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A KICK EXPERIENCED BY A PARTICLE OBLIQUELY TRAVERSING A BUNCH*

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ABSTRACT

Crossing of bunches at an angle in a storage ring is studied. A case when a particle trajectory intersects the opposing bunch off-center and at an angle to the main axis of the bunch charge distribution is considered. Formulae for a kick experienced by a particle are derived for crossing in both horizontal and vertical planes. This kick is a function of the off-center distance in the longitudinal direction and the crossing angle. In such a geometry, synchro-betatron resonances can be excited. The resonance width is proportional to the derivative of the kick over the longitudinal distance from the bunch center. The magnitude of this derivative is evaluated in detail.

1. INTRODUCTION

A crab-crossing scheme suggested by R. Palmer[1] is aimed to increase the integrated luminosity of future multibunch linear colliders. As shown in Ref. 2, this

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idea may be utilized to control synchro-betatron resonances in storage rings where bunched beams intersect each other at the interaction point at an angle unequal to zero. Recently, Oide and Voss[3] suggested testing the scheme in PEP.

To estimate the strength of synchro-betatron coupling in a crossing beam geometry without crab crossing, it is necessary to calculate the overall kick α a particle experiences when its trajectory intersects the opposing bunch off-center at a distance z_0.

Here we provide a formula which gives such an estimate. Of particular interest is the value $(d\alpha/dz_0)$ at $z_0 = 0$, which plays the role of the driving force for the synchro-betatron resonance. The magnitude of this derivative is evaluated here in detail.

2. THE ELECTROMAGNETIC FIELD OF THE BUNCH

Consider a particle which traverses an oncoming bunch along the trajectory, which we assume for simplicity to be a straight line (Fig. 1). In general, this line is slanted at an angle δ to the main axis of the bunch charge distribution. The crossing angle of the bunch trajectories in the laboratory coordinate system is defined as $\delta/2$. As a result of going through the bunch, the particle experiences a kick which depends both on the angle δ and on the position of the particle with respect to the center of gravity of the charge distribution. We characterize the latter by a displacement z_0 at which the particle trajectory crosses the main axis of the bunch.

We assume that the distribution of the charge in the oncoming bunch is Gaussian in all three dimensions. In the rest frame of the bunch with coordinates x', y', z', the charge density is then given by:

$$\rho\left(x',y',z'\right) = \frac{Ne}{\pi^{3/2}a'b'd'}\exp\left(-\frac{x'^2}{a'^2}-\frac{y'^2}{b'^2}-\frac{z'^2}{d'^2}\right) \quad , \tag{1}$$

where Ne is the full charge of the bunch, and a', b' and d' are $\sqrt{2}$ times the horizontal, vertical and longitudinal standard deviations (σ) of the charge distribution, respectively. Here and everywhere below, the prime denotes a value calculated in the rest frame of the bunch.

The electric field of such a bunch can be found from the potential derived in the last century by Houssais:[4]

$$U(x',y',z') = \frac{Ne}{\sqrt{\pi}}\int_0^\infty dq\frac{\exp\left(-\frac{x'^2}{a'^2+q}-\frac{y'^2}{b'^2+q}-\frac{z'^2}{d'^2+q}\right)}{\sqrt{(a'^2+q)(b'^2+q)(d'^2+q)}} \tag{2}$$

$$E_x' \equiv -\frac{\partial U}{\partial x'} = \frac{2Ne}{\sqrt{\pi}}\int_0^\infty dq\frac{x'\exp\left(-\frac{x'^2}{a'^2+q}-\frac{y'^2}{b'^2+q}-\frac{z'^2}{d'^2+q}\right)}{(a'^2+q)\sqrt{(a'^2+q)(b'^2+q)(d'^2+q)}} \tag{3}$$

$$E_y' \equiv -\frac{\partial U}{\partial y'} = \frac{2Ne}{\sqrt{\pi}}\int_0^\infty dq\frac{y'\exp\left(-\frac{x'^2}{a'^2+q}-\frac{y'^2}{b'^2+q}-\frac{z'^2}{d'^2+q}\right)}{(b'^2+q)\sqrt{(a'^2+q)(b'^2+q)(d'^2+q)}} \tag{4}$$

$$E_z' \equiv -\frac{\partial U}{\partial z'} = \frac{2Ne}{\sqrt{\pi}}\int_0^\infty dq\frac{z'\exp\left(-\frac{x'^2}{a'^2+q}-\frac{y'^2}{b'^2+q}-\frac{z'^2}{d'^2+q}\right)}{(d'^2+q)\sqrt{(a'^2+q)(b'^2+q)(d'^2+q)}} \quad . \tag{5}$$

Now we go back to the laboratory frame which moves with the velocity $-V$ along the z-axis. Defining $\beta = V/c$, $\gamma = 1/\sqrt{1-\beta^2}$, we find:

$$E_x = \frac{2Ne}{\sqrt{\pi}}\gamma x \int_0^\infty dq \frac{\exp\left(-\frac{x^2}{a^2+q} - \frac{y^2}{b^2+q} - \frac{\gamma^2(z+Vt)^2}{\gamma^2 d^2+q}\right)}{(a^2+q)\sqrt{(a^2+q)(b^2+q)(d^2\gamma^2+q)}} \tag{6}$$

$$E_y = \frac{2Ne}{\sqrt{\pi}}\gamma y \int_0^\infty dq \frac{\exp\left(-\frac{x^2}{a^2+q} - \frac{y^2}{b^2+q} - \frac{\gamma^2(z+Vt)^2}{\gamma^2 d^2+q}\right)}{(b^2+q)\sqrt{(a^2+q)(b^2+q)(d^2\gamma^2+q)}} \tag{7}$$

$$H_x = \beta E_y \quad , \tag{8}$$

$$H_y = -\beta E_x \quad . \tag{9}$$

From these expressions, it is easy to calculate the transverse kicks by integrating the corresponding component of the Lorentz force over time t:

$$\alpha_x \equiv \Delta P_x/P = \int_{-\infty}^{\infty} dte(E_x - \beta H_y)/P \approx (2e/\gamma mc)\int_{-\infty}^{\infty} dtE_x(\mathbf{r}(t)) \tag{10}$$

$$\alpha_y \equiv \Delta P_y/P = \int_{-\infty}^{\infty} dte(E_y + \beta H_x)/P \approx (2e/\gamma mc)\int_{-\infty}^{\infty} dtE_y(\mathbf{r}(t)) \tag{11}$$

In these formulae, the fields should be integrated along the particle trajectory $\mathbf{r} = \mathbf{r}(t)$.

For the further calculations, we assume that the particle trajectory lies in the vertical plane $x = 0$ for the case of the vertical crossing and, correspondingly, in the horizontal plane $y = 0$ for the case of the horizontal crossing.

3. THE VERTICAL COMPONENT OF THE ELECTRICAL FIELD

It is instructive to see how our expression produces known results for the vertical component of the electrical field for a flat bunch in its rest frame (we omit here the primes which denote the rest frame). We rewrite Eq. (4) in the following form:

$$E_y(x,y,z) = \frac{2Ne}{\sqrt{\pi}} y \int_0^\infty dq \frac{\exp\left(- \frac{x^2}{a^2+q} - \frac{y^2}{b^2+q} - \frac{z^2}{d^2+q} \right)}{(a^2+q)^{1/2}(b^2+q)^{3/2}(d^2+q)^{1/2}} \quad . \tag{12}$$

If we assume now $b \ll a$, $x \ll a$, and $z \ll d$, the integration in (12) is easily performed, and we obtain:

$$E_y(x,y,z) = \frac{2Ne}{d\sqrt{a^2-b^2}} \cdot sign(y) \left[\Phi\left(\frac{a|y|}{b\sqrt{a^2-b^2}} \right) - \Phi\left(\frac{|y|}{\sqrt{a^2-b^2}} \right) \right] \exp\left[\frac{y^2}{a^2-b^2} \right] \quad ,$$

where $\Phi(x)$ is the error function[5] with the asymptotic behavior $\Phi(x) \to 1$ for $x \to \infty$, and $sign(y)$ is the step function $sign(y) = +1$ for $y > 0$, $sign(y) = -1$ for $y < 0$.

In the region $b \ll y \ll a$, this formula simplifies to the following expression:

$$E_y(x,y,z) = \frac{2Ne}{ad} \cdot sign(y) \cdot \left[\Phi\left(\frac{|y|}{b} \right) - \Phi\left(\frac{|y|}{a} \right) \right] \quad . \tag{13}$$

Expression (13) gives an electric field of a plate with a Gaussian charge distribution in its transverse direction y. Outside the plate, for $y \gg b$, the field is uniform and changes sign from one side to another. Deep inside the plate, where $y \ll b$, the field is proportional to the distance y from its midplane.

4. THE VERTICAL KICK

From now on, we assume the following:

(a) the beam is flat: $a > b$;

(b) the particle energy of both beams is large: $\gamma \gg 1$;

(c) the angle δ is small: $\tan\,\delta \approx \delta$.

Under these assumptions, we obtain the vertical kick of the trajectory by integrating (11) along the line

$$x(t) \;=\; 0 \quad ; \qquad y(t) \;=\; c\delta t \quad ; \qquad z(t) \;=\; z_0 + ct \qquad . \tag{14}$$

After integration over q, we get:

$$\alpha_y \;=\; \frac{8Nr_0c}{\sqrt{\pi}\gamma d}\int_0^{|y|/b} dt\cdot sign[y(t)]\;\exp\left\{-\left[\frac{(z_0+2ct)^2}{d^2}\right]\right\}\int_0^{|y|/b}\frac{u\,du\,\exp(-u^2)}{\sqrt{u^2(a^2-b^2)+y(t)^2}} \;\; . \tag{15}$$

Here, r_0 is the classical electron radius $r_0 = e^2/mc^2$. The function $sign(y)$ has the value $+1$ for $y \geq 0$ and -1 for $y < 0$.

The second integral in Eq. (15) can be expressed in terms of the error function:

$$\alpha_y \;=\; \frac{2Nr_0}{\gamma\sqrt{a^2-b^2}}F_y(s,\tau) \quad , \tag{16}$$

where

$$F_y(s,\tau) \;=\; \exp(-s^2)\int_0^{\infty} dv\;\exp[-(1-\tau^2)v]\;\frac{\sinh(2s\sqrt{v})}{\sqrt{v}}\;\left[\Phi\left(\frac{a\tau}{b}\sqrt{v}\right)-\Phi(\tau\sqrt{v})\right] \quad , \tag{17}$$

$$s = z_0/d \tag{18}$$

and

$$\tau = \frac{d\delta}{2\sqrt{a^2 - b^2}} \quad . \tag{19}$$

5. THE HORIZONTAL KICK

Similarly, we obtain the horizontal kick of the trajectory by integrating (10) along the line

$$x(t) = c\delta t \quad ; \qquad y(t) = 0 \quad ; \qquad z(t) = z_0 + ct \quad . \tag{20}$$

After integration over q, we get:

$$\alpha_x = \frac{8Nr_0c}{\sqrt{\pi}\gamma d} \int\limits_0^{|y|/b} dt \cdot sign[x(t)] \, \exp\left\{ - \left[\frac{(z_0 + 2ct)^2}{d^2} \right] \right\} \int\limits_0^{|x|/a} \frac{udu \exp(-u^2)}{\sqrt{x^2(t) - u^2(a^2 - b^2)}} \quad . \tag{21}$$

The second integral in Eq. (21) can be expressed in terms of the Dawson's function

$$D(x) = e^{-x^2} \int\limits_0^x dte^{t^2} \quad . \tag{22}$$

We get

$$\alpha_x = \frac{2Nr_0}{\gamma\sqrt{a^2 - b^2}} F_x(s, \tau) \quad , \tag{23}$$

where

$$F_x(s, \tau) = e^{-s^2} \int\limits_0^\infty dve^{-v} \frac{\sinh(2s\sqrt{v})}{\sqrt{v}} \left\{ D(\tau\sqrt{v}) - e^{-\tau^2(1-b^2/a^2)v} D\left(\frac{b}{a}\tau\sqrt{v}\right) \right\} \quad . \tag{24}$$

The quantities s and τ are defined in Eqs. (18) and (19).

6. THE RESONANCE DRIVING FORCE

As was mentioned before, the derivative of the kick α over $z_0, \Delta \equiv [d\alpha/dz_0]_{z_0}$ evaluated at the point $z_0 = 0$ plays the role of a synchro-betatron resonance driving force. From Eqs. (16) and (23), one obtains the following derivatives for the vertical and horizontal planes:

$$\Delta_y \equiv [d\alpha_y/dz_0]_{z_0} = \frac{4Nr_0}{\gamma\sqrt{a^2-b^2}} G_y(\tau) \quad , \tag{25}$$

where

$$G_y(\tau) = \int_0^\infty dv \exp[-(1-\tau^2)v]\left[\Phi\left(\frac{a\tau}{b}\sqrt{v}\right) - \Phi(\tau\sqrt{v})\right] \quad ; \tag{26}$$

$$\Delta_x \equiv [d\alpha_x/dz_0]_{z_0} = \frac{4Nr_0}{\gamma\sqrt{a^2-b^2}} G_x(\tau) \quad , \tag{27}$$

where

$$G_x(\tau) = \frac{2}{\sqrt{\pi}} \int_0^\infty dv \exp(-v)\left\{D(\tau\sqrt{v}) - \exp[-\tau^2(1-b^2/a^2)v] D\left(\frac{b}{a}\tau\sqrt{v}\right)\right\} \quad . \tag{28}$$

The integrals in Eqs. (26) and (28) can be evaluated analytically in some limiting cases.

(*a*) In the case of not-too-large aspect ratio of the beam $a \geq b$ and for very small crossing angles $\delta \ll 1$, the arguments of both the error functions in Eq. (26) and the Dawson's functions in Eq. (28) are small. Since for small x, $\Phi(x) \approx 2x/\sqrt{\pi}$, we get

$$\Delta_y = \frac{2Nr_0 d}{\gamma b(a+b)}\delta \qquad \text{for} \quad \delta \to 0 \quad . \tag{29}$$

Similarly, since $D(x) \to x$ for $x \to 0$, we get in this case

$$\Delta_x = \frac{2Nr_0 d}{\gamma a(a+b)}\delta \qquad \text{for} \quad \delta \to 0 \quad . \tag{30}$$

(*b*) In the opposite case of large δ, when the arguments of both error function are large,

$$\Phi(x) \to 1 - \exp(-x^2)/x\pi \qquad \text{for} \quad x \gg 1 \quad .$$

Correspondingly, for such values of δ for which $\tau \gg 1$

$$\Delta_y = \frac{8Nr_0}{\sqrt{\pi}\gamma d\delta}\left(1 - \frac{2b^2}{ad\delta}\right) \quad . \tag{31}$$

(*c*) For a particular case of a flat beam $b \ll a$, in the region of the values of δ where τ is small, the argument of the second error function in Eq. (26) is typically small, while the argument of the first error function is large. In this case, formula (25) gives

$$\Delta_y = \frac{4Nr_0}{\gamma\sqrt{a^2 - b^2}} \quad . \tag{32}$$

Note that in this case the kick does not depend on the crossing angle δ; that is the direct consequence of the homogeneity of the field in the region which contributes to the kick.

For intermediate values of δ, the integration in formulae (26) and (28) should be performed numerically.

7. NUMERICAL EXAMPLE

To get the feeling of the magnitude of the kicks, consider the following numerical example for a beam configuration typical for a test on crab crossing in PEP.[3]

$$\begin{aligned}
N &= 10^{10} \quad , & I &= 0.2 \text{ mA} \quad ; \\
\gamma &= 2 \cdot 10^{4} \quad , & P &= 10 \text{ GeV/c} \quad ; \\
\sigma_z &= 1 \text{ cm} \quad , & d &= \sqrt{2} \text{ cm} \quad ; \\
\sigma_x &= 0.274 \text{ mm} \quad , & a &= 0.387 \text{ mm} \quad ; \\
\sigma_y &\ll \sigma_x \quad , & b &\ll a \quad .
\end{aligned}$$

The solid curves in Fig. 2 represent the derivatives Δ_y and Δ_x as functions of the crossing angles δ_y and δ_x for the vertical and horizontal crossing, respectively. The curves are obtained by direct numerical integration in Eqs. (26) and (28). The dashed curves represent the estimates Eqs. (29) and (30).

Formula (32) gives, in this case, $\alpha_y = 14.6 z_0/d$ μrad.

In Fig. 3, the kicks α_x and α_y are presented as functions of z_0/d for three different values of the crossing angle δ_x and δ_y, respectively.

REFERENCES

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FIGURE CAPTIONS

1. Crossing geometry in (y, z) plane. The position of the particle is shown at the time $t = 0$.

2. The derivatives $\Delta \equiv d\alpha/dz_0$ at $z_0 = 0$ [see Eqs. (25) and (27) in text]. In each case, the solid lines give the result of numerical integration in Eqs. (26) and 28, respectively. The dashed lines represent the approximation given by Eqs. (29) and (30). Calculations are done for the following values of parameters: $\sigma_x = 372\ \mu$m, $\sigma_y = 8.1\ \mu$m, $\sigma_z = 1$ cm, $\gamma = 2 \cdot 10^4$, and $N = 10^{10}$.

3. Horizontal and vertical kicks α_y and α_x as functions of the ratio z_0/d [see Eqs. (25) and (27) in text]. Numerical integration in Eqs. (17) and (24) is performed for the same bunch parameters as in Fig. 2 for three values of the crossing angle δ_y and δ_x, respectively: (a) $\delta = 4$ mrad; (b) $\delta = 8$ mrad; (c) $\delta = 12$ mrad.