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The Spectrum Of Topological Quantum Field Theories On N–Manifolds *

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ABSTRACT

The topological particle is first quantized and its spectrum is found to be given by the de Rahm cohomogy ring of the target manifold. Morse theoretic arguments are used to show how this topological spectrum depends on the gauge fixing. A supermanifold formalism is introduced which leads to the generalized Donaldson invariants. It is found that the BRST symmetry cannot be spontaneously or dynamically broken by conventional methods.

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I. Introduction

Approximately one year ago, Witten [1] wrote a collection of three papers on topological quantum field theories (TQFT's). These are theories which only exist quantum mechanically; as there is no classical physics associated with them. Up to a possible surface term, the entire Lagrangian is obtained from the gauge fixing of a local symmetry. The only observables in these theories are topological invariants. By definition, these invariants are independent of the choice of points on the manifold. Thus the quantum field theories from which they are obtained should arise as the second quantization of a particle whose coordinates may be arbitrarily chosen. This particle will be called a topological particle (TP). Its existence is only quantum mechanical and will be derived from a very simple BRST gauge fixing of a topological symmetry (TOPSY).

To verify that the second quantized TP gives the TQFT's on n-manifolds, the complete procedure must be carried out. However, the primary stage in that process, namely the first quantization, will quickly lead to the identification. The physical states will be shown to be given by $H^*(M)$, the de Rahm cohomology ring on the space-time manifold, M. Two points must be kept in mind. First, for the cohomology arguments used here, we will assume that M is compact and orientable. Should M not be compact, the wavefunctions must be compactly supported. Secondly, the phrase "physical states" will be used euphemistically. They are only physical in the sense of BRST–cohomology. Unless the TOPSY is broken they are not to be associated with real propagating degrees of freedom.

The (anti–)ghosts and TP coordinates extend the target manifold to a space over which the analogs of superfields may be constructed. These will be field representations of the TOPSY algebra. They will have $D/2$-forms and the complete set of ghost fields of the D-dimensional space-time TQFT as their components. The components of the polynomials in the fields on this extended space may then be used to compute the topological invariants.

So the framework of first quantization immediately leads to a generalization of all even-dimensional TQFT's; futhermore, as will be discussed below, also to odd-dimensional TQFT's.

As topological theories, they contain no local physics. Nevertheless, their renormalizability properties and relationship to supersymmetry may prove to be interesting from the physical point of view. If there is to be local physics, the topological symmetry from which they are constructed (via BRST gauge fixing), must be broken. We will also address this possibility here. Rather than directly examining TOPSY breaking in the space-time theories (topological Yang-Mills, gravity, *etc.*), we will address the question in terms of the first quantization. As the TOPSY is a BRST symmetry, we will see that spontaneous TOPSY breakdown is not possible. Apart from the question of space-time TOPSY breaking, the procedure used may offer a novel way of studying spontaneous supersymmetry breaking, when applied to the super-particle (as opposed to spinning particle).

The action for topological quantum mechanics (TQM) was given in ref. [2] in the context of stochastic quantization. Its relationship to supersymmetric quantum mechanics was pointed out and a potential was introduced. A related treatment for the non-linear sigma model of the TP was given in refs. [3,4]. The spectrum of the theory with a one-dimensional target space was given in ref. [5].

In section II, the action and global symmetry algebras of the TP action will be given. We will find that the TOPSY algebra is a twisting of the supersymmetry algebra of the N=2 spinning particle. As all of the information we will need may be obtained from the theory on a flat target manifold, M, we will assume the latter only for simplicity.[1] The space-time spectrum of states obtained from the first quantization will be given in section III. We will see that the physical states are in one to one corresponsence with the states

[1] Although this fixes the topology of M, the calculations to follow may also be carried out on a curved target manifold.

of the space-time TQFT's. Thus the quantization suggests a classification scheme for TQFT's in arbitrary space-time dimensions. The ghost field of the TP multiplet will be used as a Grassmann coordinate in a manner similar to the use of the spinor coordinates of superspace. However, this coordinate will be an anti-commuting vector, not a spinor. Fields over this "topospace" will be constructed out of component fields which contain the space-time TQFT multiplets. This will be done in section IV. It will also be shown there that this extended space provides a convenient way of constructing the topological invariants. A Morse theoretic counting of the states will be given in section V. Such a construction explicitly allows us to see how the topological spectrum depends on the gauge fixing function. A discussion of TOPSY breaking will be given in section VI. It will be argued there that spontaneous TOPSY breaking is not possible. However, breaking the symmetry via topology changing mechanisms may be possible. Conclusions are given in section VII.

II. Topological Particle and the TOPSY Algebra

To begin, we must gauge fix the local, topological symmetry whose BRST transformation is

$$\hat{\delta} x^a = \lambda^a \quad , \tag{2.1}$$

where $\hat{\delta}$ is the BRST operator realized as $\hat{\delta} = [iQ, \ ]$ in terms of the BRST charge, Q. The object x^a is the space-time coordinate of the particle mapping the world-line, Σ, to space-time, $x : \Sigma \rightarrow M$. It is assumed that Σ is connected so that it is either R or S^1. The object λ^a is a real anti-commuting ghost field. Both x^a and λ^a are functions over the world-line. The one-cycle $\oint dx^a A_a(x) = \int d\tau \dot{x}^a A_a$ is invariant under eqn. (2.1) *iff* $dA = 0$. A simple gauge fixing is $\dot{x}^a \equiv 0$, where the dot denotes differentiation with respect to the world-line's coordinate, τ. We would expect that the canonical momentum conjugate to x^a is $p^a \propto \dot{x}^a$. Heuristically, the gauge fixing constraint would then imply that $p^a|\Omega\rangle = 0$, where $|\Omega\rangle$ is an element of the wavefunction sector and will be some space-time tensor. On fields this would imply that $d\Omega = 0$ (where d is the exterior derivative on M) and/or the adjoint to this equation.

As we will eventually study TOPSY breaking we will need a mechanism for introducing a potential term into the gauge fixed Lagrangian. To this aim, a more interesting gauge slice is

$$\dot{x}^a \ + \ A^a(x) \ = \ 0 \ . \tag{2.2}$$

This was used in the construction of the non-linear sigma model in refs. [3,4]. We may think of $A^a(x)$ as a vector field on M, *i.e.* $A \in T(M)$. Then the slice defines an integral curve on M. Once again, the left-hand-side of eqn. (2.2) is the familiar conjugate momentum. We expect that all physical fields will be closed and/or co-closed. One could also imagine other gauge fixings; to S^D for example. However, this would not lead to any propagating fields in the Lagrangian. Any reasonable gauge fixing must include a derivative with respect to the Σ coordinate. A single such derivative will lead to the usual kinetic terms. Notice that the statement $P_a = 0$ means that the little group is the full Lorentz group. In this section, we will keep A arbitrary (requiring $dA \neq 0$ means starting with a zero Lagrangian). In the next section, we will set $A = 0$ for simplicity. For the discussion of TOPSY breaking, it will be taken to be $A = dW$, for some scalar function $W(x)$. Let us now proceed with the required analysis.

Follow the BRST gauge fixing procedure and define a gauge fixing plus Faddeev-Popov ghost lagrangian

$$\begin{aligned} \mathcal{L} \ &= \ -i\hat{\delta}[\beta^a(\dot{x}_a \ + \ A_a(x) \ + \ \tfrac{1}{2}\alpha_0 B_a)] \\ &= B^a(\dot{x}_a \ + \ A_a(x) \ + \ \tfrac{1}{2}\alpha_0 B_a) \ + \ i\beta^a(\dot{\lambda}_a \ + \ \lambda^b A_{a,b}) \ . \end{aligned} \tag{2.3}$$

Here β^a is the real anti-ghost, B_a is the real BRST auxiliary field and α_0 is the gauge fixing constant. In general, a space-time metric must be introduced in order that the inner product be well defined. However, a flat target manifold will be assumed for calculational simplicity. In reading off the second line, the additional relations $\hat{\delta}\beta^a = iB^a$, $\hat{\delta}B_a = 0$ and eqn. (2.1) have been used, along with $A_{a,b} \equiv \frac{\partial A_a}{\partial x^b}$. When B is integrated out of eqn. (2.3), we quickly obtain

$$\mathcal{L} \ = \ -\frac{1}{2\alpha_0}[\dot{x}^2 \ + \ A^a A_a] \ + \ i\beta^a \dot{\lambda}_a \ + \ i\beta^a \lambda^b A_{a,b} \ - \ \frac{1}{\alpha_0}\dot{x}^a A_a \ . \tag{2.4}$$

This is suggestive of a spinning particle. Notice that the last term is a surface term when $A = dW$. If the classical Lagrangian is $\mathcal{L}_0 = \frac{1}{\alpha_0}\dot{x}^a A_a$, then only the first three terms above contribute to $\mathcal{L}_T \equiv \mathcal{L}_0 + \mathcal{L}$. In this section, we will primarily work with eqn. (2.3).

With the canonical momenta $P_a = B_a$ conjugate to x_a and $\Pi^a = -i\beta^a$ conjugate to λ_a, we find the Hamiltonian to be

$$H \ = \ -\tfrac{1}{2}\alpha_0 P^2 \ - \ A \cdot P \ - \ i\beta^a \lambda^b A_{a,b} \ . \tag{2.5}$$

This suggests that the canonical choice for α_0 is $\alpha_0 = -1$. The BRST charge is

$$Q \ = \ \lambda^a B_a \ . \tag{2.6}$$

Its action on the fields in eqn. (2.3) is given by

$$\begin{aligned} [iQ, x^a\} \ &= \ \lambda^a \ , \qquad [iQ, \lambda^a\} \ = \ 0 \ , \\ [iQ, \beta^a\} \ &= \ iB^a \ , \qquad [iQ, B^a\} \ = \ 0 \ . \end{aligned} \tag{2.7}$$

To see that eqn. (2.6) indeed gives this result compute and replace the Poisson brackets by the usual (anti-)commutators to find $[x^a, P^b] = i\eta^{ab}$ and $\{\lambda^a, \beta^b\} = \eta^{ab}$. With these relations, eqn. (2.7) follows immediately. Also notice that the reality of the fields implies that Q is real.

In principle, it is possible to add terms of the form $\hat{\delta}$(something) to eqn. (2.3). The "something" here contributes a potential, for example. The fields β^a and x^a are the only ones which transform under Q. Contributions to the charge Q can come only from the appearance of world-line derivatives of these fields. Whenever there is an $\dot{x}^a$ factor in the Lagrangian, it contributes isomorphically to both P_a and Q. Introducing a $\dot{\beta}^a$ term will mean that B_a is no longer auxiliary. The BRST procedure is invalidated by this. So Q will always be proportional to P. The importance of this point will be manifest in the discussion of spontaneous TOPSY breaking (see section VI).

There is an additional charge we will need. This is the hermitean ghost number charge given by

$$\mathcal{G} \;=\; i\tfrac{1}{2}(\beta^a \lambda_a \;-\; \lambda^a \beta_a) \;\; . \tag{2.8}$$

It assigns ghost numbers (eigenvalues of $i\mathcal{G}$) to the fields via

$$\begin{aligned} [i\mathcal{G}, x_a\} \;&=\; 0 \;\; , \qquad [i\mathcal{G}, \lambda_a\} \;=\; \lambda_a \;\; , \\ [i\mathcal{G}, \beta_a\} \;&=\; -\beta_a \;\; , \qquad [i\mathcal{G}, B_a\} \;=\; 0 \;\; . \end{aligned} \tag{2.9}$$

Look at the first line of eqn. (2.3). Covariantize it with respect to a background world-line metric. Take its variation with respect to this field. This then implies that the Hamiltonian is

$$H \;=\; [iQ, \Lambda] \;\; , \qquad \Lambda \;\equiv\; i\beta^a(\tfrac{1}{2}\alpha_0 P_a \;+\; A_a) \;\; . \tag{2.10}$$

This can be quickly checked by explicitly using eqn. (2.7). Define $\bar{Q} \equiv i2\Lambda$ with $\bar{Q}^\star = \bar{Q}$, then $[Q, \bar{Q}\} = 2H$ along with $Q^2 = \bar{Q}^2 = 0$. Make the "twistings" $Q_1 \equiv \frac{1}{\sqrt{2}}(Q + \bar{Q})$ and $Q_2 \equiv i\frac{1}{\sqrt{2}}(Q - \bar{Q})$. Then the TOPSY algebra of the Q_i, H and $\mathcal{G}$ operators is

$$\begin{aligned} [\mathcal{G}, H\} \;&=\; 0 \;\; , \qquad [Q_i, H\} \;=\; 0 \;\; , \\ [Q_i, Q_j\} \;&=\; 2H\delta_{ij} \;\; , \qquad [\mathcal{G}, Q_i\} \;=\; -\epsilon_{ij} Q_j \;\; . \end{aligned} \tag{2.11}$$

Equation (2.11) is the algebra of N=2 supersymmetry in one-dimension. The ghost number charge $\mathcal{G}$ is to be identified with the $O(2)$ charge of the N=2 theory. This result is in agreement with the demonstrations [6] that the D=4 topological Yang-Mills theory may be obtained from twisting the D=4, N=2 super Yang-Mills theory. It has also been shown [7] that the β–function (in particluar, the coefficient) of the two theories are identical. For purposes of comparison with the N=2 theory, it is useful to define $\psi_1^a \equiv \frac{1}{\sqrt{2}}(\lambda^a - \alpha_0\beta^a)$ and $\psi_2^a \equiv i\frac{1}{\sqrt{2}}(\lambda^a + \alpha_0\beta^a)$. Then, for example, the $O(2)$ generator $M \propto \psi_1^a\psi_{2a}$ may be identified with $\mathcal{G}$ quite readily.

We have learned that Q indeed has the correct ghost number assignment and that $\bar{Q}$ has the ghost number of an anti-BRST charge. Furthermore, as $(B = P) \rightarrow -(B = P)$, $\lambda \rightarrow \beta$ and $\beta \rightarrow -\lambda$ under reflection, it follows that $Q \rightarrow -\bar{Q}$ and $\bar{Q} \rightarrow Q$ if $\alpha_0 = -1$. Henceforth, this choice for α_0 will be taken.

With the algebra (2.11), we are now in a position to use the Hodge theory arguments of ref. [1] relating the Q-cohomology to the Hamiltonian ground-states. The following results are useful to keep in mind: Hamiltonian eigenstates $(\{|\Omega\rangle; H|\Omega\rangle = E|\Omega\rangle \text{ and } Q|\Omega\rangle = 0\})$ are trivial in cohomology unless $E = 0$. Consequently, the physical subspace must satisfy $H|\text{phys}\rangle = 0$. Furhermore, $\bar{Q}|\text{phys}\rangle = 0$ up to zero-norm states if the physical subspace is of positive definite metric. Observe that these results are all consequences of the TOPSY algebra.

Yet another interesting feature of eqn. (2.11) is that the TOPSY algebras of the space-time TQFT's are also given by this equation [1]. Thus we may interpret eqn. (2.3) as either the Lagrangian for the TP or as a one-dimensional realization of the symmetry algebra of TQFT's. In the same way, the superparticle algebra (as opposed to the spinning particle's algebra) is the same as that of the space-time theory.

To further characterize the structure of TQFT's, it is useful to study the global space-time symmetries (apart from the BRST symmetry) of the Lagrangian (2.3). The construction of the Poincaré algebra is standard. For the Lorentz generator, one readily finds

$$J^{ab} = x^{[a}P^{b]} - i\beta^{[a}\lambda^{b]} \ , \tag{2.12}$$

from $\delta_\omega x^a = \omega^{ab}x_b$ and $\delta_\omega\lambda^a = \omega^{ab}\lambda_b$. The transformation on λ^a was chosen so that δ_ω commutes with the TOPSY transformations. Keep in mind that P_a vanishes only on physical states. These generators act on the TP multiplet as

$$\begin{aligned}
&[P_a, x_b] = -i\eta_{ab} \ , \qquad [P_a, \lambda_b] = 0 \ , \qquad [P_a, \beta_b] = 0 \ , \\
&[J_{ab}, x_c] = i\eta_{c[a}x_{b]} \ , \qquad [J_{ab}, \lambda_c] = i\eta_{c[a}\lambda_{b]} \ , \\
&[J_{ab}, \beta_c] = i\eta_{c[a}\beta_{b]} \ .
\end{aligned} \tag{2.13}$$

The Poincaré algebra is then easily computed and found to be the usual one. One can also show that the elements of the TOPSY algebra commute with the P and J generators.

As a representation of the D=4 Clifford Algebra of the (anti-)ghosts, take the set of Γ–matrices of ref. [7]:

$$\begin{aligned}
&\beta^a = \frac{1}{\sqrt{2}}\bar{\Gamma}^a \ , \qquad \lambda^a = \frac{1}{\sqrt{2}}\Gamma^a \ , \\
&\bar{\Gamma}^a = (\Gamma^0, -\Gamma^i) \ , \\
&\Gamma^0 = \mathbf{1} \ , \qquad \Gamma^1 = \sigma^3 \otimes i\sigma^2 \ , \\
&\Gamma^2 = i\sigma^2 \otimes \mathbf{1} \ , \qquad \Gamma^3 = \sigma^1 \otimes i\sigma^2 \ , \\
&\{\Gamma^i, \Gamma^j\} = -2\delta^{ij} \ , \qquad [\Gamma^i, \Gamma^j] = 2\epsilon^{ijk}\Gamma^k \ .
\end{aligned} \tag{2.14}$$

The σ^i are the Pauli matrices. With this, the "Lorentz spin" generator of the D=4 TQFT has as its components:

$$\mathcal{M}^{ij} = i\epsilon^{ijk}\Gamma^k \ , \qquad \mathcal{M}^{0i} = -i\Gamma^i \ . \tag{2.15}$$

There is clearly a close relationship between the $N = 2$ spinning particle and the TP. However, an important difference lies in the fact that the TOPSY charge is independent of A. This will lead to a major difference in the respective symmetry breakdowns. Furthermore, it must be stressed that the TOPSY algebra of the space-time TQFT is the same as that of the TP. The on-shell spectra of the space-time TQFT's is given by the first quantization of the TP. Let us now look at the Hilbert space of this particle.

III. Topological Spectrum

To investigate the spectrum of the theory, we turn to the seminal work of Kugo and Ojima [8]. Physical states are defined by the condition $Q|\mathrm{phys}\rangle \equiv 0$ mod null (Q-exact) states. As follows from ref. [8], $Q|\mathrm{phys}\rangle = 0$ imposes the gauge fixing condition $B_a = P_a = 0$ on the wavefunctions of physical states since $0 = \langle\mathrm{phys}|[Q, \beta_a\}|\mathrm{phys}\rangle = \langle\mathrm{phys}|B_a|\mathrm{phys}\rangle$. As we saw in the previous section, we must also impose $H|\mathrm{phys}\rangle = 0$. Let us now see what states satisfy these constraints. In this and the next section, we will take $A = 0$ for clarity.

In principle, the quantization is similar to that of the N=2 spinning particle [9,10,11b]. We will follow the analysis for that theory. There are, of course, important differences between the two theories. For example, the vacua and constraints differ. The "fermions" here carry non-zero ghost number. The space-time physics drastically differs between the two theories.

Define [11] the ghost vacua $|0\rangle_\lambda$ and $|0\rangle_\beta$ by the conditions

$$\begin{aligned} \lambda_a|0\rangle_\lambda &= \beta^a|0\rangle_\beta = 0 \ , \\ {}_\lambda\langle 0|0\rangle_\lambda &= {}_\beta\langle 0|0\rangle_\beta = 0 \ , \\ {}_\beta\langle 0|0\rangle_\lambda &= 1 \ . \end{aligned} \tag{3.1}$$

The ghost numbers of these vacua are

$$i\mathcal{G}|0\rangle_\lambda = \tfrac{1}{2}D|0\rangle_\lambda \ , \qquad i\mathcal{G}|0\rangle_\beta = -\tfrac{1}{2}D|0\rangle_\beta \ , \tag{3.2}$$

where D is the dimension of space-time. The ghost sector contains states built from either of the following:

$$
\begin{aligned}
|N_{\mathcal{G}} = \tfrac{1}{2}D - k\rangle^{a_1\cdots a_k} &= \beta^{a_1}\cdots\beta^{a_k}|0\rangle_\lambda \ , \\
|N_{\mathcal{G}} = -\tfrac{1}{2}D + k\rangle_{a_1\cdots a_k} &= \lambda_{a_1}\cdots\lambda_{a_k}|0\rangle_\beta \ ,
\end{aligned}
\tag{3.3}
$$

where $N_{\mathcal{G}}$ labels the ghost number.

The wavefunction sector ($|\Psi\rangle$) is built out of the coordinate and momentum vacua $|0\rangle_x$ and $|0\rangle_P$. These must satisfy the conditions

$$
\begin{aligned}
x_a|0\rangle_x &= P_a|0\rangle_P = 0 \ , \\
{}_x\langle 0|0\rangle_x &= {}_P\langle 0|0\rangle_P = \delta^D(0) \ , \\
{}_x\langle 0|0\rangle_P &= 1 \ .
\end{aligned}
\tag{3.4}
$$

An element of the state vector space is obtained from the tensor product of states given by eqn. (3.3) and wavefunction states built out of the x and P vacua. They will be labelled as $|\Omega, N_{\mathcal{G}}\rangle$, where Ω labels the wavefunction. Schematically, a state is of the from $|\Omega, N_{\mathcal{G}}\rangle = \Omega(x)|0\rangle_P|N_{\mathcal{G}}\rangle$. Its bra state is built out of the opposite ghost vacua and the ${}_x\langle 0|$ bra. This assures a finite inner product [11]. The physical subspace is given by the BRST cohomology class (ker$Q$/im$Q$) modulo zero norm states.

A zero ghost number state in D=4 space-time dimensions is

$$
|F,0\rangle = F_{ab}(x)|0\rangle_P\beta^a\beta^b|0\rangle_\lambda \ . \tag{3.5}
$$

Due to the anti-symmetry of the β's, F is a 2-form. The condition $Q|F,0\rangle = 0$ leads to

$$
\partial^a F_{ab} = 0 \ . \tag{3.6a}
$$

The anti–BRST charge is realized on $|F,0\rangle$ as $\bar{Q}|F,0\rangle = -i\frac{1}{3!}\partial_{[a}F_{bc]}|0\rangle_P\beta^a\beta^b\beta^c|0\rangle_\lambda$. Requiring $\langle 0,F|\bar{Q}|F,0\rangle$ to be zero yields

$$
\partial_{[a}F_{bc]} = 0 \ . \tag{3.6b}
$$

If F were exact, this would be interpreted as the Bianchi Identity. One can generalize these expressions to k–forms. It is then learned that Q acts on a wavefunction as δ and $\bar{Q}$ as d. So $H = \frac{1}{2}[Q, \bar{Q}\} = \frac{1}{2}(\delta d + d\delta)$ gives the realization of the Hamiltonian on wavefunctions to be half the Laplacian, Δ, on k–forms. Imposing $H|F,0\rangle = 0$ as the Hamiltonian physical state constraint requires

$$\Delta F \ = \ 0 \ . \tag{3.6c}$$

Clearly, (3.6a) is the sourceless Maxwell equation if F is exact. On the other hand, for eqn. (3.5) not to be a null state, F must not be exact. In addition to $F = 0$ as a solution of these two equations, we may also have $F = \pm^\star F$. In fact, on a compact, oriented four-manifold with only one solution to these equations, F must be self-dual by Poincaré duality. The D=4 topological Yang-Mills theory [1] is obtained by gauge fixing to (anti-)self-dual solutions. Another interesting gauge fixing is the $F = 0$ (only if $H^1(M, R) \neq 0$ as in the Bohm-Aharanov Effect) condition [12]. The other $N_{\mathcal{G}} = 0$ state is obtained from eqn. (3.5) with $\beta \leftrightarrow \lambda$. It is dual to the state above. Consequently, it also leads to eqn. (3.6) but with $(a) \leftrightarrow (b)$.

For even D dimensions (and curved manifolds in general), the condition that H and Q annihilate physical states is satisfied by a zero ghost number $D/2$–form, $\omega_{D/2}$, which is harmonic and co-closed[2]

$$\Delta\omega_{D/2} \ = \ \delta\omega_{D/2} \ = \ 0 \ . \tag{3.7}$$

The δ here is the adjoint of d and should not be confused with the BRST's $\hat{\delta}$.[3] Now on a compact and oriented manifold, a form is harmonic if and only if it is closed and co-closed. So $H = 0$ already implies $Q = 0$. Furthermore, for $\omega_{D/2}$ to be a wavefunction

[2] If we were using the $|0\rangle_\beta$ ghost vacuum, $\omega_{D/2}$ would have to be closed. The BRST charge is realized as $Q \approx d$, on wavefunctions in this sector.

[3] Although the former is a realization of the latter on wavefunctions in the $|0\rangle_\lambda$ vacuum sector.

in the BRST cohomology, it must not be co–exact. Had we worked in the β–vacuum, $\omega_{D/2}$ would be harmonic, closed and not exact. Hence it must be a member of the $D/2$–de Rahm cohomology class. The number of such forms is equal to the k^{th} Betti number $\equiv b_k = \dim H^k(M)$. Only those $D/2$–form states exist for which the $H^{D/2}(M)$ is non-trivial. From Poincaré duality it may be inferred that $\omega_{D/2}$ is self-dual when $b_{D/2} = 1$.

An interesting result of this construction is found when we look at the $N_{\mathcal{G}} \neq 0$ sectors. Once again, let us restrict ourselves to $D = 4$. The $N_{\mathcal{G}} = +1$ state is

$$|\Psi, +1\rangle = \Psi_a(x)|0\rangle_P \beta^a |0\rangle_\lambda \ , \tag{3.8}$$

with Ψ being an anti-commuting 1-form. Application of the spin generator from eqn. (2.12) to this state yields that it is indeed a Lorentz vector. The condition $Q|\Psi, +1\rangle = 0$ leads to eqn. (a) below and $H|\Psi, +1\rangle = 0$ leads to (b):

$$(a)\ \delta\Psi = 0 \ , \qquad (b)\ \Delta\Psi = 0 \ . \tag{3.9}$$

For Ψ not to be a null state, it must not be co-exact. Equation (3.9b) implies (3.9a) along with

$$\partial_{[a}\Psi_{b]} = 0 \ . \tag{3.9c}$$

We recognize this as the primary ghost equation in the linearized topological theory of flat connections. Equation (3.9a) is the gauge fixing slice for the D=4 ghost symmetry. The solution to $\bar{Q}|\Psi, +1\rangle = 0$ is eqn. (3.9c). The $N_{\mathcal{G}} = 2$ state,

$$|\Phi, +2\rangle = \Phi(x)|0\rangle_P |0\rangle_\lambda \ , \tag{3.10}$$

unconditionally satisfies $Q|\Phi, +2\rangle = 0$. Of course, this is a consequence of the fact that $\delta(0 - \text{form}) = 0$. The vanishing of the Hamiltonian leads to

$$\Box\Phi = 0 \ . \tag{3.11}$$

This is to be identified as the equation of motion of the secondary ghost field of the D=4 linearized topological theory. The statement that Φ is harmonic must be supplemented by the condition that it not be co-exact.

These results may be generalized to arbitrary even D dimensions. In general, the ghost tower will consist of graded k-forms of degree $k = \frac{D-2N_{\mathcal{G}}}{2}$ (where $N_{\mathcal{G}}$ is also the space-time ghost number) which are elements of $H^k(M)$. Put differently, all form states of degree $k = 1, \ldots, D$ carry a ghost number charge $N_{\mathcal{G}} = \frac{1}{2}D - k$. There are also states dual to these, but with opposite ghost number.

States from the N=2 spinning particle similar to the $N_{\mathcal{G}} \neq 0$ sector arise when the global $O(2)$ symmetry of the supersymmetric theory is not gauged [13]. However, the anti-commuting field there is a spin$-\frac{1}{2}$ field. Also, F may be exact there. The physical spectrum of a spinning particle theory is not given by a super-charge cohomology. The ground states of the ungauged supersymmetric theory only has to satisfy $Q = H = 0$. However, the topological spectrum is composed of those states which are elements of the BRST cohomology. The Hodge Decomposition Theorem tells us that on compact manifolds, there is an isomorphism between the harmonic k–forms and $H^k(M, R)$. So that the zero-energy spectra of the two theories are isomorphic. This isomorphism is true only for the ground state. As the BRST charge, Q, annihilates all physical states and $H = [Q, \Lambda\}$, all of the physical states of a TQFT are "zero-energy" states. Supersymmetric states go beyond the zero-energy state to include excited states for which $Q|\text{boson}\rangle = |\text{fermion}\rangle$ and vice-versa.

IV. Topospace and TQFT Invariants

There is another way to realize the physical states of the previous section. This is in terms of fields which are functions over the space of the x's and λ's. In analogy with superfields, these fields will be called *topofields*. Topofields will give a field representation of the TOPSY algebra. In this section, we will assume an extension of the linearized results of the previous section to a full non-linear theory. Then if $D=4$, the 2–form F is a curvature on a G-bundle with gauge group G and covariant exterior derivative, $\mathcal{D}$.

The coordinates (x^a, λ^b) will be the coordinates of *topospace*, a supermanifold. Consider the Taylor series expansion of a scalar topofield, $\mathcal{S}(x,\lambda)$ in four dimensions:

$$\mathcal{S}(x,\lambda) \;=\; \Phi(x) \;+\; i\lambda^a\Psi_a(x) \;+\; i\lambda^a\lambda^b F_{ab}(x) \;+\; \dots \quad . \tag{4.1}$$

The ghost number of $\mathcal{S}$ is given by Φ to be 2. Successive component fields in the expansion decrease in ghost number by 1. Under a BRST transformation, $Q = -i\lambda^a\mathcal{D}_a$, this topofield transforms as $\hat{\delta}_T\mathcal{S} = [iQ, \mathcal{S}\}$:

$$[iQ, \mathcal{S}(x,\lambda)\} \;=\; \lambda^a\mathcal{D}_a\Phi \;+\; i\lambda^a\lambda^b\mathcal{D}_a\Psi_b \;+\; i\lambda^a\lambda^b\lambda^c\mathcal{D}_aF_{bc} \;+\; \dots \quad . \tag{4.2}$$

We then read off, when $\mathcal{S}$ is off-shell,

$$\begin{aligned} \hat{\delta}_T\Phi \; &= 0 \;\; , \\ \hat{\delta}_T\Psi_a \; &= \mathcal{D}_a\Phi \;\; , \\ \hat{\delta}_T F_{ab} \; &= \mathcal{D}_{[a}\Psi_{b]} \;\; , \qquad \mathit{etc.} \end{aligned} \tag{4.3}$$

The object $\hat{\delta}_T$ denotes the space-time TOPSY transformation. Observe that $\hat{\delta}_T$ is not nilpotent. The algebra closes up to a gauge transformation with parameter Φ.

In four dimensions, the expansion in eqn. (4.1) should be taken to terminate at the 2–form. At present, the only justification for this is the following observation. Consider $\mathcal{S}$ to be on-shell. Poincaré duality (on compact and orientable manifolds) tells us that if the expansion were continued to include the next two forms, they would be dual to the first two. For example, in D=4, the 3–form G would be given by ${}^*d$ acting on a 0–form.

(This bears close resemblance to supersymmetry where placing a chiral constraint on a scalar superfield means that some of the upper components in the superfield are defined in terms of space-time derivatives of the lower components.) Off-shell, the component fields are not required to be elements of $H^k(M)$ and so Poincaré duality no longer holds. Thus, in principle the expansion continues up to a D–form. However, terminating the expansion at the $D/2$–form gives the known transformation laws, *etc.* In fact, eqn. (4.3) gives the BRST transformations of the positive ghost number multiplet of the topological Yang-Mills theory [1]. Presumably, the forms of degree greater than $D/2$ are auxiliary. In the fully gauged fixed path integral they may be integrated out and replaced by their equations of motion.

The general form of the space-time BRST transformations on the k–form fields of the topological multiplet in even D–dimensions is

$$\hat{\delta}_T \omega_k \; = \; \mathcal{D}\omega_{k-1} \quad . \tag{4.4}$$

The 0–form, ω_0, is BRST invariant. As discussed in section III, the ghost number of the k–form is $N_{\mathcal{G}} = \frac{D}{2} - k$. From the structure of eqn. (4.4), it is obvious why ω_0 is invariant. It is interesting to note that the original TOPSY [1] which started the BRST analyzes: $\hat{\delta}A_a(x) = \Psi_a(x)$, does not directly appear in this sequence. It would, however, appear indirectly if F were an exact form. Although F must be an element of $H^2(M)$ on-shell, there is no such restriction off-shell.

Recall the anti-commutation relation: $[\beta_a, \lambda^b\} = \delta_a{}^b$, from which we realize the anti-ghost β_a as $\beta_a \equiv \frac{\partial}{\partial \lambda^a} \equiv -iD_a$. This Grassmann derivative provides for a more convenient way of defining the component fields. This is by the projection technique

$$\begin{aligned} \Phi(x) \; &\equiv \mathcal{S}(x, \lambda)| \quad , \\ \Psi_a(x) \; &\equiv D_a \mathcal{S}| \quad , \\ iF_{ab}(x) \; &\equiv \tfrac{1}{2!} D_{[a} D_{b]} \mathcal{S}| \quad , \qquad \textit{etc.}, \end{aligned} \tag{4.5}$$

in even dimensions. The slash, $|$, means take $\lambda = 0$ after the operation is performed. With this, let us see how the descendants of the $(0, D)$–form[4] given by polynomials in Φ arise in the topospace formalism. Consider a polynomial, $\mathcal{W}(\mathcal{S})$. Differentiate it with respect to λ_a and define

$$\mathcal{W}_k \equiv \tfrac{1}{k!} dx^{a_1} \wedge dx^{a_2} \wedge \cdots \wedge dx^{a_k} D_{a_1} D_{a_2} \cdots D_{a_k} \mathcal{W} \quad . \tag{4.6}$$

By construction, it is a k–form on M. One can show that

$$\begin{aligned} [Q, \mathcal{W}_{k+1}\}| &= i^k d\mathcal{W}_k| \quad , \\ [Q, \int_{C_k} \mathcal{W}_k\}| &= 0 \quad , \end{aligned} \tag{4.7}$$

where C_k is a homology k–cycle. Take $\mathcal{W}(\mathcal{S}) = \frac{1}{2} Tr(\mathcal{S}^2)$ as an example. Project it to components to find that on four-manifolds ($D^k \equiv D_{[a_1} D_{a_2} \cdots D_{a_k]}$):

$$\begin{aligned} W_0^4 &\equiv \mathcal{W}| = Tr(\tfrac{1}{2}\Phi \wedge \Phi) \quad , \\ W_1^3 &\equiv D\mathcal{W}| = Tr(\Phi \wedge \Psi) \quad , \\ W_2^2 &\equiv \tfrac{1}{2!} D^2 \mathcal{W}| = Tr(\tfrac{1}{2}\Psi \wedge \Psi + i\Phi \wedge F) \quad , \\ W_3^1 &\equiv \tfrac{1}{3!} D^3 \mathcal{W}| = Tr(i\Psi \wedge F) \quad , \\ W_4^0 &\equiv \tfrac{1}{4!} D^4 \mathcal{W}| = Tr(\tfrac{1}{2} F \wedge F) \quad . \end{aligned} \tag{4.8}$$

These k-forms, $W_k^{N_{\mathcal{G}}}$, carry a ghost number charge $N_{\mathcal{G}}$. Their k–cycle integrals are the Donaldson maps: $H_k(M) \rightarrow H^{4-k}(\mathcal{M})$ as constructed in ref. [1]. One can construct the generalized Donaldson invariants on any even dimensional, compact and oriented manifold from eqn. (4.6), $\mathcal{W}_N \equiv \frac{1}{N} Tr(\mathcal{S}^N)$, eqn. (4.5) and then computing the correlation functions of the resulting expressions. Whether or not these are zero would have to be ascertained on an individual basis. For the general case, Tr denotes a suitable inner product which is independent of the metric on M. To construct the invariants of the odd D-manifolds, one

[4] The notation $(k, N_{\mathcal{G}})$ indicates that the form is of degree k on M and its degree on the gauge orbit space is $N_{\mathcal{G}}$, the ghost number. When the second quantized theory is gauge fixed to a particular field configuration, the gauge orbit space becomes the moduli space, $\mathcal{M}$.

must first carry out the construction for the even $(D+1)$-manifold and then dimensionally reduce them. In this way, for example, the W's of the three dimensional Yang-Mills-Higgs [3,14] and supersymmetric Yang-Mills Higgs [15] theories, are easily realized.

Now let us instead work on the dual topospace with coordinates (x^a, β^b). Consider a Taylor series expansion of a $D/2$–form topofield in terms of the Grassmann anti-ghost coordinate, β^a:

$$\mathcal{F}(x,\beta) \;=\; F(x) \;+\; i\beta\wedge\Psi \;+\; i\beta\wedge\beta\wedge\Phi \;+\; \ldots \;, \tag{4.9}$$

where the expansion terminates at the $\mathcal{O}(\beta^{D/2})$ term whose coefficient is a 0-form. The D=4 expansion terminates at the 0-form Φ, the secondary ghost field with $N_{\mathcal{G}} = 2$. As the order in β increases, the ghost number of the component field increases. The topofield $\mathcal{F}$ is the curvature of the Weil algebra system of ref. [3,16]. In the context of the first work in ref. [16], the dual topospace is the product manifold $\tilde{\mathcal{T}} \equiv M \times \mathcal{A}/G$. Here $\mathcal{A}/G$ is the gauge orbit space with gauge group G. By the Kunneth formula, the de Rahm cohomology on $\tilde{\mathcal{T}}$ is given by $H^*(\tilde{\mathcal{T}}) \approx H^*(M)\otimes H^*(\mathcal{A}/G)$ with $H^k(\tilde{\mathcal{T}}) \approx \bigoplus_{l+m=k} H^l(M)\otimes H^m(\mathcal{A}/G)$. In a second quantized theory, with the TOPSY gauge fixed, $\mathcal{A}/G$ is the moduli space $\mathcal{M}$. The $W_k^{N_{\mathcal{G}}}$'s of eqn. (4.2) are found by the projection of $\mathcal{W}(\mathcal{F}) = \frac{1}{2}Tr(\mathcal{F}^2)$ in a manner similar to their derivation in the (x^a, λ^b) topospace.

It would be of interest to determine the relationship between the topospace construction introduced here and the extended moduli space of the second reference in [1]. This will not be investigated here. However, the following observations may prove useful. Assume that in the second quantized theory, one has gauge fixed to a certain field configuration. Consider a basis, ${e_I}^a$ $(I = 1,\ldots,N_{\mathcal{M}})$ for the deformations of the gauge fixing function of the TP. Write the "zero modes" of the ghost field, λ^a, as an expansion in this basis:

$$\lambda^a \;\equiv\; \sum_{I=1}^{N_{\mathcal{M}}} \zeta^I {e_I}^a \;+\; (\mathrm{non-zero\ modes}) \;. \tag{4.10}$$

Then as ref. [1], the ${e_I}^a$ are a basis for the tangent space of the moduli space. With this, $D_a = i{e_a}^I \frac{\partial}{\partial\zeta^I}$, where an inverse for e_I has been assumed: ${e_a}^I {e_I}^b \equiv {\delta_a}^b$. Consequently it is easy to see that eqn. (4.6) becomes

$$\mathcal{W}_k = \frac{i^k}{k!} da^{I_1} \wedge \cdots \wedge da^{I_k} \frac{\partial}{\partial\zeta^{I_1}} \cdots \frac{\partial}{\partial\zeta^{I_k}} \mathcal{W} \ , \tag{4.11}$$

where $da^I \equiv dx^a {e_a}^I$ with the a^I being the bosonic coordinates of the moduli space. Together, the coordinates a^I and ζ^I are the even and odd coordinates of extended moduli space. The measure on this space is [1] $d\mu = d^{N_\mathcal{M}} a d^{N_\mathcal{M}} \zeta$ and we interpret $N_\mathcal{M}$ as the dimension of moduli space.

The $(4,0)$–form W_4^0 in $D = 4$ is the Pontryagin density. It is well known that it may be written as an exterior derivative of the Chern-Simons (CS) 3–form. Then on the boundary of the 4–manifold one has a CS density. An action in $2+1$ dimensions built out of the CS density is an *alter idem* of the TQFT's. It is metric independent but not invariant under the TOPSY: $\delta A = \psi$. Whatsmore, it exists classically. As is well known, it is straightforward to construct higher (odd) dimensional analogs of them. One starts with the n^{th} Chern character: $\Omega_{2n} = Tr(F^n)$ which may be written as $\Omega_{2n} = d\omega_{2n-1}$. However, the ω_{2n-1} are no longer quadratic in fields (even in the $U(1)$ theory). They are higher order in derivatives, also. If one were to use them as a Lagrangian, they would be interacting field theories without a kinetic term. With the results of this section, it is easy to obtain higher dimensional analogs of the D=3 CS action. They are simply realized by assuming a (D+1) oriented manifold with a boundary and taking the surface term from

$$S_{odd}^{Top} = \int_N I \wedge I = \int_{\partial N} m \wedge I \ . \tag{4.12}$$

Here $I \equiv dm$ is an exact $(D+1)/2$–form. The action is invariant under $\delta m = d\Lambda$. For example, the D=5 version is $S_{D=5} = \int d^5x B \wedge dB$, where B is a 2–form. Notice that these higher dimensional extensions are not constructed from connections on a G–bundle.

They are free field theories, independent of the metric on the manifold and may be exactly soluble (see ref. [17] and references therein).

V. Morse Theoretic Counting of Physical States

Morse theory offers a simple counting of the physical states given in section III. De Rahm cohomology tells us the number of states, but we will see how Morse theory relates this counting to the function $W(x)$. In this section, we will take eqn. (2.4) with $A = dW$ for our Lagrangian.

According to ref. [8], an observable, $\mathcal{O}$, must satisfy

$$[Q, \mathcal{O}\} = 0 \ . \tag{5.1}$$

The Hamiltonian and BRST charges are examples of trivial observables, since they both (anti)-commute with the BRST charge and vanish on physical states. In fact, it is easy to convince oneself that the only non-trivial observable in the one-dimensional field theory is the identity operator: $\mathcal{O} = \mathbf{1}$. The vacuum expectation value of this operator is given by the partition function. It is the analog of the Witten index and was shown by standard Morse theoretic arguments to be given by the Euler Characteristic of M [3,4]. It is computed [4] from eqn. (2.4) to be the sign of the determinant of the Hessian of W, $[H_p(W)]_{ab} = \partial_a \partial_b W|_p$, summed over all the non-degenerate critical points, p:

$$\mathcal{Z} = \sum_p \frac{\det H_p(W)}{|\det H_p(W)|} = \chi(M) \ . \tag{5.2}$$

Path integral methods based on eqn. (2.4), with $A = dW$, were earlier used to compute this index [18].

Just as $Tr(-1)^F$ is a natural index in the supersymmetric particle theory, there is such an index here. It measures the difference between the number of states of even and odd ghost number and arises from the following identity:

$$Tr(e^{\pm\pi\mathcal{G}}) = Tr(-1)^{N_\mathcal{G}} = n_{\text{even}} - n_{\text{odd}} \equiv \Delta \ . \tag{5.3}$$

This Δ is not to be confused with the Laplacian. The quantity n_{even} (n_{odd}) is the number of physical states with even (odd) ghost number. From section III we know that the number of physical states (which are also all zero-energy states) of a given ghost number, $N_\mathcal{G}$, is equal to the k^{th} Betti number of the manifold . The wavefunctions of these states are forms of degree $k = \frac{1}{2}(D - 2N_\mathcal{G})$, for some even space-time dimension D. It then follows that

$$\Delta = Tr(-1)^{N_\mathcal{G}} = (-1)^{D/2}\sum_k (-1)^k b_k = (-1)^{D/2}\chi(M) \ , \tag{5.4}$$

where b_k is the k^{th} Betti number of the compact and oriented manifold M. The last equality follows from de Rahm cohomology theory. Consequently, the index is given by

$$\Delta = (-1)^{D/2}\mathcal{Z} = (-1)^{D/2}\sum_p \frac{\det H_p(W)}{|\det H_p(W)|} = (-1)^{D/2}\sum_p (-1)^{i_p} \ . \tag{5.5}$$

The object i_p is called the index of the critical point, p, it is the number of negative eigenvalues of $H_p(W)$ and is invariant under coordinate changes.

It is useful to introduce some results from Morse theory [19] which underly the equations above. The Poincaré Polynomial, $P(M,t)$, and the Morse Series

$$P(M,t) \equiv \sum_k t^k b_k \ , \qquad \mathcal{M}(W,t) = \sum_p t^{i_p} \ , \tag{5.6}$$

are related by

$$P(M,t) - \mathcal{M}(W,t) = (1 + t)Q(t) \ , \tag{5.7}$$

where $Q(t)$ is a polynomial with non-negative coefficients. This is a remarkable formula as it relates cohomology ($P(M,t)$) to the properties of some non-degenerate function, W, on

M. It gives the Morse Inequalities. A function for which $P(M,t)$ and $\mathcal{M}(W,t)$ are equal is called a perfect Morse function. Examples of $P(M,t)$ have been given by Bott [19a]:

$$
\begin{aligned}
P(R^D,t) &= 1 \ , \qquad P(S^D,t) \ = \ 1 \ + \ t^D \ , \\
P(CP^D,t) &= \sum_{m=0}^{D} t^{2m} \ , \\
P(M \times N,t) &= P(M,t)P(N,t) \ .
\end{aligned}
\tag{5.8}
$$

Clearly,

$$
P(M,-1) \ = \ \mathcal{M}(W,-1) \ = \ (-1)^{D/2}\Delta \ , \tag{5.9}
$$

is manifested in eqns. (5.4) and (5.5). Consider the case when $t \ = \ +1$. The Poincaré Polynomial evaluates to

$$
P(M,+1) \ = \ \sum_k b_k \ = \ N \ , \tag{5.10}
$$

the total number of physical states: $N \equiv n_{even} + n_{odd}$. As N is not a topological invariant, it is expected to depend on topology. The number of even and odd physical states depend separately on $P(M,+1)$ and the Euler Characteristic. Morse theory tells us some more useful information. Consider the case where W is a perfect Morse function. There is a principle which gives the criteria under which W is such a function. It is called the Lucunary Principle. It simply says that if $M(W,t)$ is such that the product of any two consecutive coefficients in its expansion is zero, then W is a perfect Morse function for the manifold, M. Under these circumstances, the numbers of even and odd ghost number states are simply given by

$$
n_{\binom{\text{even}}{\text{odd}}} \ = \ \begin{cases} \#_{\binom{\text{even}}{\text{odd}}} & \text{if } D = 4l \ , \\ \#_{\binom{\text{odd}}{\text{even}}} & \text{if } D = 4l+2 \ . \end{cases} \tag{5.11}
$$

The quantity $\#_{\text{even}}$ ($\#_{\text{odd}}$) is the number of critical points of W for which i_p is even (odd) and $l \in Z^+$.

For most potentials of physical interest (such as the *sombrero* potential), there exists a submanifold of critical points. For example, suppose W defined on R^D is such that

$dW = 0$ implies that $x_p^2 = 1$ (properly normalized). Then there is a connected submanifold, $\mathcal{N} \subset M$, of critical points given by S^{D-1}. Such a submanifold is called [19a] a non-degenerate critical manifold for W if and only if $dW = 0$ along $\mathcal{N}$ and $\det H_{\mathcal{N}}(W) \neq 0$. Here the Hessian is computed with respect to coordinates normal to $\mathcal{N}$. Choose a set of points $y^A \equiv (y^{\hat{A}}, y^{\tilde{A}})$ so that $\mathcal{N}$ is locally given by $y^{\tilde{A}} = 0$ for $\tilde{A} = \dim(\mathcal{N}) + 1, \ldots, D-1$. Evaluate the Hessian at the y^A. The index of this critical manifold, $i_{\mathcal{N}}$, which is the number of negative eigenvalues of $H_{\mathcal{N}}(W)$, is constant along $\mathcal{N}$. The Morse Series is then defined by a sum over the critical manifolds:

$$M(W,t) \;=\; \sum_{\mathcal{N}} t^{i_{\mathcal{N}}} P(\mathcal{N},t) \;=\; \sum_{\mathcal{N}} \sum_{k} t^{(i_{\mathcal{N}}+k)} b_k(\mathcal{N}) \quad . \tag{5.12}$$

Here $b_k(\mathcal{N})$ denotes the Betti number of the compactly supported cohomology on $\mathcal{N}$ with real coefficients. The Morse Inequalities hold under a technicality which is that the normal bundle, $\nu(\mathcal{N})$, of $\mathcal{N}$ is such that its restriction to negatives eigenvalues ($\nu(\mathcal{N}) \equiv \nu^+(\mathcal{N}) \oplus \nu^-(\mathcal{N})$) is orientable. Under these circumstances and if W is a perfect Morse function, the number of even and odd ghost number states is given less transparently by

$$n_{\binom{\text{even}}{\text{odd}}} \;=\; \tfrac{1}{2} \sum_{\mathcal{N}} \sum_{k} [1 \;\pm\; (-1)^{(D/2+i_{\mathcal{N}}+k)}] b_k(\mathcal{N}) \quad . \tag{5.13}$$

Due to Lorentz invariance, this is not the complete story. The function W is a scalar under space-time Lorentz transformations; but the x's transform. In computing the critical points of W, one must be careful to mod out by $SO(D-1,1)$. Such a counting is the realm of Equivariant Morse Theory [19a]. The solution to this problem is to restrict the computation to the space $M/SO(D-1,1)$ and then carry out the Morse counting. This is a simplified result which is valid only for those cases in which there is no stabalizing subgroup.

V. Constraints on TOPSY Breaking

Spontaneous and dynamical supersymmetry breaking in supersymmetric quantum mechanics were studied in ref. [20]. The critical points of the potential were used to determine whether or not supersymmetry was broken. We have seen the relationship between the TP and the N=2 spinning particle. It is then natural to ask if it is possible to break the TOPSY. This question was asked before [3] for one-dimensional target manifolds Equation (2.4) will be the Lagrangian for this section.

A symmetry is broken if an operator, $\mathcal{O}$, exists for which $\langle\delta\mathcal{O}\rangle \neq 0$. Any observable is BRST invariant by definition. Let us also look at the other operators which transform under the fermionic symmetry. For example, a Goldstone fermion [21] arises from spontaneous symmetry breaking because the term into which the spinor field transforms has a *vev*. Our vacuum is given by the bra ${}_{\beta}\langle 0|_{x}\langle 0|$ and the ket $|0\rangle_P|0\rangle_\lambda$. It is translationally invariant: $P_a|0\rangle_P|0\rangle_\lambda = 0$. Now recall that $Q = \lambda^a P_a$ is the TOPSY charge. If $\langle 0|[Q,\mathcal{O}\}|0\rangle \neq 0$ then $Q|0\rangle \neq 0$ and the vacuum cannot be translationally invariant. The spontaneously broken theory would be ill-defined. Consider some examples keeping in mind that the *vev* of the β's and λ's is zero since $\lambda^a|0\rangle_P|0\rangle_\lambda = 0$ and ${}_{\beta}\langle 0|_{x}\langle 0|\beta^a = 0$. Now the only operators which are not BRST invariant are β^a, $\bar{Q}$, $\mathcal{G}$ and x^a. The BRST anti-commutator of $\bar{Q}$ is $[Q,\bar{Q}\} = 2H$. With the properties of the vacuum, one has $\langle H\rangle = 0$. Similarly, $[Q,\mathcal{G}\}$ is proportional to Q with a *vev* $\langle Q\rangle = \langle\lambda^a P_a\rangle = 0$. So we are left with $[Q,\beta^a\} = B^a$. Recall that the BRST auxiliary field is given by $B_a = P_a$, so once again $\langle[Q,\beta^a\}\rangle = \langle B^a\rangle = 0$. Given an operator $\mathcal{O}(x)$, the *vev* of its BRST commutator is zero since λ annihilates the vacuum ket. TOPSY cannot be spontaneously broken. This is understandable since it is a BRST symmetry.

We saw that the TOPSY algebra is a twisted D=1, N=2 supersymmetry algebra. Then how is it that TOPSY cannot be spontaneously broken, but the supersymmetry can be broken? The answer is that Q and $\bar{Q}$ are not supersymmetry charges. They can be combined to give supercharges by: $Q_k \equiv -(-1)^{1/k}\frac{1}{\sqrt{2}}(\bar{Q} - (-)^k Q)$. Make a similar defintion for the supersymmetric "fermions": $\psi_k \equiv -(-1)^{1/k}\frac{1}{\sqrt{2}}(\beta^a - (-)^k \lambda^a)$. Then the supersymmetry transformations are:

$$\begin{aligned} [iQ_i, x^a\} &= \psi_i \quad , \\ [iQ_i, \psi_j^a\} &= i\delta_{ij}(P^a - \partial^a W) \; + \; \epsilon_{ij}\partial^a W \quad . \end{aligned} \tag{6.1}$$

Thus $\langle [Q_i, \psi_j^a\}\rangle \neq 0$ if $\langle dW \rangle \neq 0$ at the minimum of the "potential" in eqn. (2.4). Ordinarily, Lorentz invariance would prevent a *vev* for dW. For a one-dimensional space-time, Lorentz invariance is not a problem and in that case supersymmetry is spontaneously broken.

Although it is not possible to spontaneously break the TOPSY symmetry, it is possible to break the anti-BRST symmtery under the same conditions for supersymmetry breaking. To see this, consider the anti-commutator

$$[\bar{Q}, \lambda^a\} \;=\; P^a \;-\; 2\partial^a W \quad . \tag{6.2}$$

Take the target manifold to be a product, $M = Y \times S^1$ where Y is $(D-1)$–dimensional. If only the derivative of W in the S^1 direction is non-zero at the minumum of the potential, then the *vev* of the anti-commutator is non-zero and Lorentz invariant.

Given that it is not possible to spontaneously break TOPSY, we must search for other mechanisms to break the symmetry if the hope for local physics is to be kept alive. It is well known that for judicious choices of the function W, the supersymmetry of supersymmetric quantum mechanics can be dynamically broken [20]. This is because the condition $Q\Psi = 0$ (Q is a supercharge and Ψ is a supersymmetric wavefunction) has the solution (in the case D=1) $\Psi(x) = \Psi(0)\exp\left[\frac{1}{\hbar}\int_0^x dy W(y)\sigma_3\right]$. This is a state only if Ψ is normalizable. Otherwise, no such state exists and supersymmetry is dynamically broken by quantum corrections. In the TOPSY case, we have that the BRST charge must vanish on any physical state. However, as we have seen above, there is no potential contribution to this charge, even beyond tree level. So we simply have the condition of section III, namely the wavefunction must be co-closed. The independence of the BRST charge on the potential implies that perturbative effects cannot break the TOPSY.

There is another form of symmetry breaking through which one may imagine breaking the TOPSY. It is known [22] that topology change breaks global symmetries. The TOPSY is a BRST symmetry. BRST symmetries are global symmetries of a gauge fixed action. However, it appears that if one global symmetry of a field theory is broken by topology change, all of its global symmetries are so broken [23]. Apart from the BRST symmetry, TQFT's also have global ghost number and scaling symmetries. Breaking the last of these symmetries may be a undesireable feature for the resulting field theory. Nevertheless, we will neglect the latter in the discussion to follow.

First, let us clarify the meaning of topology change in the context of a theory whose observables are topological invariants. It is useful to recall some definitions. Topological spaces are *topologically equivalent* if and only if they are homeomorphic to each other. Given a property which is endemic to all of these spaces, that property is called a *topological invariant.* In distinguishing between spaces which are or are not topologically equivalent, it is sufficient to find a topological invariant of one of the spaces which is not an invariant for the other [24]. After all, topology characterizes the properties of spaces which are invariant under homeomorphisms. By definition, topology change subverts the topological class of the manifold. A typical example is taking a sphere and putting a handle (wormhole) on it. Such a transformation falls outside of the TOPSY discussed above. Thus the possibility of the breaking of TOPSY via topology change should be thought of from the point of view that the symmetry is just another global symmetry of a theory.

If there is a classical Lagrangian for a given TQFT, it must be metric independent. Thus any topology change cannot occur classically. The manifestation of wormholes as quantum fluctuations is apparent in wormhole physics [22]. In a TQFT, a metric is introduced in order to define inner products in the gauge fixing procedure. In a topological gravity theory [1,7], one would have to integrate over this metric. The path integral includes a sum over all topologies.

A wormhole, which connects two space-time points, x and x', contributes an additional term to the effective Lagrangian. Then by the arguments of ref. [22], that term violates any continuous global symmetries of the original Lagrangian. The respective conserved currents may flow through the wormhole; therebye becoming non-conserving. The TOPSY may be broken in ths way.

Removing the constraint that the BRST charge vanish on physical states would lead to a spectrum of propagating particles which are no longer constrained to be elements of the de Rahm cohomology classes. As there was no classical physics in the unbroken TQFT, these real particles would appear from a spectrum of virtual particles as though they were spontaneously created from nothing. There would be something from nothing. Breaking the TOPSY by this mechanism is intriguing and its detailed study may be rewarding. In fact, if the symmetry breaking effects of wormholes does not alter the renormalizability of a theory, then it is conceivable that the broken topological gravity theory would yield a renormalizable quantum gravity theory.

VII. Conclusion

To summarize, the topological particle has been first quantized for even dimensional manifolds. The symmetry algebra is that of a twisted N=2 spinning particle. The spectrum is given by the de Rahm cohomology ring of the compact and oriented manifold. A topospace, whose coordinates are the x's of the manifold and the Grassmann ghosts of the topological symmetry (TOPSY), was used to deduce the BRST transformations of the space-time fields in the spectrum. Polynomials in a $(0, N/2)$–form on the space $M \times \mathcal{A}/G$, where $\mathcal{A}/G$ is the moduli space in the second quantized theory, immediately led to a generalization of the Donaldson invariants on compact and orientable N–manifolds. Some of the results of Morse theory were used to count the number of even and odd ghost number states. It was found that if a perfect Morse function is used in the gauge fixing of the TOPSY, there is a simple counting of these states. Finally, a discussion of TOPSY breaking was presented. As the symmetry is a BRST symmetry, it cannot be spontaneously broken. Furthermore, TOPSY breaking akin to the dynamical supersymmetry breaking of ref. [20], fails. However, in summing over all topologies in the partition function of a topological gravity theory, it is conceivable that a topology change can break the TOPSY; as it is a global (BRST) symmetry.

Based on the results of this work, there are many questions still to be answered. Chief among these is the role of topology change in TOPSY breaking. The connection between Weil homorphisms, topospace and the moduli spaces [25] is worthy of further study. Also, it may be of interest to first quantize the topological string of ref. [26] and look for TOPSY breaking there. A Morse theoretic counting similar to that of section V may offer some physical insight into the work of ref. [27]. There, a counting of the number of critical points on symplectic manifolds (whose coordinates are maps from $R \times [0, 1]$) was given.

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Note Added: Upon completion of this work, two preprints [28] were received which briefly discuss TQFT's in arbitrary dimensions.

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