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N=2 Superconformal Symmetry and $SO(2,1)$ Current Algebra

LANCE J. DIXON,^{*} MICHAEL E. PESKIN,[†]

*Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309*

and

JOSEPH LYKKEN[‡]

*Santa Cruz Institute for Particle Physics
University of California, Santa Cruz, California 95064*

ABSTRACT

We demonstrate that all unitary representations of the N=2 superconformal algebra (for $c > 3$) may be obtained from representations of $SO(2,1)$ current algebra by subtracting and then adding back a free boson. This construction gives insight into the unitarity domains for N=2 representations. It may also have a deeper structural significance.

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1. Introduction

The search for underlying principles which classify the variety of possible string compactifications leads us naturally to study the basic symmetry algebras of the string world surface. In the viewpoint on string compactification that is now generally accepted, the world surface of the string carries a conformally invariant two-dimensional quantum field theory. The excitation spectrum of this theory determines the spectrum of particles in space-time, and, in this way, the geometry of space-time itself. To determine the possibilities for space-time geometry, then, we must understand the representations of two-dimensional conformal symmetry. It is clear that any complete two-dimensional conformal field theory can be decomposed into irreducible representations of the conformal algebra. And it is equally clear that, for the purpose of constructing constraints of real generality, this is the most general structure that any two disparate compactifications share.

We may obtain a more constrained classification problem by considering two-dimensional theories that are invariant under extended conformal algebras. Conformal supersymmetry, as one example of such an extension, appears naturally in theories of superstrings. In the heterotic string theory, for example, the right-moving sector of fields must be $N=1$ supersymmetric for simple consistency. Specific schemes for obtaining realistic compactifications, including compactification on Calabi-Yau manifolds, produced world-surface theories in which this algebra was further extended to $N=2$.^[1,2] Subsequently a general argument was given^[3] that this extension always appears if the spacetime theory that the string produces is at least $N=1$ supersymmetric in four dimensions. Irreducible representations of the $N=2$ superconformal algebra, then, provide at least some of the ingredients needed to construct a realistic string model of Nature.

The unitary irreducible representations of the $N=2$ superconformal algebra were characterized by Boucher, Friedan and Kent (BFK),^[4] using results on the determinant of inner products in an $N=2$ module derived by themselves and by several other authors.^[5-7] BFK showed that, for central charge $c < 3$, there exists

only a discrete series of unitary representations, with

$$c = \frac{3n}{n+2}, \quad n = 1, 2, \dots, \quad (1.1)$$

in precise analogy to the unitary discrete series of the simple conformal algebra. But BFK also found unitarity constraints on the representations for $c > 3$, in the form of an allowed domain in the space of conformal dimension and $U(1)$ charge. (We will present the precise restrictions in Section 5.) Subsequently, Zamolodchikov and Fateev^[8] and Qiu^[9] gave a manifestly unitary construction of the discrete series, building the generators by combining a free boson with fields belonging to the $SU(2)$ parafermion algebra characterized earlier by Zamolodchikov and Fateev.^[10]

There has been no similarly general construction of the representations with $c > 3$, though many specific examples of such representations have been analyzed.^[11–13] However, Lykken^[14] did discover a generalization of the parafermion construction of the discrete series which led to new discrete series, with central charges of the form

$$c = \frac{3k}{k-2}, \quad (1.2)$$

where the parameter k ran over a series of rational numbers. These new series converged on the limiting value $c = 3$ from above.

In this paper, we will explain the new series discovered by Lykken, and place these and the $c < 3$ unitary series within a general picture of the $N=2$ superconformal representations. We will show that, for $c > 3$, the generators of the $N=2$ superconformal algebra can be used to construct the two-dimensional current algebra of the group $SO(2, 1)$ (or, equivalently, of the group $SU(1, 1)$ or $SL(2, R)$ ^{*}). We will find that the rational number k in Lykken's construction is replaced by a real number k which may vary continuously subject to some simple inequalities.

* $SL(2, R)$ current algebra has also played a key role in recent analysis of two-dimensional quantum gravity.^[15]

We can then understand the various classes of $N=2$ superconformal representations as derived from different classes of affine $SO(2,1)$ representations. The unitarity domains found by BFK also follow straightforwardly from the $SO(2,1)$ picture. For $c < 3$, the global group of the current algebra becomes $SO(3)$, and we recover the well-known relation between the $N=2$ discrete series and the representations of $SU(2)$ current algebra which follows from the fact that both may be built from Zamolodchikov-Fateev parafermions. In essence, our analysis simply generalizes this connection to the remaining $N=2$ representations.

Our arguments will be laid out as follows: In Section 2, we will review the $N=2$ superconformal algebra and derive the associated current algebra. In Section 3, we will study the representations of $SO(2,1)$ current algebra; we will show that, while the complete current algebra cannot have unitary representations, we can identify conditions under which a coset construction $SO(2,1)/U(1)$ can be unitary.[†] We will then give an elementary proof of the unitarity which applies to most classes of $SO(2,1)/U(1)$ representations. In Section 4, we will complete our demonstration of the unitarity of the coset representations, by applying a determinant formula for current algebra representations due to Kac and Kazhdan.^[17] In Section 5, we will relate the conditions for unitarity of the modules constructed in Section 3 to the results of BFK on $N=2$ superconformal representations. Finally, in Section 6, we will set out some ideas on the significance of this current algebra picture for the problem of classifying $N=2$ superconformal field theories.

We would like to note one novel feature of our analysis, which we feel should have many generalizations in the study of conformal field theory. In ordinary field theory, one does not normally encounter non-compact internal symmetry groups. For such groups, the finite-dimensional representations are necessarily non-unitary; to work with unitary representations, one must accept that these are infinite-dimensional. In two-dimensional conformal models, this is hardly an objection,

[†] A different coset construction, the quotient of $SO(2,1)$ current algebra by its Virasoro subalgebra, has recently been studied by Balog, *et al.*^[16] These authors showed that their construction is generally non-unitary.

since the representations are already infinite-dimensional. The fact that the adjoint representation of a noncompact group is not unitary implies that the current algebra representations all contain states of negative norm. However, it is highly plausible that these states can be eliminated by a coset construction G/H , where H is the maximal compact subgroup of the noncompact group G . We conjecture that for any such G/H model, there are unitary representations for a continuous range of values of the central charge. This conclusion generalizes one for which we supply the proof in Sections 3 and 4 of this paper; however, the resolution of the more general question is beyond the scope of our analysis here.

2. How N=2 superconformal symmetry creates currents

Let us begin by relating the N=2 superconformal algebra to a current algebra. This is most easily done by studying the operator products of holomorphic fields. The N=2 algebra is generated by the stress tensor $T(z)$ and two supersymmetry currents $T_F^\pm(z)$ distinguished by their charges under a $U(1)$ current $J(z)$. T_F^\pm and J are primary conformal fields of dimension $\frac{3}{2}$ and 1 which obey among themselves the operator product relations

$$\begin{aligned}
 J(z)J(w) &= \frac{c/3}{(z-w)^2} + \dots \\
 J(z)T_F^\pm(w) &= \pm \frac{1}{(z-w)} T_F^\pm(w) + \dots \\
 T_F^+(z)T_F^-(w) &= \frac{2c/3}{(z-w)^3} + \frac{2}{(z-w)^2} J(w) + \frac{1}{(z-w)} (2T(w) + \partial_w J) + \dots \\
 T_F^\pm(z)T_F^\pm(w) &\sim \mathcal{O}(z-w).
 \end{aligned}
 \tag{2.1}$$

Up to an overall normalization, the operator product of J with itself is exactly that of $\partial_z \varphi$, where $\varphi(z)$ is a free scalar field. This identification implies the equivalence of these two operators. Thus, the N=2 generators decompose into two

mutually commuting sectors, one of which contains the free field φ . Then

$$J(z) = i\sqrt{\frac{c}{3}} \partial_z \varphi; \quad T(z) = -\frac{1}{2}(\partial_z \varphi)^2 + T_\psi(z). \quad (2.2)$$

T_ψ is an energy-momentum tensor with central charge $c_\psi = c - 1$. In order to reproduce the N=2 algebra (2.1), the remaining generators must decompose as follows:

$$T_F^+(z) = \left(\frac{2c}{3}\right)^{\frac{1}{2}} \psi(z) e^{i\sqrt{\frac{3}{c}}\varphi}, \quad T_F^-(z) = \left(\frac{2c}{3}\right)^{\frac{1}{2}} \psi^\dagger(z) e^{-i\sqrt{\frac{3}{c}}\varphi}, \quad (2.3)$$

where ψ, ψ^\dagger are holomorphic fields, primary with respect to T_ψ and of dimension $\Delta_\psi = \frac{3}{2}(c-1)/c$, with the operator product relations

$$\psi(z)\psi^\dagger(w) = (z-w)^{-2\Delta_\psi} \left[1 + (z-w)^2 \cdot \frac{2\Delta_\psi}{c_\psi} T_\psi(w) + \dots \right] \quad (2.4)$$

$$\psi(z)\psi(w) \sim \mathcal{O}((z-w)^{1-3/c}).$$

It is natural to think of ψ, ψ^\dagger as a generalized form of the parafermions defined by Zamolodchikov and Fateev.^[10] In fact, (2.4) is precisely the generalized parafermion algebra identified by Lykken in ref. 14. This algebra makes sense independently of its origin in the N=2 algebra, since the detailed form of the first line of (2.4) follows just from the assumption that T_ψ is the leading nontrivial operator appearing in this operator product.

Zamolodchikov and Fateev showed that, for $c < 3$, one could convert the parafermion algebra to a current algebra of $SU(2)$ by adding back a new free boson. Let us follow an analogous procedure here. Let $\phi(z)$ be a free scalar field, and define

$$J^3(z) = -\sqrt{\frac{k}{2}} \partial_z \phi, \quad \text{where } k = \frac{2c}{c-3}. \quad (2.5)$$

Then let

$$J^+(z) = \sqrt{k} \psi(z) e^{\sqrt{\frac{2}{k}}\phi}, \quad J^-(z) = \sqrt{k} \psi^\dagger(z) e^{-\sqrt{\frac{2}{k}}\phi}, \quad (2.6)$$

These relations clearly make sense for $c > 3$; their continuation to $c < 3$ (with

$\sqrt{k} \rightarrow -i\sqrt{|k|}$) gives the construction of Zamolodchikov and Fateev. The fields defined in (2.5), (2.6) are holomorphic fields of dimension 1—currents! They obey the operator product relations

$$\begin{aligned}
J^3(z)J^3(w) &= \frac{-\frac{1}{2}k}{(z-w)^2} + \dots \\
J^3(z)J^\pm(w) &= \pm \frac{1}{(z-w)} J^\pm(w) + \dots \\
J^+(z)J^-(w) &= \frac{k}{(z-w)^2} - \frac{2}{(z-w)} J^3(w) + \dots \\
J^\pm(z)J^\pm(w) &\sim \mathcal{O}(1).
\end{aligned} \tag{2.7}$$

It is useful to compare these relations to the commutation relations of the global $SO(2,1)$ algebra. This algebra has three generators J^1, J^2, J^3 , with metric

$$g_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \tag{2.8}$$

and commutation relations

$$[J^i, J^j] = i\epsilon^{ij}_k J^k, \tag{2.9}$$

with $\epsilon^{123} = +1$. If we set

$$J^\pm = J^1 \pm iJ^2, \tag{2.10}$$

the commutation relations take the form

$$[J^3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = -2J^3. \tag{2.11}$$

We can see, then, that (2.7) is of the standard form

$$J^i(z)J^j(w) = \frac{\frac{1}{2}kg^{ij}}{(z-w)^2} + \frac{if^{ij}_k J^k(w)}{(z-w)} + \dots \tag{2.12}$$

of a Kac-Moody algebra based on the group $SO(2,1)$. The constant k defined in (2.5) may be recognized as the central charge of this algebra.

For $c < 3$, the sign of the $-k$ in (2.5) changes; if we redefine $k = 2c/(3 - c)$, we revert to the current algebra of $SU(2)$. The $N=2$ discrete series corresponds to positive integer values of k for the $SU(2)$ algebra; these are the only values for which that algebra has unitary representations.^[18]

3. Unitarity of $SO(2,1)/U(1)$ modules

If we are to make use of this current algebra construction in building unitary representations of the $N=2$ algebra, we must first learn how to build unitary representations of the current algebra generators. There are in fact no unitary representations of the full $SO(2,1)$ current algebra, since, as we will see below, the moments of J^3 create states with negative norm. This problem is the result of the appearance of the indefinite metric (2.8) in the operator product algebra, and so is a general difficulty associated with non-compact groups. But, fortunately, the analysis of $N=2$ superconformal representations requires only the unitarity of coset representations in which the action of J^3 is set equal to zero. In this section, we will construct these representations and find the conditions under which they are unitary.

For this discussion, it will be useful to define the moments of $J^i(z)$

$$J_n^i = \oint \frac{dz}{2\pi i} z^n J^i(z); \quad J^i(z) = \sum_{n=-\infty}^{\infty} z^{-n-1} J_n^i. \quad (3.1)$$

By standard manipulations, we can convert (2.7) into commutation relations for the moments

$$\begin{aligned} [J_n^3, J_m^3] &= -\frac{1}{2}kn \delta(n+m) \\ [J_n^3, J_m^\pm] &= \pm J_{n+m}^\pm \\ [J_n^+, J_m^-] &= kn \delta(n+m) - 2J_{n+m}^3. \end{aligned} \quad (3.2)$$

Using the Sugawara construction, we can form bilinears in the J_n^i which generate

a Virasoro algebra. The Virasoro generators are moments of the stress tensor

$$T(z) = \frac{1}{k-2} g_{ij} : J^i(z) J^j(z) : . \quad (3.3)$$

The factor (-2) in the denominator is the value of the $SO(2,1)$ Casimir operator: $\epsilon^{imn} \epsilon^j_{mn} = -2g^{ij}$. The central charge of the Virasoro algebra is

$$c = \frac{3k}{k-2} . \quad (3.4)$$

This value of c should be the same as that for the corresponding $N=2$ algebra, since our construction subtracts and then adds back a free boson; this is confirmed by the relation in eq. (2.5) between k and the $N=2$ central charge. We will need the explicit form only for L_0 :

$$L_0 = \frac{1}{k-2} \left[\frac{1}{2} (J_0^+ J_0^- + J_0^- J_0^+) - (J_0^3)^2 + \sum_{m=1}^{\infty} (J_{-m}^+ J_m^- + J_{-m}^- J_m^+ - 2J_{-m}^3 J_m^3) \right] . \quad (3.5)$$

Using these elements, we can build up a representation of the $SO(2,1)$ current algebra by the following standard procedure: Begin with a unitary irreducible representation of the global group $SO(2,1)$ at the lowest L_0 level of the representation, then form the higher levels by acting in all possible ways with the J_{-n}^i , $n > 0$. The states at the lowest L_0 level satisfy $J_n^i |\psi\rangle = 0$, for $n > 0$. Using this statement and (3.5), we can compute the value of L_0 for these states:

$$L_0 = \frac{\mathbf{J}^2}{k-2} , \quad (3.6)$$

where

$$\mathbf{J}^2 = \frac{1}{2} (J_0^+ J_0^- + J_0^- J_0^+) - (J_0^3)^2 \quad (3.7)$$

is the quadratic Casimir operator of the global group.

The classification of states at the lowest level is given by the representation theory of $SO(2,1)$; let us briefly review that theory here.^[19] If we are concerned only with the representations of $SO(2,1)$ (or its double cover $SU(1,1)$), we should restrict the eigenvalues of J_0^3 to be integers (or half-integers). Most references on $SO(2,1)$ consider only these special cases. However, it will be important for us to work more generally in the universal cover of $SO(2,1)$ and allow J_0^3 to have an arbitrary fractional part. This fractional part is not changed by the action of the $SO(2,1)$ generators. To classify $SO(2,1)$ representations, we will consider separately states with $J_0^3 = \phi_0 \pmod{1}$ for each value ϕ_0 in the interval $0 \leq \phi_0 < 1$.

Because $SO(2,1)$ is noncompact, its unitary representations are infinite-dimensional. Nevertheless, they are straightforward to construct. As in $SU(2)$, the operators J_0^+ , J_0^- are ladder operators which change the eigenvalue of J_0^3 by one unit. Let us first consider representations containing a state $|m\rangle$ with $J_0^3 = m$, such that $m > 0$ and $|m\rangle$ has positive norm. Then the next state up in the ladder is $J_0^+ |m\rangle$, and this state has norm

$$\begin{aligned} \|J_0^+ |m\rangle\|^2 &= \langle m | J_0^- J_0^+ |m\rangle = \langle m | J_0^+ J_0^- + 2J_0^3 |m\rangle \\ &\geq 2m \langle m | m\rangle > 0 . \end{aligned} \tag{3.8}$$

The last line follows from the fact that $J_0^+ J_0^-$ is a positive operator. Thus, this next state and, by induction, all states in the ladder with $J_0^3 > m$, have positive norm. A similar argument implies that, if we had started with $m < 0$, then all states below m on the ladder would have positive norm.

Now return to the state $|m\rangle$, $m > 0$, and consider the lower states on the ladder which are obtained by repeatedly applying J_0^- . There are two possibilities: Either some state in the sequence has zero norm, terminating the ladder, or the sequence continues indefinitely. A zero norm state appears if $J_0^- |\ell\rangle = 0$ for some state $|\ell\rangle$ in the ladder. Then

$$\mathbf{J}^2 |\ell\rangle = \left[J_0^+ J_0^- - J_0^3 (J_0^3 - 1) \right] |\ell\rangle = -\ell(\ell - 1) |\ell\rangle . \tag{3.9}$$

Since $\ell = n + \phi_0$ for some integer n , the terminating representations for fixed ϕ_0 appear at discrete values of \mathbf{J}^2 . Similarly, we can identify representations with negative J_0^3 that terminate at their upper boundary; these also have discrete \mathbf{J}^2 . The last family of unitary representations are those which do not terminate in either direction. For these representations, \mathbf{J}^2 is not quantized. However, the requirement that no states in the representation have negative norm implies a set of inequalities for \mathbf{J}^2 , of which the strongest is obtained at the state $|\phi_0\rangle$, corresponding to the smallest positive value of J_0^3 on the ladder. We must insist that

$$\begin{aligned} 0 < \|J_0^- |\phi_0\rangle\|^2 &= \langle \phi_0 | J_0^+ J_0^- |\phi_0\rangle \\ &= \langle \phi_0 | \mathbf{J}^2 + J_0^3 (J_0^3 - 1) |\phi_0\rangle = \left[\mathbf{J}^2 + \phi_0(\phi_0 - 1) \right] \langle \phi_0 | \phi_0\rangle . \end{aligned} \tag{3.10}$$

This implies a lower bound $\mathbf{J}^2 > \phi_0(1 - \phi_0)$.

By this analysis, the irreducible unitary representations of $SO(2,1)$ fall into four classes:

1. Identity: The trivial representation $|0\rangle$. This representation has $\mathbf{J}^2 = 0$ and $J_0^\pm |0\rangle = 0$.
2. Discrete Series \mathcal{D}_n^+ : Representations of the form $\{|k + \phi_0\rangle\}$, $k = n, (n + 1), \dots$, $n \geq 0$, such that $J_0^- |n + \phi_0\rangle = 0$. These representations have $\mathbf{J}^2 = -(n + \phi_0)(n - 1 + \phi_0)$.
3. Discrete Series \mathcal{D}_n^- : Representations of the form $\{|k + \phi_0\rangle\}$, $k = -n, -(n + 1), \dots$, $n \geq 1$, such that $J_0^+ |-n + \phi_0\rangle = 0$. These representations have $\mathbf{J}^2 = -(n - \phi_0)(n - 1 - \phi_0)$.
4. Continuous Series \mathcal{C} : Representations of the form $\{|k + \phi_0\rangle\}$, $k = -\infty, \dots, \infty$. These representations have $\mathbf{J}^2 > \phi_0(1 - \phi_0)$.

The spectrum of \mathbf{J}^2 for fixed ϕ_0 is shown in Fig. 1.

We now use each of these unitary irreducible representations as a base to build up a representation of the $SO(2,1)$ current algebra. Let us begin by discussing the

module built on a base in the discrete series \mathcal{D}_n^+ . The analysis is quite similar for all of the other cases.

Consider, then, applying the L_0 raising generators J_{-n}^+ to states in a representation in \mathcal{D}_n^+ . We label this representation by ℓ , the J_0^3 value of the lowest-weight state in the base. We must assume that the state $|\ell\rangle$, the lowest-weight state in the base, has positive norm. The module is unitary if it follows that all other states have positive norm. We showed above that all states in the base have positive norm if $\ell > 0$. A second elementary condition is given by considering

$$\|J_{-1}^- |\ell\rangle\|^2 = \langle \ell | J_1^+ J_{-1}^- | \ell \rangle = (k - 2\ell) \langle \ell | \ell \rangle . \quad (3.11)$$

Thus, the state at the first L_0 excited level with smallest J_0^3 has positive norm only if $k > 2\ell$. The chain of states $(J_{-1}^-)^n |\ell\rangle$ form the left-hand boundary of the current algebra representation, as shown in Fig. 2. One can easily compute the norms of the higher states in this series and verify that all are positive if the condition $k > 2\ell$ is met. Actually, the whole set of conclusions follows from the positivity of norms in the base representation by application of the Weyl symmetry

$$J_n^+ \rightarrow J_{n-1}^- ; \quad J_n^- \rightarrow J_{n+1}^+ ; \quad J_n^3 \rightarrow -J_n^3 + \frac{1}{2}k \delta(n) \quad (3.12)$$

of the $SO(2,1)$ current algebra.

However, even if the boundary states all have positive norm, the interior of the current algebra representation always contains states of negative norm. The simplest example is

$$\|J_{-1}^3 |\ell\rangle\|^2 = \langle \ell | J_1^3 J_{-1}^3 | \ell \rangle = -\frac{1}{2}k \langle \ell | \ell \rangle . \quad (3.13)$$

The minus sign here comes directly from the indefinite sign of the metric (2.8).

Since the minus sign in (3.13) is associated with the $U(1)$ part of the current algebra, we might hope to obtain a unitary module if we remove all states created

by moments of the $U(1)$ current; this defines a module for the coset $SO(2,1)/U(1)$. Let us test this idea in the first nontrivial case: $J_0^3 = \ell$, at the first excited level in L_0 . The complete current algebra contains two states with these quantum numbers: $J_{-1}^3 |\ell\rangle$ and $J_{-1}^- J_0^+ |\ell\rangle$. The coset is defined to contain only states of highest weight under the $U(1)$ current algebra, that is, only states $|\psi\rangle$ such that $J_n^3 |\psi\rangle = 0$ for all $n > 0$. The one linear combination satisfying this restriction is

$$|\psi\rangle = 4\ell J_{-1}^3 |\ell\rangle - k J_{-1}^- J_0^+ |\ell\rangle ; \quad \|\psi\|^2 = 2k\ell(k - 2\ell)(k - 2) . \quad (3.14)$$

Thus, the one state that lies in the coset module does have positive norm, provided that we impose the further restriction $k > 2$.

We will now demonstrate that the three conditions $\ell > 0$, $k > 2\ell$, $k > 2$ suffice to establish the unitarity of the coset module. Note that this result stands in clear contrast to the situation for coset current algebras built from a compact Lie group such as $SU(2)$. In that case, both k and 2ℓ are restricted to integer values. In the case we consider here, k and ℓ can vary continuously and independently, as long as the three basic inequalities are satisfied. In the compact case, the quantization of k can be understood in an alternative way by representing the current algebra as a two-dimensional sigma model and identifying k with the quantized coefficient of a Wess-Zumino term. But in the noncompact case that we consider here, the corresponding Wess-Zumino coefficient is not quantized, since $\pi_3(SO(2,1)) = 0$.

Our argument is based on the trick used by Kac to prove the unitarity of the standard $SU(2)$ current algebra modules.^[18] Let $|x\rangle$ be a state with $J_0^3 = m$ located on the N^{th} level of L_0 above the base. This state obeys

$$\langle x | L_0 | x \rangle = \left\{ -\frac{\ell(\ell - 1)}{k - 2} + N \right\} \langle x | x \rangle . \quad (3.15)$$

By inserting the explicit form for L_0 given by (3.5), we can rearrange (3.15) into

the identity

$$\langle x | x \rangle = \frac{\langle x | \left[J_0^+ J_0^- + \sum_{p \geq 1} (J_{-p}^+ J_p^- + J_{-p}^- J_p^+ - 2J_{-p}^3 J_p^3) \right] | x \rangle}{\left[(k-2)N + m(m-1) - \ell(\ell-1) \right]} . \quad (3.16)$$

We will now show that the norm of $|x\rangle$ is positive by proving that both the numerator and the denominator of (3.16) are positive.

Let us first analyze the denominator. The case $N = 0$, $m = \ell$ is the lowest-weight state $|\ell\rangle$ itself, which has positive norm by assumption. Aside from that case, the denominator is manifestly positive if $k > 2$ and $m \geq \ell$. For states with $m < \ell$, we may apply the Weyl symmetry (3.12) to relate them to states with $m > \ell$ in a module where ℓ is replaced by $(\frac{1}{2}k - \ell)$, or we may argue more directly as follows: The structure of the current algebra representation (Fig. 2) implies that if $m < \ell$, then $\ell - m \leq N$. With this in mind, let us rewrite the denominator in terms of $p = (\ell - m)$. It becomes

$$\left[(k-2)(N-p) + p(p-1) + (k-2\ell)p \right] , \quad (3.17)$$

which is manifestly positive under these conditions if $p > 0$. Thus, the denominator is positive for all states (except $|\ell\rangle$) in the representation.

To prove that the numerator in (3.16) is positive, we proceed by induction. Of course, because of the minus sign that appears, we can prove positivity only for states that are highest-weight under the $U(1)$ current algebra, $J_n^3 |x\rangle = 0$ for all $n > 0$. We will show that, for states of this type at L_0 level N , the numerator is positive if it is positive for all such states at all lower levels, and for all smaller values of J_0^3 . To begin, use the highest-weight condition to rewrite the quantity as

$$\langle x | \left[\sum_{p \geq 0} J_{-p}^+ J_p^- + \sum_{p \geq 1} J_{-p}^- J_p^+ \right] | x \rangle . \quad (3.18)$$

This expression would be manifestly positive if we could insert between each J_{-p} and J_p a complete set of positive-norm states. Since possible intermediate states

lie either at lower L_0 levels or at lower J_0^3 (and since, for fixed L_0 in this module, the J_0^3 eigenvalues are bounded from below), this manipulation would prove the induction step as stated above. However, some possible intermediate states are not highest-weight under the $U(1)$ current algebra and thus need not have positive norm.

We can remedy this difficulty by inserting projection operators that restrict the intermediate states to be positive. Let \mathbf{P} be the projection operator onto highest-weight states of the $U(1)$ current algebra. Since all states of the theory can be built by applying operators J_{-n}^3 to the highest-weight states, and since these operators obey the simple algebra of raising and lowering operators, we have the completeness relation

$$1 = \mathbf{P} + \sum_{n>0} \left(-\frac{kn}{2}\right)^{-1} J_{-n}^3 \mathbf{P} J_n^3 + \frac{1}{2!} \sum_{n_1, n_2 > 0} \left(-\frac{kn_1}{2}\right)^{-1} \left(-\frac{kn_2}{2}\right)^{-1} J_{-n_1}^3 J_{-n_2}^3 \mathbf{P} J_{n_2}^3 J_{n_1}^3 + \dots \quad (3.19)$$

We may insert this expression for the sum over intermediate states in (3.18). This produces an infinite sum of terms in which a typical one has the form

$$\langle x | \left[J_{-p}^+ \frac{1}{m!} \left(-\frac{2}{k}\right)^m \frac{1}{n_1 \dots n_m} J_{-n_1}^3 \dots J_{-n_m}^3 \mathbf{P} J_{n_m}^3 \dots J_{n_1}^3 J_p^- \right] | x \rangle . \quad (3.20)$$

This expression can be simplified by commuting the factors of J_n^3 to the left and right, and annihilating them against the states $|x\rangle$, to produce

$$\langle x | \left[\frac{1}{m!} \left(-\frac{2}{k}\right)^m \frac{1}{n_1 \dots n_m} J_{-(p+n_1+\dots+n_m)}^+ \mathbf{P} J_{p+n_1+\dots+n_m}^- \right] | x \rangle . \quad (3.21)$$

In this way, (3.18) can be brought into the form

$$\langle x | \left[\sum_{p \geq 0} F_p(y) J_{-p}^+ \mathbf{P} J_p^- + \sum_{p \geq 1} F_{p-1}(y) J_{-p}^- \mathbf{P} J_p^+ \right] | x \rangle , \quad (3.22)$$

where $y = 2/k$ and

$$F_p(y) = \sum_{q=0}^p f_q(y), \quad (3.23)$$

for $f_0(y) = 1$ and

$$f_q(y) = \sum_{m \geq 0} \frac{(-y)^m}{m!} \sum_{n_i > 0} \frac{1}{n_1 \cdots n_m} \cdot \delta(n_1 + n_2 + \cdots + n_m - q). \quad (3.24)$$

If $F_p(y)$ is positive, we can now carry out the induction argument in the manner explained above.

It is easy to put $F_p(y)$ into a more explicit form. We first compute $f_q(y)$, using the generating function

$$\begin{aligned} f(y, z) &= \sum_{q=0}^{\infty} f_q(y) z^q \\ &= \sum_{m=0}^{\infty} \frac{(-y)^m}{m!} \left[-\log(1-z) \right]^m \\ &= (1-z)^y. \end{aligned} \quad (3.25)$$

Now we may reexpand in z to find ($q > 0$)

$$f_q(y) = -\frac{y}{q} \cdot (1-y) \left(1 - \frac{y}{2}\right) \cdots \left(1 - \frac{y}{q-1}\right). \quad (3.26)$$

Thus, by induction on p ,

$$F_p(y) = (1-y) \left(1 - \frac{y}{2}\right) \cdots \left(1 - \frac{y}{p}\right). \quad (3.27)$$

This is positive for $y < 1$, that is, $k > 2$. Our inductive argument is now complete. Note that all of the correction terms, $f_q(y)$ for $q > 0$, are negative, so the outcome of our analysis was actually rather delicate until the last stage.

We have now proved that an $SO(2,1)/U(1)$ coset module built on a \mathcal{D}_n^+ discrete series representation is unitary for continuous values of k and ℓ . We required only that $k > 2$, plus the additional restrictions $\ell > 0$, $k > 2\ell$. For \mathcal{D}_n^- discrete series representations, the proof goes in exactly the same way and demonstrates unitarity in the region $\ell < 0$, $k > -2\ell$. The coset modules built on the identity representation ($\ell = 0$) and on the $\ell = k/2$ representation also can be shown to be unitary by this method.

Finally, we must consider coset modules built on continuous series representations. The inductive argument given above breaks down, because for fixed L_0 , these modules have no lower bound on J_0^3 , so $J_0^+ J_0^-$ terms in the numerator are not under control. The argument is only sufficient to demonstrate that, if the module has positive norm states for sufficiently large positive or negative values of J_0^3 , then the entire module has positive norms. We have not found a simple way to demonstrate this last point. In the next section, we will apply some heavier technical machinery to this question and prove that modules built on continuous series representations of $SO(2,1)$ are also unitary for $k > 2$.

4. An $SO(2,1)$ determinant formula

In this section, we will take a different approach to the proof of unitarity for coset modules of $SO(2,1)/U(1)$. The standard method for analyzing the unitarity of representations of the conformal and superconformal algebras is to study the determinant of inner products of states at a given L_0 level. That method can be applied also to $SO(2,1)$ current algebra and its cosets. Kac and Kazhdan^[17] have given a straightforward general determinant formula for representations of current algebra. From this formula, we can see easily that there are no states of negative norm in the coset representations built from continuous series representations of $SO(2,1)$. This resolves the problem left at the end of the previous section.

To begin the analysis, let us give a precise definition of the determinants we will investigate and compute their explicit form at some low levels. We label by

\mathbf{D}_N the determinant of inner products of all states in an $SO(2,1)$ current algebra representation with $J_0^3 = m$ and lying at an L_0 level N above the base. It is useful to think of these of all of these states as being built by applying products of the J_{-n}^i and J_0^\pm to the state $|m\rangle$ in the base, and to compute their inner products in terms of $\langle m | m \rangle$ (which we set equal to 1).

The individual inner products clearly depend on m and k ; they also depend on the value of \mathbf{J}^2 in the base representation, through relations of the form:

$$\langle m | J_0^+ J_0^- | m \rangle = \mathbf{J}^2 + m(m-1) . \quad (4.1)$$

Let us define, for $n > 0$,

$$\mathcal{J}_{-n} = \mathbf{J}^2 + (m-n+1)(m-n) , \quad \mathcal{J}_{+n} = \mathbf{J}^2 + (m+n-1)(m+n) . \quad (4.2)$$

These give, respectively, the value of $(J_0^+ J_0^-)$ acting on the state $|m-n+1\rangle$ and the value of $(J_0^- J_0^+)$ acting on $|m+n-1\rangle$. These quantities vanish when the next state along the chain, in the direction they indicate, is null. Of course they are positive when \mathbf{J}^2 lies in the range of the continuous series and vanish only when \mathbf{J}^2 is continued to the range of the discrete series.

As an example, consider the case of level $N = 1$. At this level, there are three states, which we write:

$$J_{-1}^+ J_0^- |m\rangle , \quad J_{-1}^- J_0^+ |m\rangle , \quad J_{-1}^3 |m\rangle . \quad (4.3)$$

Their inner products are readily computed using the algebra (3.2), and one finds for the determinant

$$\mathbf{D}_1 = -2(k-2)\mathcal{J}_{-1}\mathcal{J}_{+1} \left(\mathbf{J}^2 + \frac{k}{2} \left(\frac{k}{2} - 1 \right) \right) . \quad (4.4)$$

To apply this result to a unitarity argument, we must pull the formula apart into contributions from specific levels of the $U(1)$ current algebra. Among the three

states in (4.3), two linear combinations are annihilated by J_1^3 , and one state may be written as a current algebra descendant: $J_{-1}^3 |m\rangle$. These states are at L_0 level 0 and 1, respectively, with respect to the $U(1)$ current algebra. It is apparent that the state at level 1 is orthogonal to the two states at level 0. More generally, any two states are orthogonal if they lie at different levels of the $U(1)$ current algebra. Thus the determinant \mathbf{D}_N will factorize: Let $\mathbf{D}_N^{(q)}$ be the determinant of inner products of the states at level N of the $SO(2,1)$ representation which are at level q of the $U(1)$ current algebra. Then

$$\mathbf{D}_N = \prod_{q=0}^N \mathbf{D}_N^{(q)}. \quad (4.5)$$

The term $\mathbf{D}_N^{(0)}$ is the determinant of inner products of the states in the coset module $SO(2,1)/U(1)$, and so this is the object of primary interest to us here. For the level 1 example above, $\mathbf{D}_1^{(1)} = \|J_{-1}^3 |m\rangle\|^2 = -\frac{1}{2}k$. Dividing (4.4) by this factor, we see that $\mathbf{D}_1^{(0)}$ is a product of factors each of which is positive when $k > 2$ and \mathbf{J}^2 is in the range of the continuous series.

This example illustrates the general strategy of our unitarity argument. It is apparent by inspection of (3.2) that as $k \rightarrow \infty$ with \mathbf{J}^2 , m held fixed, all states of the coset module have positive norm. Thus, if there exists a region of (k, \mathbf{J}^2) extending to large k in which the determinants $\mathbf{D}_N^{(0)}$ are strictly positive, the norm of each individual state of the coset module must remain positive in this region. This is just the same strategy applied by BFK^[4] to the determinant formula for the $N=2$ supersymmetry algebra to demonstrate unitarity for $c > 3$. We will show explicitly that $\mathbf{D}_N^{(0)}$ is positive in the region $k > 2$, $\mathbf{J}^2 > \phi_0(1 - \phi_0)$, that is, the region of $k > 2$ and continuous series $SO(2,1)$ representations at the base. This will close the gap left in the the unitarity argument given above. The reader may verify that the zeros of our determinant formula indicate further unitary coset modules containing null states, and that these are precisely the coset representations built on \mathcal{D}_n^+ and \mathcal{D}_n^- representations which we analyzed in the previous section.

As a preliminary, we should simplify further the decomposition (4.5). The states of level N and $U(1)$ level q are created by applying operators J_{-n}^3 to the states of level $(N - q)$ and $U(1)$ level 0. Thus, $\mathbf{D}_N^{(q)}$ can be assembled from $\mathbf{D}_{N-q}^{(0)}$ and the determinant \mathbf{d}_q of inner products at of states at level q built on a normalized primary state in the $U(1)$ current algebra generated by J^3 . To be more precise, we must define a set of multiplicity functions. Let $p(n)$ be the number of partitions of the integer n . This number-theoretic function is well-known to give the number of states at level n in the theory of one free boson. More generally, let us denote by $p_k(n)$ the number of states at level n in a theory of k free bosons. There are $p(n)$ states at level n of a $U(1)$ current algebra representation. In an $SO(2,1)$ representation, barring the presence of null states, the states may be counted as deriving from moments of the three fields J^+ , J^- , J^3 , and so the number of states at level n is $p_3(n)$. Similarly, we may count the states at level n and $U(1)$ level 0 by dropping all terms involving moments of J^3 ; this leaves $p_2(n)$ states. Thus, applying all possible level q combinations of moments of J^3 to the states at level $(N - q)$ and $U(1)$ level 0 gives

$$\mathbf{D}_N^{(q)} = (\mathbf{d}_q)^{p_2(N-q)} (\mathbf{D}_{N-q}^{(0)})^{p(q)}. \quad (4.6)$$

The factor \mathbf{d}_q is very simple:

$$\mathbf{d}_q = C(-k)^{r(q)}, \quad (4.7)$$

where C is a positive numerical constant and $r(n)$ gives the number of factors of J_{-p}^3 in all states at level n . The generalization $r_k(n)$ to a system with k generators will appear below. Thus we can use (4.6) together with (4.5), to compute the reduced determinant $\mathbf{D}_N^{(0)}$ recursively level by level.

At a general level, the determinant \mathbf{D}_N is given by the following formula:

$$\mathbf{D}_N = (-1)^{r_3(N)} C_N (k-2)^{r_3(N)} \prod_{n=1}^N (\mathcal{J}_{-n} \mathcal{J}_{+n})^{P_3(N,n)} \prod_{\substack{r,s=1 \\ r+s \leq N}}^N \left[\mathbf{J}^2 + \left(\frac{r}{2}(k-2) + \frac{s+1}{2} \right) \left(\frac{r}{2}(k-2) + \frac{s-1}{2} \right) \right]^{p_3(N-rs)}, \quad (4.8)$$

where C_N is a positive numerical constant and the exponent $P_3(N, n)$ is defined implicitly below. This formula can be obtained as a special case of the formula for the level N determinant in current algebra representations obtained by Kac and Kazhdan as Theorem 1 of ref. 17. We have made two simple modifications of their result. These account for the fact that Kac and Kazhdan have stated their result for a highest-weight representation, in which the base terminates, while we need the result for a continuous representation in the base. First, Kac and Kazhdan normalize the highest weight state, $|\ell\rangle$ in a \mathcal{D}_n^+ representation, to 1, while we prefer to normalize the state $|m\rangle = (J_0^+)^{m-\ell} |\ell\rangle$ to 1. The factors of $\mathcal{J}_{-n} \mathcal{J}_{+n}$ in (4.8) correct this convention. Specifically, we need one factor of \mathcal{J}_{+n} for every appearance of the state $(J_0^+)^n |m\rangle$ in enumerating the states contributing to \mathbf{D}_N , and similarly for \mathcal{J}_{-n} . We may then compare the inner products directly for \mathcal{D}_n^+ and \mathcal{C} current algebra modules. In general, a different number of states contribute to \mathbf{D}_N in the two cases. However, the counting of states at level N and $J_0^3 = m$ is identical between the two cases if, in the \mathcal{D}_n^+ case, $m \geq (N + \ell)$. When this condition is met, the exponents in the Kac-Kazhdan formula become independent of m and the whole formula depends on \mathbf{J}^2 , k , and m simply as a polynomial. In addition, the states contributing to \mathbf{D}_N can be matched one-to-one between the \mathcal{D}_n^+ and \mathcal{C} cases, and the individual inner products are identical functions when written as polynomials in \mathbf{J}^2 , k , and m . Thus, the evaluation of the Kac-Kazhdan result for $m \geq (N + \ell)$, after the change in normalizations, gives directly the determinant of inner products in a continuous $SO(2, 1)$ coset module. This is the formula (4.8).

The various determinants $\mathbf{D}_N^{(0)}$ can be determined recursively from (4.8) by the

strategy that we have outlined above. The result is

$$\mathbf{D}_N^{(0)} = D_N k^{-r_2(N)} (k-2)^{r_2(N)} \prod_{n=1}^N (\mathcal{J}_{-n} \mathcal{J}_{+n})^{P_2(N,n)}$$

$$\cdot \prod_{\substack{r,s=1 \\ r \cdot s \leq N}}^N \left[\mathbf{J}^2 + \left(\frac{r}{2}(k-2) + \frac{s+1}{2} \right) \left(\frac{r}{2}(k-2) + \frac{s-1}{2} \right) \right]^{P_2(N-rs)},$$
(4.9)

where D_N is a positive numerical constant. The factors in the first line are positive when $k > 2$ and \mathbf{J}^2 is in the region of the continuous series. The factors in the second line are positive when $k > 2$ and $\mathbf{J}^2 > 0$. Thus, the determinant of inner products of coset modules $SO(2,1)/U(1)$ is always positive for continuous series representations under the condition $k > 2$. As we have argued above, this implies the unitarity of these representations and completes our general analysis of the unitarity domains. Note that for each type of global $SO(2,1)$ representation that gives rise to a unitary coset module (except for the trivial cases $\ell = 0, k/2$), all of the primary states in the module have strictly positive norm; there are no nontrivial null states.

5. Unitary representations of the N=2 algebra

Now that we have constructed unitary modules for the coset $SO(2,1)/U(1)$, we can combine these with the state space of a free boson to construct unitary representations of the N=2 superconformal algebra. The arguments of Section 2 imply that the most general representation of the local N=2 current algebra with $c > 3$ can be obtained in this way. Thus, we should obtain a complete picture of the domain of unitary N=2 representations for $c > 3$.

The representation theory of the superconformal algebra is often discussed in terms of the moments of the operators T , T_F^\pm , and J ; these are usually named L_n , G_r^\pm , and J_k , respectively. In order to specify the commutation relations of the moments, one must specify the boundary conditions on T_F^\pm . Taking T_F to be

antiperiodic around the origin defines the Ramond algebra. In that case, n , r , and k are all integers. Taking T_F to be periodic gives the Neveu-Schwarz algebra; in this case, n and k are integers, but r runs over half-integers. Fortunately, these two algebras are not really distinct, since if L_n , G_r^\pm , J_k satisfy the Neveu-Schwarz algebra, we can construct an algebra corresponding to an arbitrary (complementary) moding of the G^\pm by writing:^[20]

$$\begin{aligned}\hat{G}_{r+a}^+ &= G_r^+, & \hat{G}_{r-a}^- &= G_r^-, \\ \hat{L}_n &= L_n - aJ_n + \frac{c}{6}a^2\delta(n), & \hat{J}_n &= J_n - \frac{c}{3}\delta(n).\end{aligned}\tag{5.1}$$

We will discuss our results below in the language appropriate to the Neveu-Schwarz algebra. The analogous results for the Ramond algebra and other modings of this class may be obtained by applying the transformation (5.1). BFK^[4] also discuss a ‘twisted’ algebra in which one supercharge is moded by integers and the other by half-integers. Our analysis does not generalize simply to that case.

Our general method will be to begin with the local operators that create highest-weight states of an $SO(2,1)$ representation, remove the dependence on the boson ϕ , and restore a dependence on the boson φ in such a way as to produce an operator that creates highest-weight states of the $N=2$ superconformal algebra. Let $\Phi(z)$ be an operator which creates a state in the base $SO(2,1)$ representation of the current algebra module. Such a state is annihilated by J_n^i for all $n > 0$. With the expansion of $J^i(z)$ in (3.1), this implies the operator product relations

$$J^+(z)\Phi(0) \sim \frac{1}{z}\Phi_+(0), \quad J^-(z)\Phi(0) \sim \frac{1}{z}\Phi_-(0), \quad J^3(z)\Phi(0) \sim \frac{1}{z}J_0^3\Phi(0).\tag{5.2}$$

The last of these relations implies that, if Φ creates a state with $J_0^3 = m$, this operator depends on ϕ through

$$\Phi(z) = \hat{\Phi}(z)e^{m\sqrt{\frac{2}{k}}\phi},\tag{5.3}$$

where the operator $\hat{\Phi}$ lives entirely in the (generalized) parafermion theory. Using (2.6), we see that the singularities in (5.2) correspond to the following singularity

structure of parafermion operators:

$$\psi(z)\widehat{\Phi}(0) \sim z^{-1+\frac{2m}{k}}, \quad \psi^\dagger(z)\widehat{\Phi}(0) \sim z^{-1-\frac{2m}{k}}. \quad (5.4)$$

On the other hand, an operator $\Phi(z)$ which creates a highest-weight state of the N=2 algebra obeys the operator product relations

$$T_F^+(z)\Phi(0) \sim \frac{1}{z}\Phi_+(0), \quad T_F^-(z)\Phi(0) \sim \frac{1}{z}\Phi_-(0), \quad J(z)\Phi(0) \sim \frac{1}{z}J_0\Phi(0), \quad (5.5)$$

by virtue of the fact that the moments G_r^\pm and J_n of T_F^\pm and J annihilate the highest-weight state for $r, n > 0$. To restore (5.4) to the form (5.5), we define

$$\Phi(z) = \widehat{\Phi}(z)e^{iq\sqrt{\frac{3}{c}}\varphi}, \quad (5.6)$$

with

$$\frac{3}{c}q = -\frac{2}{k}m. \quad (5.7)$$

By (2.2), q is the $U(1)$ charge of the highest-weight state of the N=2 representation. Thus, each individual state in the base $SO(2,1)$ representation of a current algebra module gives rise to an irreducible representation of the N=2 algebra. If the current algebra coset module is unitary, the N=2 representation is also unitary.

Let us work through this construction more explicitly for the base states of a current algebra representation built on a representation in the discrete series \mathcal{D}_n^+ . Let the lowest-weight state on the base be $|\ell\rangle$, and label the higher states as $|\ell+n\rangle$. For each n , we should find a unitary irreducible representation of the N=2 algebra. Following the procedure sketched above, we begin with the operator $\Phi(z)$ which creates the state $|\ell+n\rangle$. We divide out the dependence on ϕ , as in (5.3), and then add back an exponential of φ , with charge given by (5.7):

$$q = -\frac{c}{3} \cdot \frac{2}{k}(\ell+n) = -\left[\frac{c}{3}\left(\frac{2\ell}{k}\right) + n\left(\frac{c}{3}-1\right) \right]. \quad (5.8)$$

Note that q is negative for these representations. According to (3.6), the original operator Φ created states with $L_0 = -\ell(\ell-1)/(k-2)$. The final operator Φ thus

creates states with the L_0 value

$$\begin{aligned} h &= -\frac{\ell(\ell-1)}{k-2} + \frac{1}{k}(\ell+n)^2 + \frac{1}{2} \frac{3}{c} q^2 \\ &= -\frac{\ell(\ell-1)}{k-2} + \frac{(\ell+n)^2}{k-2}. \end{aligned} \quad (5.9)$$

This formula for h can be simplified to

$$h = (n + \frac{1}{2})|q| - \frac{1}{2}(\frac{c}{3} - 1)(n^2 + n). \quad (5.10)$$

For each fixed $c > 3$, this formula gives a line segment in the (q, h) plane. Because $0 < (2\ell/k) < 1$, this segment is bounded by the points

$$\text{from : } \quad q = -n(\frac{c}{3} - 1), \quad h = \frac{n^2}{2}(\frac{c}{3} - 1) \quad (5.11)$$

$$\text{to : } \quad q = -\frac{c}{3} - n(\frac{c}{3} - 1), \quad h = (n + \frac{1}{2})\frac{c}{3} + \frac{n^2}{2}(\frac{c}{3} - 1).$$

In Fig. 3, we plot this set of line segments, together with their reflection $(-q) \rightarrow q$. These segments with $q > 0$ give the location of the representations formed in an analogous way from the base states of \mathcal{D}_n^- discrete series representations.

These results are in precise agreement with the general constraints derived by BFK.^[4] Those authors found that, for $c > 3$, unitary representations of the $N=2$ algebra lie in a domain in the (q, h) plane bounded by just the line segments (5.11). The boundary segments correspond to modules built on $|h, q\rangle$ in which the leading state in the multiple operator product

$$q < 0 : \quad (T_F^+)^{n+1} |h, q\rangle \quad ; \quad q > 0 : \quad (T_F^-)^{n+1} |h, q\rangle \quad (5.12)$$

is null. In our construction, this condition follows from the fact that the base representation of the original $SO(2, 1)$ module terminates at $|\ell\rangle$. BFK found that

modules with such null states were unitary for q values beginning at the intersection of each segment with the previous one and ending on the parabola

$$2\left(\frac{c}{3} - 1\right)h + \frac{c}{3} - q^2 = 0. \quad (5.13)$$

These are precisely the limits we have derived in (5.11).

The Weyl symmetry (3.12) implies that the boundary N=2 representations can be derived in another way. We begin with the operators Φ which create the states $(J_{-1}^-)^n |\ell\rangle$ on the left-hand boundary of a \mathcal{D}_n^+ representation. These states are annihilated by J_0^- but not by J_1^+ , so the operator products of the J^i with Φ have the singularities

$$J^+(z)\Phi(0) \sim z^{-2}\Phi_+(0), \quad J^-(z)\Phi(0) \sim z^0\Phi_-(0), \quad J^3(z)\Phi(0) \sim \frac{(\ell-n)}{z}\Phi(0). \quad (5.14)$$

Now define

$$\Phi(z) = \left(\Phi e^{-(\ell-n)\sqrt{\frac{2}{k}}\phi}\right) \cdot e^{iq\sqrt{\frac{3}{c}}}, \quad (5.15)$$

with

$$\frac{2}{k}(\ell-n) + \frac{3}{c}q = 1. \quad (5.16)$$

These operators Φ satisfy the N=2 highest weight conditions (5.5). They create states with $J_0 = q$ and $L_0 = h$, where

$$h = \left[-\frac{\ell(\ell-1)}{k-2} + n\right] + \frac{1}{k}(\ell-n)^2 + \frac{1}{2}\frac{3}{c}q^2. \quad (5.17)$$

It is not difficult to show that these relations reproduce the $q > 0$ branches of (5.10), with the endpoints again correctly given by (5.11).

Finally, let us consider N=2 representations built from the coset modules with continuous series $SO(2,1)$ representations at the base. These \mathcal{C} representations are characterized by the value of the Casimir operator \mathbf{J}^2 and the fractional part ϕ_0 of

the J_0^3 eigenvalues. Choose the original operator Φ in our construction to be the operator which creates the base state with $J_0^3 = m = (n + \phi_0)$. This operator can be carried through the steps from (5.2) to (5.7); this produces an N=2 primary field which creates states of $J_0 = q$ and $L_0 = h$, where h is given, analogously to (5.9), by

$$h = \frac{1}{k-2}(\mathbf{J}^2 + m^2). \quad (5.18)$$

For each value of q , \mathbf{J}^2 (and therefore also h) varies continuously from a lower bound to infinity. To determine the region of the (q, h) plane covered by such continuous representations, we need to compute the lower bound on h as a function of q . Since the base $SO(2, 1)$ representation is unitary, the lower bound on \mathbf{J}^2 depends on the fractional part of m : $\mathbf{J}^2 > \phi_0(1 - \phi_0)$. Thus

$$\begin{aligned} h &> \frac{1}{2}\left(\frac{c}{3} - 1\right) \left[\phi_0(1 - \phi_0) + (n + \phi_0)^2 \right] \\ &= \left(\frac{c}{3} - 1\right) \left[\left(n + \frac{1}{2}\right)(n + \phi_0) - \frac{1}{2}(n^2 + n) \right], \end{aligned} \quad (5.19)$$

while q is given by

$$q = -\left(\frac{c}{3} - 1\right)(n + \phi_0). \quad (5.20)$$

The boundary of (5.19) is just the line segment (5.10). As ϕ_0 is varied from 0 to 1, q sweeps over the region from the beginning of the n^{th} segment to the beginning of the $(n + 1)^{\text{st}}$ segment. That is, the N=2 representations formed from the continuous series of $SO(2, 1)$ representations cover completely the interior of the boundary shown in Fig. 3. Thus, our correspondence gives a manifestly unitary construction of all of the unitary representations of the N=2 algebra identified by BFK for $c > 3$.

6. Is N=2 superconformal symmetry solvable?

We have seen in the previous sections how $SO(2,1)$ current algebra provides a unified construction of the unitary representations of the N=2 algebra with $c > 3$. However, we have not yet discussed how to combine such representations into modular invariant, crossing symmetric, conformal field theories. In this section we explore whether the connection to $SO(2,1)$ can help in this regard. At present we have no definite answer to the question, and parts of this section will be rather speculative.

We have shown that the $SO(2,1)$ current algebra, the algebra of $SO(2,1)$ parafermions defined by (2.4), and the N=2 superconformal algebra for $c > 3$ are all related by addition or subtraction of a free boson, so a solution to any one of the three would solve them all. Correlation functions of highest-weight fields for the three algebras differ only by multiplication by correlation functions of free boson exponentials. The analogous observation for $c < 3$ representations has been exploited extensively by Zamolodchikov and Fateev,^[8] Qiu,^[22] Gepner^[12,21] and others,^[23,24] who have used the solution of $SU(2)$ current algebra due to Knizhnik and Zamolodchikov,^[25] and Fateev and Zamolodchikov^[26] to calculate quantities of interest in the N=2 discrete series. So one might try to solve $SO(2,1)$ current algebra with the same goals in mind.

One possible approach to $SO(2,1)$ current algebra would be to construct the four-point correlation functions of highest weight fields. For $SU(2)$ current algebra, this calculation can be done in two quite different ways. Unfortunately, both seem to fail in the noncompact case. The method of refs. 25, 26 uses the Sugawara expression for the stress tensor to derive a differential equation for the four-point functions which is a matrix in the group representation space. In our case, this matrix equation becomes infinite-dimensional. An alternative approach is to study the four-point function of parafermion highest-weight fields, and to apply the stress-tensor method to derive a differential equation for this object. This method has been used by Mussardo, *et al.*,^[27] in the Ising model (Z_2 parafermions),

and the method generalizes to the higher level $SU(2)$ parafermions. However, in all of these cases, the stress tensor method gives a closed system of equations only by virtue of the fact that the $SU(2)$ parafermion algebra contains nontrivial null states, which follow from the null states of global $SU(2)$ representations. We have seen that these null states do not appear in $SO(2,1)$. We believe that these difficulties are essential and stem from the fact that the tensor products of highest- and lowest-weight $SO(2,1)$ representations that contain the identity also contain infinitely many continuous-series representations. Thus, new methods are necessary to compute correlation functions in $SO(2,1)$ current algebra, though we have some hope that these methods may be found through the group theory of $SO(2,1)$.

An alternative approach to $c > 3$ superconformal field theory would be to construct modular-invariant partition functions from combinations of the characters of the supersymmetry algebra. It is important to note that for $c > 3$ one cannot build a modular-invariant partition function from a finite number of $N=2$ representations. This result is a generalization (due to E. Verlinde^[28]) of Cardy's result^[29] that for $c > 1$ one needs an infinite number of Virasoro primary fields. One compares the $\tau \rightarrow i\infty$ behavior of the partition function (dominated by the contribution of the identity operator)

$$Z \sim e^{\pi c \text{Im}\tau/6}, \quad \tau \rightarrow i\infty, \quad (6.1)$$

with that obtained by a modular transformation from the behavior as $\tau \rightarrow 0$. The latter is controlled by the growth of the characters of the representations of the chiral algebra, if there is a finite number of representations. For the $N=2$ algebra, the Neveu-Schwarz characters are^[4] (if there are no null states)

$$\chi(q) = q^{h-c/24} \prod_{n=1}^{\infty} \frac{(1+q^{n-1/2})^2}{(1-q^n)^2} = q^{h-c/24+1/8} \frac{\vartheta_3(q)}{\eta^3(q)}, \quad (6.2)$$

where $q = e^{2\pi i\tau}$. So their maximum growth is $\sim e^{\pi i/4\tau}$ as $\tau \rightarrow 0$. Including the antiholomorphic characters and transforming to the $\tau \rightarrow i\infty$ region, the maximum growth is $Z \sim e^{\pi \text{Im}\tau/2}$, which is incompatible with eq. (6.1) unless $c < 3$.

In a rational conformal field theory,^[30] the partition function is organized into a finite number of characters by some chiral algebra. From the above argument the $N=2$ superconformal algebra is not large enough to provide such an organization for $c > 3$. So one approach to defining solvable $N=2$ superconformal field theories for $c > 3$ is to find larger chiral algebras containing the $N=2$ algebra. Kazama and Suzuki^[13] have recently provided a wealth of such theories using coset (G/H) constructions. Tensor products of $N=2$ theories with $c < 3$ ^[12] also provide solvable $N=2$ theories with $c > 3$. We would like to ask whether there are theories solved by extensions of the $N=2$ algebra that are connected in a natural way with $SO(2,1)$.

One possible clue in this direction is provided by the recent relations found^[31,32] between $N=2$ superconformal invariance and the theory of quasi-homogeneous complex analytic singularities.^[33] One may consider each possible singularity as a polynomial in fields forming the superpotential for a class of $N=2$ supersymmetric Landau-Ginsburg theories. Since the superpotential is not renormalized, renormalization-group flows stay within this class of Lagrangians. For an appropriate choice of the kinetic term, one can find a renormalization-group fixed point, which then corresponds to an $N=2$ superconformal field theory. Many of the properties of this superconformal theory, such as the central charge c and the $U(1)$ charges q of the chiral $N=2$ superfields, may be inferred directly from the superpotential.^[31,32] Since chiral superfields are just those for which $G_{-\frac{1}{2}}^+ |\Phi\rangle$ or $G_{-\frac{1}{2}}^- |\Phi\rangle = 0$, these correspond to representations on the lowest boundaries of the BFK diagram for which $h = \pm \frac{1}{2}q$. Thus, a subset of the $N=2$ representations appearing in the theory is specified completely.

Singularities are partially characterized by their modality, which is the number of deformations preserving the singularity type, and those with modality less than three have been classified.^[33] Those of modality zero are classified by the A, D, E Dynkin diagrams of simply-laced, simple Lie algebras, or alternatively by the discrete subgroups of $SU(2)$. They correspond to the $c < 3$ modular-invariant $N=2$ superconformal theories. The modality one singularities consist of three with

$c = 3$ that have been identified^[31,32] as orbifolds of complex tori, a set of fourteen “exceptional” singularities, and an infinite set of “hyperbolic” singularities. The exceptional set is related to the existence of certain exceptional discrete subgroups of $SO(2,1)$. For the fourteen exceptional singularities one finds that c always has the form $3k/(k-2)$ with k an integer, and $|q| = 2\ell/(k-2)$ where ℓ runs over a set of integers and half-integers between 0 and $k/2$.^{*} This suggests that there might be a natural description of the associated N=2 theories in terms of $SO(2,1)$ current algebra.

Furthermore, many of the exceptional unimodal singularities are direct products of modality zero singularities, and the corresponding N=2 theories are in these cases tensor products of $c < 3$ models, which have been solved exactly.[†] Thus we know the exact partition functions of many of the $c > 3$ theories corresponding to exceptional singularities, and can describe their operator content in terms of the $SO(2,1)$ quantum numbers \mathbf{J}^2 and J_3 , using the relations found in Section 5. An N=2 primary field in the tensor product has the form $\Phi = \prod \Phi_i$, where the component fields Φ_i are not necessarily primary under the component N=2 algebras. From its conformal dimension $h = \sum h_i$ and charge $q = \sum q_i$, and eqs. (5.7), (5.18), we can compute

$$\mathbf{J}^2 = (k-2)h - [\frac{1}{2}(k-2)q]^2, \quad J_3 = -\frac{1}{2}(k-2)q. \quad (6.3)$$

To understand how an N=2 superconformal field theory might be solved using a particular singularity, we need to be able to describe the values of \mathbf{J}^2 and ϕ_0 that occur, and their multiplicities. (Each state is labelled by a left-moving and a right-moving \mathbf{J}^2 and ϕ_0 , of course.) For the tensor product examples, we have found

^{*} We thank C. Vafa and E. Martinec for pointing this out to us.

[†] It has been conjectured^[34-36] that all rational conformal field theories have G/H constructions where G, H are compact Lie groups. Tensor products of $c < 3$ models certainly fall into this class, with $G/H = (SU(2) \times U(1)/U(1))^n$. However, if one can determine the properties of the superconformal field theories associated with the other exceptional singularities, and if they turn out to be rational, it will be interesting to see whether they are consistent with the conjecture, as the choice of G, H is not obvious here.

that ϕ_0 is always an integer or half-integer, and \mathbf{J}^2 is always either an integer or an integer plus $1/4$. In fact, the values of \mathbf{J}^2 for the (infinite number of) continuous series representations that appear are all equal, modulo $(k-2)$, to \mathbf{J}^2 for one of the finite number of discrete series representations that appear, for which $\mathbf{J}^2 = \ell(1-\ell)$ with $2\ell \in \mathbf{Z}$, $0 \leq 2\ell \leq k$. We still lack the required description of the multiplicities, however.

It is easy to see that continuous series representations are always present in the tensor product partition function. For example, the linear combinations of $U(1)$ currents J^i that are orthogonal to the sum $J = \sum J_i$ are primary under the total $N=2$ algebra, with $h = 1$ and $q = 0$, hence $\mathbf{J}^2 = k - 2$. It is also worth observing that spectral flow from one Neveu-Schwarz representation to another (through the deformation (5.1)) leaves \mathbf{J}^2 fixed and corresponds to shifting J_3 by one unit within the same $SO(2, 1)$ representation.

Finally we note that it may not be necessary to organize a conformal field theory into a finite set of primary fields under some chiral algebra, in order to solve it. A free boson on a circle of irrational radius is such an example, in which the partition function contains a discrete but infinite set of characters of the Virasoro algebra. So one might try to directly construct modular-invariant partition functions from the characters (6.2) and the characters^[37,38] for the $N=2$ representations with additional null states. This type of approach was considered for the case of the $N=4$ superconformal algebra in ref. 38. Restricting to $N=2$ representations with \mathbf{J}^2 and ϕ_0 belonging to a discrete but infinite set (as in the tensor product examples described above) might simplify the problem of constructing modular invariants. We have tried such a construction in the parafermion system, but without success to date.

In summary, we have described a connection between $N=2$ superconformal field theories with $c > 3$ and $SO(2, 1)$ current algebra, which we believe will prove useful in better understanding the former, though exactly how remains to be seen.

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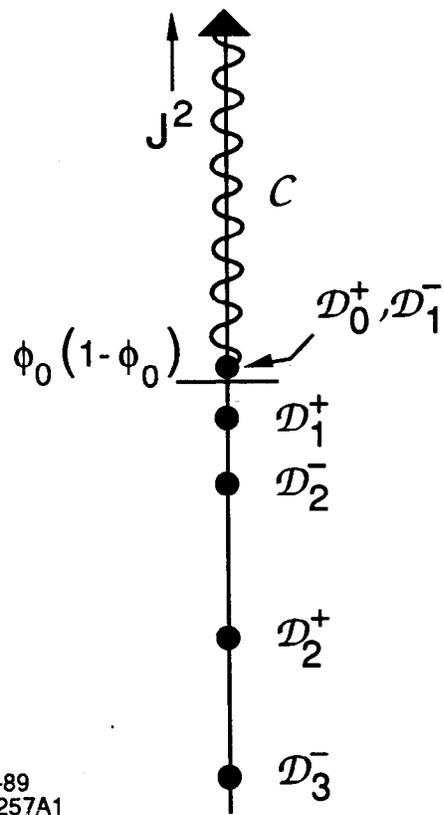
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FIGURE CAPTIONS

- 1) Spectrum of \mathbf{J}^2 for unitary representations of $SO(2,1)$ with $J_0^3 = \phi_0 \pmod{1}$.
- 2) Current algebra representation built on a discrete series representation of $SO(2,1)$. The number in each circle indicates the multiplicity of states at that value of (L_0, J_0^3) .
- 3) Unitarity boundaries on the (q, h) plane for $c = 7$.



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Fig. 1

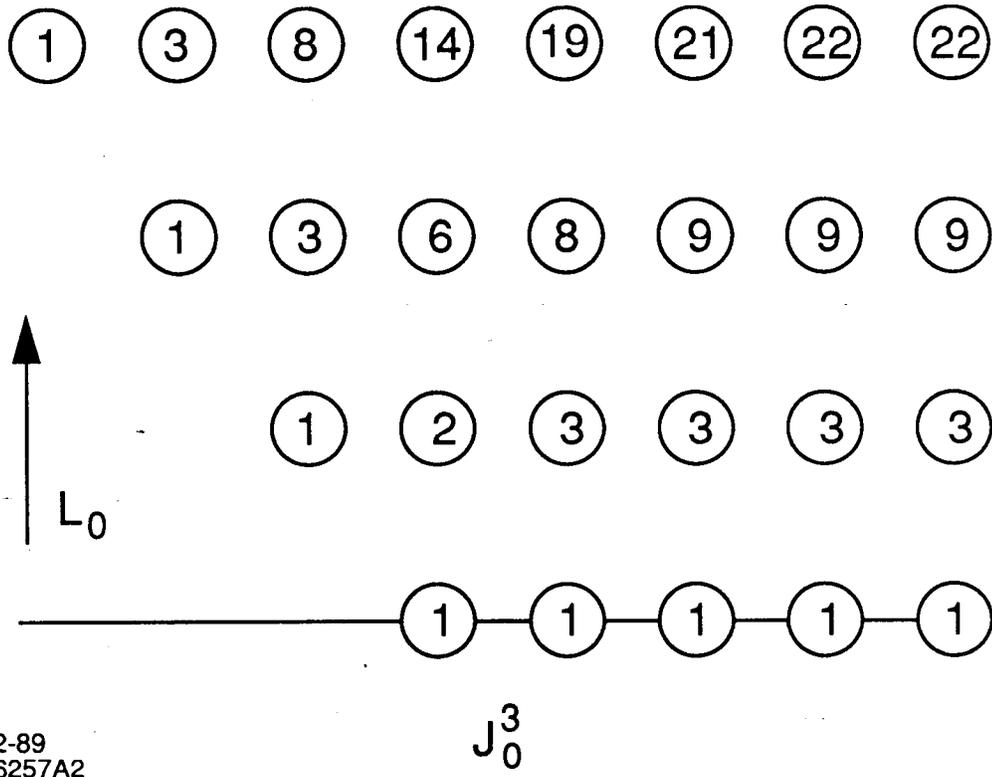
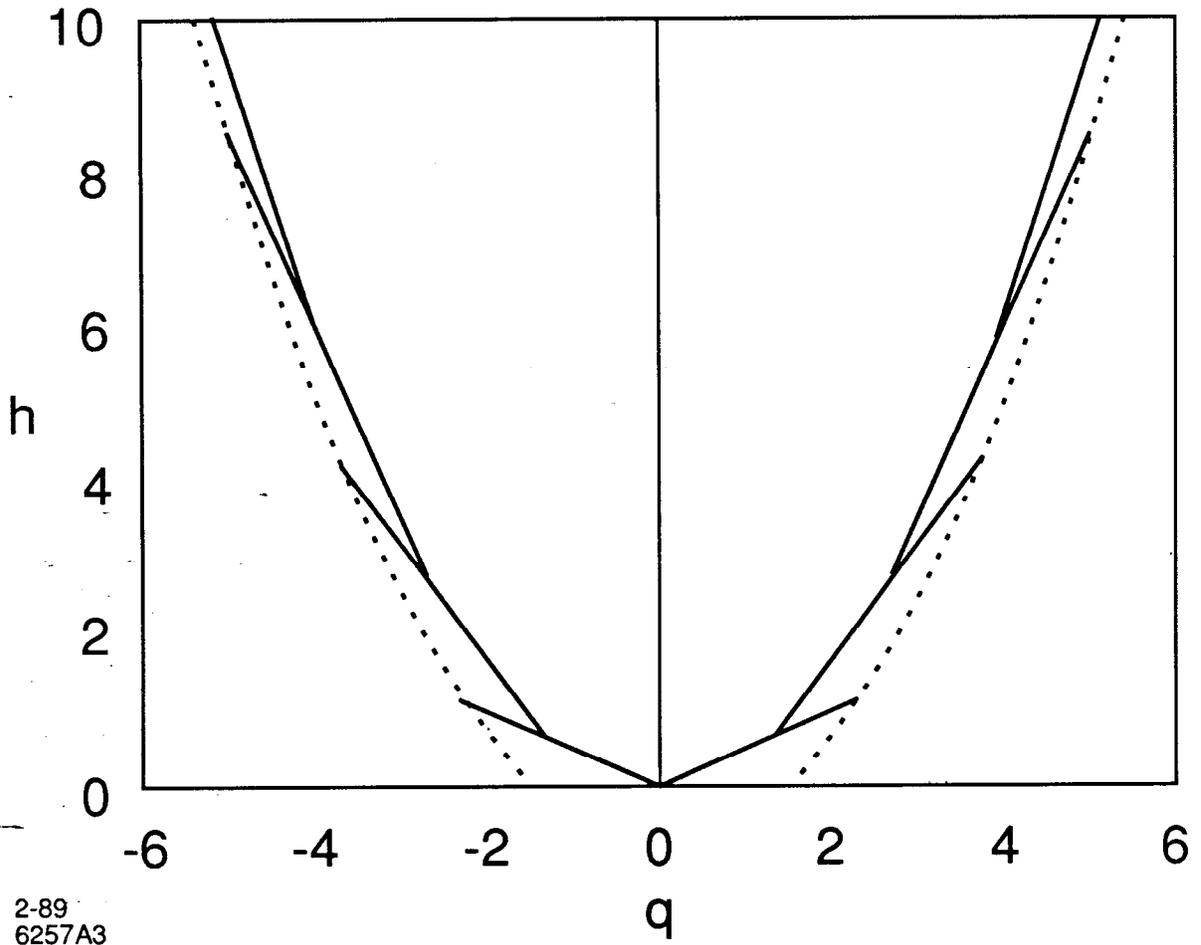


Fig. 2



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Fig. 3