$, which can also be written as $\lambda_{k_1+k_2} > \lambda_{k_1}+\lambda_{k_2}$. As the cutoff is removed, $t\to\infty$, and $e^{\lambda_{k_1+k_2}t}$ dominates over $e^{(\lambda_{k_1}+\lambda_{k_2})t}$. Our expression for the renormalized coupling (8.11) therefore only contains the leading divergence. However, we still have enough information to read off the value of the second-order coefficient in (8.1), giving $\alpha^k_{k_1k_2} = -\frac{1}{8}\delta^{(D+1)}(k_1+k_2-k)$. Its simple form makes the momentum integrals trivial, and we obtain the tachyon beta-function in position space

$$\beta^T(X) = (2+\nabla^2)\,T(X) - \frac{1}{8}\,T(X)^2 + \cdots . \quad (8.12)$$

Setting this equal to zero gives an equation of motion for the tachyon background. Near the mass-shell our results essentially agree with those of Das and Sathiapalan [41]* and those of Brustein *et al.* [42].

The above calculation only considered tachyons, whereas a more comprehensive treatment would include higher-dimension couplings such as gravitons and dilatons. Such a general approach would be quite complicated but for our present purposes it is sufficient to consider the simple case of a linear dilaton background, $\Phi = -\frac{q}{2}X^0$, in flat target-space. The tachyon beta-function is obtained in much the same way as before. The dilaton term in the two-dimensional action looks particularly simple if we write the fiducial metric as a "conformal factor" times the flat two-dimensional

⋆ The non-linear term in the tachyon equation of motion obtained in [41] has the opposite sign to ours, but the important feature, that this term has no derivatives, agrees.

metric, $\hat{\gamma}_{ab} = e^{\xi}\delta_{ab}$. Then

$$S_{dil} = \frac{1}{8\pi}\int d^2\sigma\,\Box\xi\; q\cdot X\,, \tag{8.13}$$

where $\Box$ denotes the flat space scalar Laplacian in two dimensions and $q^\mu = (q,0,\ldots,0)$. This term is linear in X^μ and its effect in the path integral is taken into account by including graphs with external legs which carry a factor of q^μ (see Figure 5). The contribution of each such external leg is given by

$$-\frac{1}{8\pi}\int d^2\sigma\;\xi(\sigma)\,ik\cdot q\,\Box_\sigma G(\sigma{-}\sigma_0) = \frac{i}{2}\,k\cdot q\,\xi(\sigma_0)\,, \tag{8.14}$$

where $G(\sigma{-}\sigma_0)$ is the two-dimensional propagator (8.5) and σ_0 is the position of the tachyon vertex from which the leg emanates.

To leading order in $T(k)$ the effective action is given by the graphs on the left in Figure 5,

$$S_{eff}[X_b] = S_0[X_b] + \frac{1}{8\pi}\int d^2\sigma\int d^{D+1}k\;\epsilon^{k^2-2}\,e^{\frac{i}{2}k\cdot q\xi(\sigma)}\,T(k)\,e^{ik\cdot X_b}\,. \tag{8.15}$$

The anomalous dimension of the renormalized coupling is determined by its scaling properties as the fiducial conformal factor ξ is varied. The cutoff ϵ is defined with reference to the fiducial metric, so it scales as $\epsilon \sim e^{\frac{\xi}{2}}$, and we find the anomalous dimension $\lambda_k = 2{-}ik{\cdot}q{-}k^2$. The condition that this vanish is the linearized tachyon equation (4.2).

The second-order contribution comes from the graphs on the right in Figure 5. The calculation involves the same steps as in the critical theory and one finds that the renormalized coupling to second order is now

$$T_t(k) = e^{(2-ik\cdot q-k^2)t}\left[T_0(k) + \frac{1}{8}\int d^{D+1}k_1\,d^{D+1}k_2\,\frac{\delta(k_1{+}k_2{-}k)}{2{+}2k_1{\cdot}k_2}T_0(k_1)T_0(k_2)\right]. \tag{8.16}$$

Apparently the only effect of the linear dilaton background is to modify the anomalous dimensions. The q-dependence cancels out of the denominator in (8.16) and we read off the same value of the second-order beta-function coefficient as before. Thus we find that the leading terms in the tachyon beta-function in non-critical string theory are

$$\beta^T(X) = \left(2 - 2\nabla\Phi\cdot\nabla + \nabla^2\right) T(X) - \frac{1}{8}\, T(X)^2 + \cdots\,, \tag{8.17}$$

in agreement with (3.7).

Note that the non-linear term has no derivatives and corresponds to a non-vanishing cubic term in the target-space potential $V(T)$. This has important consequences. In particular it implies that a uniform tachyon field will have non-trivial influence on the renormalization group flow of tachyon fluctuations with non-zero target-space momentum. This is surprising since, according to (3.4), a constant tachyon background field only contributes a c-number, $\frac{T}{8\pi}\int d^2\sigma\,\sqrt{\hat{\gamma}}$, to the two-dimensional action. This is perhaps too simple a view to take. In fact, the excluded volume of the hard sphere regularized integrals introduces non-trivial effects even when T is constant. In this case the partition function reduces to that of a simple hard sphere gas, with fugacity $f = -\frac{T}{8\pi\epsilon^2}$,

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} \frac{f^n}{n!} \int\limits_{|\sigma_i - \sigma_j| > \epsilon} d^2\sigma_1 \ldots d^2\sigma_n \\ &= 1 + f + \frac{f^2}{2}(1 - \pi\epsilon^2) + \ldots\,, \end{aligned} \tag{8.18}$$

where the integrals are over unit area and the thermodynamic limit is obtained in the limit $\epsilon \to 0$. The free energy is $F = \epsilon^2 \log Z$. The running of the coupling T with ϵ is defined by requiring the partition function to be independent of the cutoff scale. This implies

$$\begin{aligned} 0 &= \frac{\partial}{\partial\epsilon}\frac{1}{\epsilon^2}F(T) \\ &= -\frac{2}{\epsilon^3}F(T) + \frac{1}{\epsilon^2}F'(T)\frac{dT}{d\epsilon}\,. \end{aligned} \tag{8.19}$$

We define the beta-function in the usual way,

$$\beta(T) = \epsilon \frac{dT}{d\epsilon} = 2 \frac{F(T)}{F'(T)} \,. \tag{8.20}$$

To second order in T we find

$$\beta(T) = 2T - \frac{1}{8} T^2 \,, \tag{8.21}$$

in agreement with (8.17). We expect this agreement between analytic continuation and the hard sphere gas to hold to all orders in T.

Another point of interest is that statistical systems of this type typically have Lee-Yang edge type singularities at negative fugacity and therefore vanishing beta-functions at some positive values of T. However, the behavior of the beta-function is usually not analytic at these points, which suggests that cusps in $V(T)$ of the type encountered in Section 7 can occur.

To conclude this section we would like to schematically outline a regularization scheme in which the Wheeler-DeWitt amplitudes enter directly into the renormalization process. Begin by introducing a lattice on the fiducial coordinate space with lattice spacing ϵ. In each cell we define an amplitude on the boundary by integrating over the two-dimensional fields in the interior of that cell, fixing the values on the boundary. This defines an effective theory that lives on the lattice edges. The remaining integration over the boundary values of the fields yields the full path integral. The integrand of the effective theory is given by the product over all cells of the cell amplitudes. Schematically,

$$Z = \int \prod_{\text{cells}} \mathcal{D}\phi_{\text{boundary}} \, \psi(\phi_{\text{boundary}}) \,. \tag{8.22}$$

The amplitudes ψ are by construction Wheeler-DeWitt amplitudes. To introduce target-space fields we can expand $\psi(\phi_{\text{boundary}})$ in terms of string modes. Let $\hat{X}$ be the zero-mode part of X on the boundary. Then

$$\psi = (1 - T(\hat{X}) + G_{\mu\nu}(\hat{X})\tilde{a}_\mu^\dagger a_\nu^\dagger \cdots)\psi_0\,, \tag{8.23}$$

where ψ_0 is the free theory amplitude. By requiring the long wavelength behavior to be independent of the cutoff we can define beta-functions for the target space fields $T, G_{\mu\nu} \ldots$. This is certainly not a convenient scheme for beta-function calculations in the presence of general couplings, but in the special case of a constant tachyon field, we can obtain the full answer. Then the partition function is simply

$$\begin{aligned} Z &= \int \prod_{\text{cells}} (1 - T)\psi_0(\phi_b) \\ &= (1 - T)^N Z(T = 0)\,, \end{aligned} \tag{8.24}$$

where the total number of cells N is proportional to $\frac{1}{\epsilon^2}$. Requiring Z to be independent of ϵ gives the beta-function

$$\beta(T) = -2(1 - T)\ln(1 - T)\,. \tag{8.25}$$

Which precisely agrees with the zero-momentum beta-function obtained from the singular potential (7.4) (with $T_0 = 1$) in Section 7.

9. Conclusion

We conclude with some observations and speculations about 4-dimensional gravity. Unfortunately, in this case we can not choose the conformal gauge unless we restrict our attention to conformally flat geometries. Let us therefore consider such a restricted path integral over geometries with vanishing Weyl curvature. In this case we can reduce the Einstein action to a simple form. Letting $g_{\mu\nu} = \phi^2\eta_{\mu\nu}$

we find that the action has the form

$$\int d^4x \left\{-(\partial_\mu\phi)^2 + \Lambda\phi^4 + \text{matter}\right\} . \tag{9.1}$$

This is just ordinary ϕ^4 theory, but with a negative kinetic term. The original covariance requires the path integral to result in a conformally invariant field theory, thus requiring the vanishing of all beta-functions. As in the two-dimensional case, target-space fields can be introduced by allowing Λ and all other couplings to depend on ϕ. The vanishing of beta-functions defines the equations of motion in target-space. Of particular interest is the $\Lambda\phi^4$ term which would be replaced by the more general expression $T(\phi)$. Removing the restriction to conformally flat geometries makes things more complicated, but still we expect that the general logic of gauge-fixing, regulating, and restoring covariance by requiring target space field equations to be satisfied will make sense.

Can one expect the non-linearities of the target-space theory to play a similar roe in four dimensions as in two? More specifically, can the non-linearities be relevant at large scale factor? To attempt to answer this we can consider a very schematic non-linear Wheeler-DeWitt equation

$$H_{WD}\Psi = f \;\; \Psi \times \Psi\, , \tag{9.2}$$

where H_{WD} is a Wheeler-DeWitt Hamiltonian, Ψ is a functional of matter and geometry, and f is some coupling strength which depends on the scale size of the geometry. In the two-dimensional case, f is replaced by the coupling strength $e^{\Phi} = e^{-\frac{q}{2}X^0} = a^{-q^2}$. Because of tachyonic instabilities, the Wheeler-DeWitt amplitude in (9.2) increases like $e^{\frac{q}{2}X^0}$. Thus the background field increases at a sufficient rate to overcome the decrease of the coupling strength.

In four dimensions, the coupling for universes splitting and joining can be estimated from dimensional considerations. Consider a geometry with three boundaries, each with scale size $\sim a$. On dimensional grounds, the action for such a geometry is $\sim M_p^2 a^2$ where M_p is the Planck mass. Thus the amplitude for a universe splitting is likely to be of order $e^{-cM_p^2 a^2}$ where c is a numerical constant depending on dimensionless features of the geometry. The rapid decrease of the coupling strength might suggest that non-linear effects are quickly quenched with scale size. However, this is not necessarily the case. To see this, consider the linear part of the Wheeler-DeWitt equation $H_{WD}\Psi = 0$. In four dimensions, the mini-superspace Wheeler-DeWitt equation has the form

$$\left[\frac{\partial^2}{\partial a^2} - M_p^4 a^2\right] \Psi = 0\,. \tag{9.3}$$

For large a the solution is $\Psi \sim e^{M_p^2 a^2}$. Hence it is possible that the Wheeler-DeWitt amplitude grows sufficiently rapidly to overcome the decrease in coupling strength in four dimensions also. On the other hand, we have seen in Section 7 that such non-linearities do not necessarily invalidate the usual conclusions of the conventional Wheeler-DeWitt cosmology.

What the implications of such non-linearities are for large scale cosmology is a matter of speculation. It should be stressed, however, that they are not connected with the issue of summing over space-time topologies. Such sums imply quantization of the target-space theory, while our considerations have been at the classical level. Third quantization is important at small scales where the coupling strength is large. It seems likely that third quantization at small distances would provide an ensemble of classical trajectories at large scale and that only an infinitesimal fraction of these would remain at the unstable point $T = 0$ where the cosmological constant is naively zero and the non-linearities are small. The remainder would inevitably flow into the non-linear region.

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REFERENCES

1. B. S. DeWitt, *Phys. Rev.* **160** (1967), 1113;
   J. A. Wheeler, in *Batelle Recontres,* eds. C. M. DeWitt and J. A. Wheeler (W. A. Benjamin, inc., New York, 1968).
2. S. W. Hawking, D. N. Page and C. N. Pope, *Nucl. Phys.* **B170** (1980), 283;
   S. W. Hawking, *Comm. Math. Phys.* **87** (1982), 395.
3. K. Kuchar, *J. Math. Phys.* **22** (1981), 2640; also in *Quantum Gravity,* Vol. 2, C. J. Isham, R. Penrose and D. W. Sciama eds., (Clarendon 1981).
4. A. Jevicki, in *Frontiers in Particle Physics '83,* Dj. Sijacki *et al.*eds., (World Scientific 1984).
5. A. Strominger, *Phys. Rev. Lett.* **52** (1984), 1733; also in *Proceedings of the 1988 Theoretical Advanced Study Institute,* A. Jevicki and C. I. Tan eds., (World Scientific 1989).
6. D. Gross, *Nucl. Phys.* **B236** (1984), 349.
7. N. Caderni and M. Martellini, *Int. J. Theor. Phys.* **23** (1984), 23.
8. I. Moss, in *Field Theory, Quantum Gravity and Strings,* H. J. de Vega and N. Sanchez eds., (Springer 1987).
9. V. A. Rubakov, *Phys. Lett.* **B214** (1988), 503;
   G. V. Lavrelashvili, V. A. Rubakov and P. G. Tinyakov, *JETP Lett.* **46** (1987), 167, *Nucl. Phys.* **B299** (1988), 757.
10. M. McGuigan, *Phys. Rev.* **D38** (1988), 3031, *Phys. Rev.* **D39** (1989), 2229.
11. A. Anderson, *Phys. Lett.* **B212** (1989), 334, *Nucl. Phys.* **B321** (1989), 48.
12. A. Hosoya and M. Morikawa, *Phys. Rev.* **D39** (1989), 1123.
13. S. Coleman, *Nucl. Phys.* **B307** (1988), 867., *Nucl. Phys.* **B310** (1988), 643.
14. T. Banks, *Nucl. Phys.* **B309** (1988), 493.
15. E. Baum, *Phys. Lett.* **133B** (1983), 185.

16. S. W. Hawking, *Phys. Lett.* **134B** (1984), 403.

17. S. Giddings and A. Strominger, *Nucl. Phys.* **B307** (1988), 854.

18. C. Teitelboim, *Phys. Lett.* **B126** (1983), 41.

19. J. Polchinski, *Nucl. Phys.* **B324** (1989), 123.

20. T. Banks, *Physicalia Magazine,* vol 12, Special Issue in Honor of the 60th Birthday of R. Brout,(Gent 1990).

21. G. Moore, N. Seiberg and M. Staudacher, *From Loops to States in 2-D Quantum Gravity,* Rutgers preprint, RU-91-11, 1991.

22. E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, *Comm. Math. Phys.* **59** (1978), 35

23. S. R. Wadia, *Phys. Rev.* **D24** (1981), 970.

24. A. M. Polyakov, *Phys. Lett.* **163B** (1981), 207.

25. S. R. Das, S. Naik and S. R. Wadia, *Mod. Phys. Lett.* **A4** (1989), 1033;
S. R. Das, A. Dhar and S. R. Wadia, *Mod. Phys. Lett.* **A5** (1990), 799.

26. T. Banks and J. Lykken, *Nucl. Phys.* **B331** (1990), 173.

27. A. A. Tseytlin, *Int. Jour. Mod. Phys.* **A5** (1990), 1833.

28. C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, *Nucl. Phys.* **B262** (1985), 593;
E. S. Fradkin and A. A. Tseytlin, *Nucl. Phys.* **B271** (1986), 561;
C. G. Callan and Z. Gan, *Nucl. Phys.* **B272** (1986), 647.

29. A. Chodos and C. B. Thorn, *Nucl. Phys.* **B72** (1974), 509;
T. L. Curtright and C. B. Thorn, *Phys. Rev. Lett.* **48** (1982), 1309;
R. C. Myers, *Phys. Lett.* **199B** (1988), 371;
I. Antoniades, C. Bachas, J. Ellis and D. Nanopoulos, *Phys. Lett.* **211B** (1988), 393;
S. P. deAlwis, J. Polchinski and R. Schimmrigk, *Phys. Lett.* **218B** (1989), 449.

30. F. David, *Mod. Phys. Lett.* **A3** (1988), 1651.

31. J. Distler and H. Kawai, *Nucl. Phys.* **B321** (1989), 509.

32. J. Polchinski, *Nucl. Phys.* **B346** (1990), 253.

33. T. Banks *The Tachyon Potential in String Theory*, Rutgers preprint, RU-91-08, 1991.

34. A.A. Tseytlin *On The Tachyonic Terms in the String Effective Action*, Johns Hopkins preprint, JHU-TIPAC-91004, 1991.

35. N. Seiberg, *Notes on Quantum Liouville Theory and Quantum Gravity*, Rutgers preprint, RU-90-29, 1990.

36. T. Banks, *Nucl. Phys.* **B249** (1985), 332.

37. A.B. Zamolodchikov and A.B. Zamolodchikov, *Annal.Phys.* **120** (253), 1979.

38. D. Kutasov and N. Seiberg, *Number of Degrees of Freedom, Density of States and Tachyons in String Theory and CFT*, Rutgers preprint, RU-90-60, 1990.

39. D. Birmingham and C.G. Torre,, *Phys.Lett.* **B 194** (1987), 49

40. I. Klebanov and L. Susskind, *Phys. Lett.* **200B** (1988), 446.

41. S. R. Das and B. Sathiapalan, *Phys. Rev. Lett.* **56** (1986), 2664.

42. R. Brustein, D. Nemeschansky and S. Yankielowicz, *Nucl. Phys.* **B301** (1988), 224.

FIGURE CAPTIONS

1) Once a fiducial metric has been chosen, and a cutoff introduced, the short distance fluctuations will still include large physical geometries, which will affect the renormalization.

2) A Lorentzian geometry with singular points corresponding to the splitting of universes, and the emission and absorption of baby universes from the background.

3) A tachyon potential with a smooth local minimum at $T = T_0$.

4) Leading order graphs which contribute to the tachyon effective action in critical string theory.

5) Leading order graphs in the calculation of the tachyon beta function in a linear dilaton background.

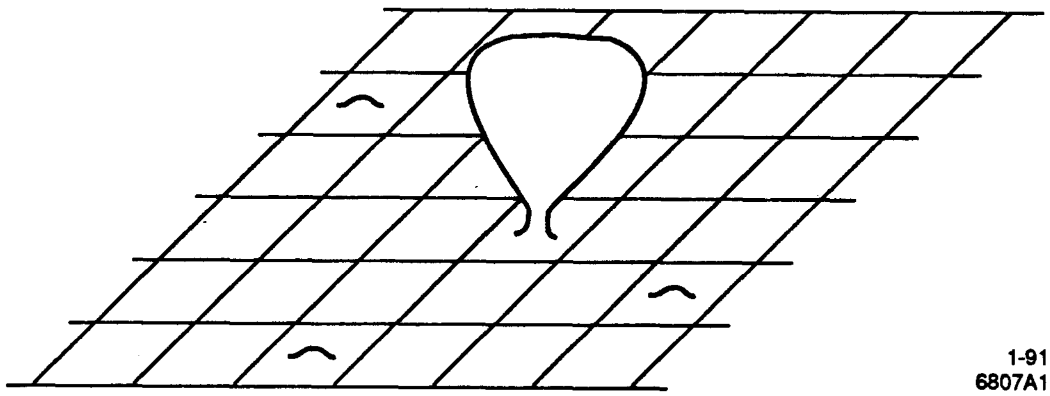

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Fig. 1

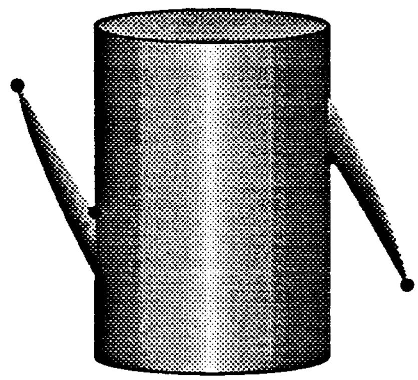

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Fig. 2

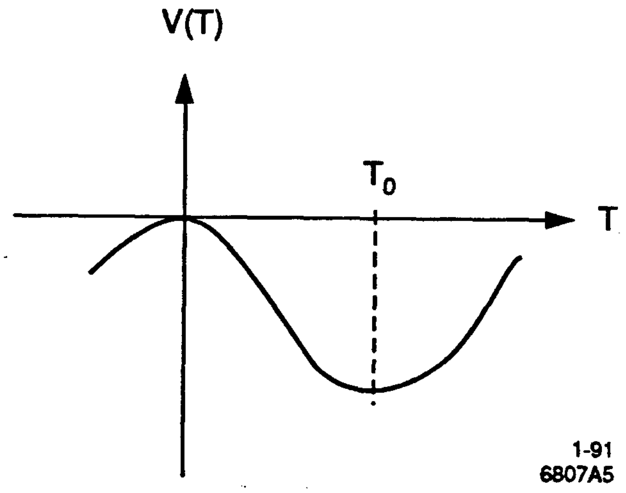

Fig. 3

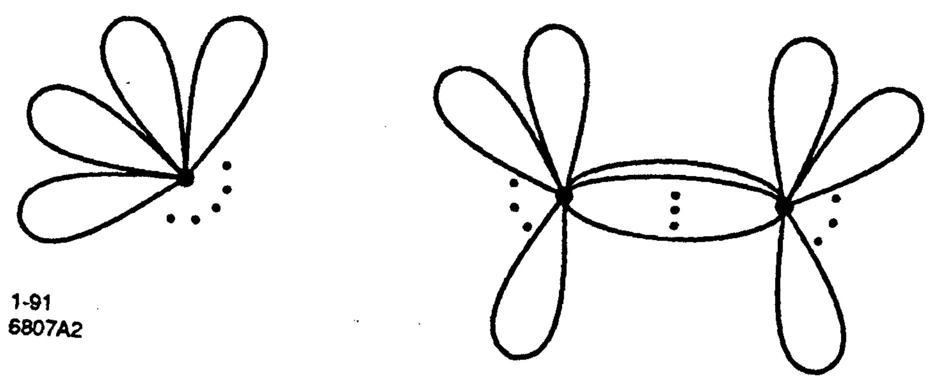

Fig. 4

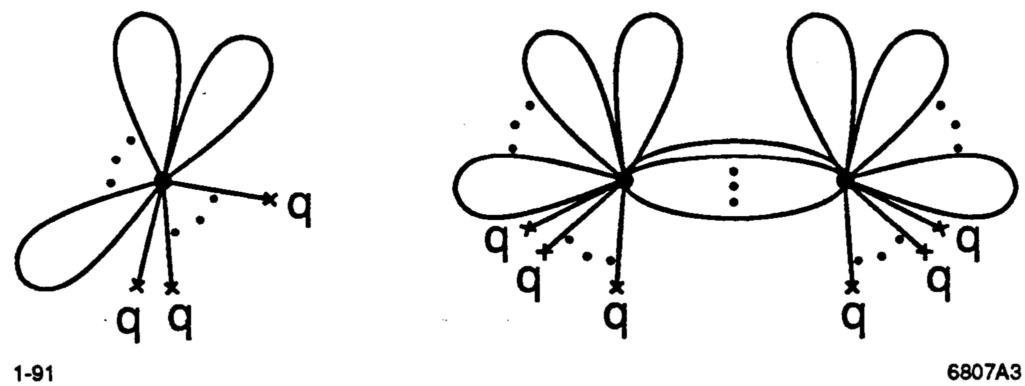

Fig. 5