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Topological Invariants And A Gauge Theory Of The Super-Poincaré Algebra In Three Dimensions †

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ABSTRACT

Various aspects of superspace topology and N=1 supersymmetry in 2+1 dimensions are explored. Non-Minimal super Yang-Mills theory is used to construct a gauge theory of the N=1 super-Poincaré algebra. A topological super Yang-Mills theory is given and used to compute elements of the cohomology classes of super-monopole moduli space.

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I. Introduction

Topology has played an important role in quantum field theory. Recently, the reverse has occurred. Witten [1] has shown how to use the path integral methods of field theory to construct certain polynomial invariants which are of interest to topologists. In four dimensions, these topological invariants are known as Donaldson invariants [2]. In the language of field theory, the topological invariants are the observables of the theory.

At first sight, these topological quantum field theories (TQFT's) may seem to be physically irrelevant. Since their observables are topological invariants, they are independent of any metric. Consequently, there is no graviton in their spectra. Furthermore, there is no notion of propagation. Hence there are no physical fields. Nevertheless, they are theories of unbroken general covariance. In principle this means that there is no longer an integral over the metric, in the partition function. From this point of view they prove to be rather interesting toy models. They are constructed by BRST gauge fixing [3,4] a topological symmetry. This procedure has two stages since the Lagrangian of the primary ghosts is invariant under a second (albeit related) symmetry; there are ghosts for ghosts.

In three dimensions, an *alter idem* of these theories is the Chern-Simons (CS) theory. The latter is generally covariant but is not invariant under the topological symmetry. Because the gauge fields (and the derivative) in the CS Lagrangian are contracted by a ϵ-tensor density, the $\frac{1}{\sqrt{g}}$ factor in the ϵ-density cancels against the $\sqrt{g}$ in the measure of the action. The result is a metric independent action. Put differently, the action is written purely as the wedge product of forms. One example in which it appears in field theory is as the parity breaking part of the effective action for fermions in a background gauge field [5]. Another example of its use is its addition to the CP^1 model. There, its $U(1)$ gauge field dresses the CP^1-bosons in such a way that the would be spin-0 fields carry fractional spin [6]. The manifold is restricted to be S^3 so that the CS action is a Hopf invariant.

Apart from these applications, the CS theory may be used to construct gauge theories of some rather unusual gauge groups.

The Lorentz generator in three dimensions may be written as a 1-form. This is done by simply taking the dual of the 2-form. Combining this with the generator of translations, the $ISO(2,1)$ algebra is obtained. Of course, this is the algebra of the Poincaré group in 2+1 dimensions. A gauge theory of this group may be written down. This was done in ref. [7] as a CS theory of the Einstein-Hilbert action. When the torsion free constraint is imposed, one obtains the usual $ISO(2,1)/SO(2,1)$ theory of gravity. The N=1 supersymmetric gauge theory of $ISO(2,1|2)$ will be constructed here. The motivation for doing this is that it is a gauge theory of a group whose generators include the D=3, N=1 supersymmetry algebra. This suggests that, among other things, it may be possible to study supersymmetry breaking in the context of a gauge theory.

A theory which is invariant under the BRST topological symmetry may also be constructed in three dimensions in the following sense. In four dimensions, the topological theory is used to construct cohomolgy classes of instanton moduli space [1,8]. There the Pontryagin density is a surface term and as such it is invariant under the topological symmetry. When a simple dimensional reduction of the Pontryagin density is done, one obtains the CS term along with a surface term in three dimensions. Whereas the CS term is not invariant under the topological symmetry, the surface term is. The latter defines a topological field theory for magnetic monopoles [9]. The integral of the surface term gives the charge of the monopole. A gauge choice [10] for fixing the symmetry is to impose the Bogomol'nyi equations [11] of D=4, static Yang-Mills-Higgs monopoles. Thus in the supersymmetrization of the topological gauge theory, one should impose the supersymmetric analogs of the Bogomol'nyi equations. Such a construction will be given in this work.

It has been shown in ref. [12] that the observables of the non-supersymmetric CS theory are Wilson Loops. For three dimensions, there is the possibility that the cycle over which one integrates, is knotted. So there is a link between knot theory and CS theory. Moreover, in computing the *vev* of the Wilson Loop operator, one encounters an object which has been known to mathematicians since Gauss. It is the self-linking number of a cycle [13]. As a phase factor, the Wilson Loop has been used to transmute the CP^1 bosons into fermions [14]. Thus it is interesting to see what implications this quantity would have on a supersymmetric theory. However, we will see that there are no fermions in the spectrum of the super Chern-Simons (SCS) theory. This means that the Wilson Loop *vev* will be the same as in the non-supersymmetric CS theory. Nevertheless, in superspace the integrand of the self-linking number contains terms involving the Grassmann coordinate. Thus, there exists the possiblility that one can have knots in superspace which do not change the self-linking number. Formulae relevant to these calculations will be given in the appendix.

We begin in section II with a review and then extension of D=3, N=1 super Yang-Mills gauge theory. This is done since the rest of the paper is composed of applications of this theory. The extension will not be to higher supersymmetries but to a non-minimal theory. The first order formulation of $ISO(2,1|2)$ theory requires such an extension. The $ISO(2,1|2)$ gauge theory will be given in section III. An example of a supersymmetric TQFT will be given in section IV. There the D=3, N=1 superspace analogs of the Donaldson invariants will be presented. This will allow us to compute elements of the cohomology classes of supersymmetric, monopole moduli space.

Although all of the calculations will be done in superspace, most of the final expressions will be reduced to components. The notations used here are generally those of ref. [15].

II. Extending Supersymmetric Chern-Simons Gauge Theory

An introduction to minimal 2+1 dimensional, N=1 rigid and local superspace may be found in chapter 2 of the book by Gates, Grisaru, Roček and Siegel [15]. Some of their results will be reviewed in the first half of this section. The second half contains a non-minimal formulation of super Yang-Mills theory (SYM). While the minimal version should be used in the construction of the super Chern-Simons (SCS) gauge theory of supergravity, the $ISO(2,1|2)$ supersymmetric gauge theory must be formulated in the non-minimal version. The latter formulation will only be used in section III.

II.1 *Minimal Theory*

As in any discussion of superspace gauge theory, a gauge superfield, Γ_A, is introduced. As usual, the 'A' index runs over both bosonic and spinor indices: $A \in (a, \alpha)$. This allows for the definition of the super-covariant derivative $\nabla_A \equiv D_A - i\Gamma_A$. Its transformation law is $\nabla'_A = e^{iK}\nabla_A e^{-iK}$ where $K \equiv K^{\hat{I}} t_{\hat{I}}$ and the $t_{\hat{I}}$ are elements of the Lie algebra of the gauge group. Superfield strengths, $\mathcal{F}_{AB}$, are defined by the graded commutators

$$[\nabla_A, \nabla_B\} \;=\; T_{AB}{}^C \nabla_C \;-\; i\mathcal{F}_{AB} \;\;, \tag{2.1}$$

where $T_{AB}{}^C$ is the torsion of the supermanifold. Constraints must be imposed. The conventional constraint, $\mathcal{F}_{\alpha\beta} = 0$, allows for the expression of the vector potential, Γ_A, in terms of the fundamental spinor potential, Γ_α, as

$$\Gamma_a \;=\; i\frac{1}{2}(\gamma_a)^{\alpha\beta} D_\alpha \Gamma_\beta \;+\; \frac{1}{4}(\gamma_a)^{\alpha\beta}[\Gamma_\alpha, \Gamma_\beta\} \;\;. \tag{2.2}$$

This gives a minimal version of supersymmetric gauge theory. There is also $\mathcal{F}_{\alpha b} = i(\gamma_b)_\alpha{}^\gamma W_\gamma$. As a constraint, it says that there is no pure spin-$\frac{3}{2}$ superfield in $\mathcal{F}_{\alpha b}$. Furthermore, it defines the spinor superfield strength, W_α. This superfield has the gaugino as its lowest ($\theta = 0$) component. Accordingly, the graded commutators and Bianchi Identity consistency conditions of minimal D=3, N=1 supersymmetric gauge theory are:

$$
\begin{aligned}
[\nabla_\alpha, \nabla_\beta\} &= i2(\gamma^a)_{\alpha\beta}\nabla_a \quad , \\
[\nabla_\alpha, \nabla_b\} &= (\gamma_b)_\alpha{}^\gamma W_\gamma \quad , \\
[\nabla_a, \nabla_b\} &= -\, i\mathcal{F}_{ab} \quad , \\
[\nabla^\alpha, W_\alpha\} &= 0 \quad , \\
\mathcal{F}_{ab} = -\frac{1}{2}(\gamma_a\gamma_b)^{\alpha\beta}[\nabla_\alpha, W_\beta\} &= i\frac{1}{2}\epsilon_{abc}(\gamma^c)^{\alpha\beta}[\nabla_\alpha, W_\beta\} \quad .
\end{aligned}
\tag{2.3}
$$

The super-Yang-Mills action is

$$
\begin{aligned}
S_{SYM} &= \frac{1}{8}\int d^3x d^2\theta \; Tr(W^2) \\
&= -\frac{1}{4}\int d^3x \; Tr[F^2 \; - \; i2\lambda^\alpha(\gamma^a)_\alpha{}^\beta \mathcal{D}_a\lambda_\beta] \quad ,
\end{aligned}
\tag{2.4}
$$

where the second line follows from the first after the Grassmann integral is performed and the projections onto components:

$$
\begin{aligned}
A_a &\equiv \Gamma_a| \quad , \qquad \lambda_\alpha \equiv W_\alpha| \quad , \\
\mathcal{D}_a &\equiv \nabla_a| = \partial_a \; - \; iA_a \quad ,
\end{aligned}
\tag{2.5}
$$

are used. As usual, F is the covariant exterior derivative of A, $F = \mathcal{D}A$, and λ is the SYM spinor.

Equation (2.4) is not the action of current interest. It is the non-abelian supersymmetric Chern-Simons (NSCS) action which is at issue. In superspace this reads

$$
\begin{aligned}
S_{NSCS} &= \int d^3x d^2\theta \; Tr(\Gamma_\alpha G^\alpha) \quad , \\
G_\alpha &\equiv W_\alpha \; - \; \frac{1}{6}(\gamma^a)_{\alpha\beta}[\Gamma^\beta, \Gamma_a\} = \frac{2}{3}W_\alpha \; - \; i\frac{1}{6}(\gamma^a)_\alpha{}^\beta D_{[\beta}\Gamma_{a)} \quad .
\end{aligned}
\tag{2.6}
$$

The coefficient of this action is proportional to the mass of the gauge field and is quantized for non-trivial π_3 of the gauge group [5].

Before proceeding with S_{NSCS} it is instructive to first learn how the supersymmetrization process works on the abelian theory. Remove the trace form S_{NSCS} and use $G_\alpha = W_\alpha$.

As S_{NSCS} or S_{SCS} (the abelian version) is a mass term for the gauge field, a mass for λ_α is expected. Furthermore, supersymmetry will require a term which is first order in derivatives but quadratic in spinor fields. In fact, one finds

$$S_{SCS} \;=\; \int d^3x \big[\epsilon^{abc} A_a F_{bc} \;-\; i2(\gamma^a)_\alpha{}^\beta \chi^\alpha \partial_a \lambda_\beta \;-\; 2\lambda^\alpha \lambda_\alpha \big] \quad , \tag{2.7}$$

as the $U(1)$ component SCS action. The lowest component of the spinor super-potential, $\chi^\alpha \equiv \Gamma^\alpha|$, is "new". Compare the second term in eqn. (2.7) with the spinor term in eqn. (2.4). As S_{SCS} and S_{SYM} are distinct actions[1] , the Dirac action for λ_α which appears in S_{SYM} cannot appear in S_{SCS}. Thus a new spinor field, χ_α, had to be "introduced". Although it was part of the superspace theory which lead to eqn. (2.4), the latter field was not required in the component action for super-Yang-Mills theory.

Equation (2.7) tells us something rather interesting. In the Wess-Zumino (W-Z) gauge we have $\chi_\alpha \equiv 0$. Since the W-Z gauge is algebraic, there are no topological obstructions to making this choice. Consequently, only the λ_α mass term appears as a spinor contribution. If S_{SCS} is not coupled to any other actions involving the photino, we must have $\lambda_\alpha = 0$, on-shell. Therefore there is no on-shell supersymmetric completion of the pure Chern-Simons theory. As to be expected, this behaviour persists in the non-abelian theory where only the A^3 term is added to eqn. (2.7) (see eqn. (2.14) below).

One can formally argue for this result in the following way. For any theory which is generally covariant, there is no energy-momentum tensor. Consequently, there is no Hamiltonian. In a supersymmetric theory, the Hamiltonian is proportional to the square of the supercharge(s). Thus with $\mathcal{H} = 0$, the supercharge must square to zero and there is no supersymmetry.

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[1] If S_{SCS} and S_{SYM} were related, the coefficient k of S_{SCS} would be determined by the coefficient in front of S_{SYM}. This is not the case.

II.2 *Non-Minimal Theory*

Later, in our treatment of Chern-Simons supergravity as a supersymmetric gauge theory, we will need a rather unconventional form of the non-abelian theory. The conventional constraint, $\mathcal{F}_{\alpha\beta} = 0$, will be shown to automatically enforce the $ISO(2,1|2)/SO(2,1)$ gauge theory of supergravity. At first, we will want to construct an $ISO(2,1|2)$ gauge theory. In such a theory, the spin-connections and dreibeins are completely independent and are in a first order formulation. The conventional constraint, however, will be shown to give the spin-connection in terms of the super-dreibein, *i.e.* the $ISO(2,1|2)/SO(2,1)$ theory. So to construct the $ISO(2,1|2)$ gauge theory, we must relax this constraint.

With this new super-geometry, both Γ_a and Γ_α are independent superfields with $\mathcal{F}_{\alpha\beta}$ now being given by

$$\mathcal{F}_{\alpha\beta} \;=\; D_{(\alpha}\Gamma_{\beta)} \;-\; i[\Gamma_\alpha, \Gamma_\beta\} \;-\; i2(\gamma^a)_{\alpha\beta}\Gamma_a \;\neq\; 0 \quad . \tag{2.8}$$

The Bianchi Identities then require

$$\begin{aligned} \nabla_{(\alpha}\mathcal{F}_{\beta\gamma)} &= 0 \quad , \\ \mathcal{F}_{bc} &= i\frac{1}{4}(\gamma_b)^{\alpha\beta}\nabla_c\mathcal{F}_{\alpha\beta} \;-\; \frac{1}{2}(\gamma_b\gamma_c)^{\alpha\beta}\nabla_\alpha W_\beta \quad , \\ \nabla^\alpha W_\alpha &= -\,i\frac{1}{6}(\gamma^c)^{\alpha\beta}\nabla_c\mathcal{F}_{\alpha\beta} \quad , \\ \nabla_\alpha \mathcal{F}_{bc} &= i(\gamma_{[c|})_\alpha{}^\beta\nabla_{|b]}W_\beta \quad , \\ \epsilon^{abc}\nabla_a\mathcal{F}_{bc} &= 0 \quad . \end{aligned} \tag{2.9}$$

Note that $\nabla^\alpha W_\alpha$ is no longer zero. In addition to these expressions, the following will prove useful for component calculations:

$$\begin{aligned} \nabla_\alpha W_\beta &= i\frac{1}{2}\epsilon^{abc}(\gamma_a)_{\alpha\beta}\mathcal{F}_{bc} \;+\; i\frac{1}{4}(\gamma^a)_\beta{}^\gamma\nabla_a\mathcal{F}_{\alpha\gamma} \\ &\quad -\; i\frac{1}{24}C_{\alpha\beta}(\gamma^a)^{\gamma\delta}\nabla_a\mathcal{F}_{\gamma\delta} \quad , \\ \nabla^2 W_\alpha &= i\frac{1}{6}(\gamma^a)^{\beta\gamma}\nabla_a\nabla_\alpha\mathcal{F}_{\beta\gamma} \;-\; i(\gamma^a)_{\alpha\beta}\nabla_a W^\beta \\ &\quad -\; i\frac{1}{6}[W^\beta, \mathcal{F}_{\alpha\beta}\} \quad , \\ \nabla_\alpha\nabla_\beta &= i(\gamma^a)_{\alpha\beta}\nabla_a \;-\; i\frac{1}{2}\mathcal{F}_{\alpha\beta} \;-\; C_{\alpha\beta}\nabla^2 \quad . \end{aligned} \tag{2.10}$$

All of this will mean that there will be more component fields than in the minimal theory. Let us now construct the super Yang-Mills action for these fields. As W_α is still a field-strength, it is gauge covariant so that $\frac{1}{4}Tr(W^2)$ is gauge invariant. Upon reducing it to components the action

$$
\begin{aligned}
S^{nm}_{SYM} \ = \ \frac{1}{4}\int d^3x \ Tr[& - F^2 \ + \ i2\lambda^\alpha(\gamma^a)_\alpha{}^\beta \mathcal{D}_a\lambda_\beta \\
& + \ \frac{4}{3}i\lambda^\alpha(\gamma^a)_\alpha{}^\beta \mathcal{D}_a\psi_\beta \ - \ \frac{2}{3}\lambda^\alpha[\lambda^\delta, V_a\}(\gamma^a)_{\alpha\delta} \\
& + \ \frac{1}{2}\mathcal{D}_a V_b \mathcal{D}^{[b}V^{a]} \ - \ \frac{2}{9}(\mathcal{D}\cdot V)^2 \ - \ 2F^{ab}\mathcal{D}_a V_b] \ ,
\end{aligned}
\tag{2.11}
$$

is found. In computing this, a new super-multiplet has been introduced with component fields defined as

$$
\mathcal{F}_{\alpha\beta}| \ \equiv \ -i2(\gamma^a)_{\alpha\beta}V_a \ , \qquad \nabla^\alpha \mathcal{F}_{\beta\gamma}| \ \equiv \ 2\delta_{(\beta}{}^\alpha\psi_{\gamma)} \ . \tag{2.12}
$$

Thus two spin-$\frac{1}{2}$ fields and two spin-1 fields are needed in the non-minimal theory, *i.e.* two vector multiplets. The supersymmetry (ϵ) transformation and Yang-Mills (K) transformation laws of the new fields are

$$
\begin{aligned}
\delta_\epsilon V_a \ &= \ - i(\gamma_a)^{\alpha\beta}\epsilon_\alpha\psi_\beta \ , \\
\delta_\epsilon \psi_\alpha \ &= \ - \frac{1}{3}\epsilon_\alpha \mathcal{D}\cdot V \ + \ i\frac{1}{2}\epsilon^{abc}(\gamma_c)_{\alpha\beta}\epsilon^\beta \mathcal{D}_a V_b \ - \ i\frac{1}{3}\epsilon_\alpha V^2 \ , \\
\delta_K V_a \ &= i[K, V_a] \ , \\
\delta_K \psi_\alpha \ &= i[K, \psi_\alpha] \ .
\end{aligned}
\tag{2.13}
$$

The V^2 term in the fermion's transformation law may be understood from the fact that the gaugino, λ_a, also has a term which is quadratic in A. There is no spin-$\frac{3}{2}$ field in the theory because of the first Bianchi Identity in eqn. (2.9), on $\mathcal{F}_{\alpha\beta}$.

Without the conventional constraint, the supersymmetric Chern-Simons action differs from eqn. (2.6) through the addition of a term proportional to $(\gamma^a)^{\alpha\beta}\Gamma_a\mathcal{F}_{\alpha\beta}$. This is a requirement of gauge invariance. The superfield and W-Z gauge component actions are

$$
\begin{aligned}
S^{nm}_{NSCS} &= \int d^3x d^2\theta Tr\big[\Gamma_\alpha G^\alpha \ - \ i\frac{1}{6}(\gamma_a)^{\alpha\beta}\Gamma^a \mathcal{F}_{\alpha\beta}\big] \\
&= \int d^3x \ Tr\big[A \wedge dA \ + \ \frac{2}{3}A \wedge A \wedge A \\
&\qquad\qquad - \ \frac{2}{3}V \wedge dV \ - \ V \wedge A \wedge V \\
&\qquad\qquad - \ 2\lambda^\alpha\lambda_\alpha \ - \ 4\psi^\alpha\lambda_\alpha \ - \ i\frac{1}{3}B[A^a, V_a]\big] \ .
\end{aligned}
\tag{2.14}
$$

Observe that ψ_α is auxiliary. Furthermore, the (ψ, λ) system vanishes by its equation of motion. The auxiliary field, B, defined by $B \equiv \nabla^\alpha\Gamma_\alpha|$, constrains $[A^a, V_a]$ to be zero.

The computation of the component action outside of the W-Z gauge is straightforward but lengthy. One finds graded-commutators of quintic order in the fields. For our purposes, nothing enlightening is obtained. Thus it won't be given here.

III. Super-Poincaré Gauge Theory

Having discussed non-abelian supersymmetric Chern-Simons gauge theory, it is now possible to construct $ISO(2,1|2)$ gauge theory following the non-supersymmetric work of Witten [7]. In so doing, a gauge theory of the super-Poincaré algebra will be given. This is a first order formulation of the Einstein-Hilbert action with the graviton, gravitino and spin-connection treated as independent fields, off-shell. The second order superconformal theory, with the spin-connection written as a functional of the graviton and gravitino was given in ref. [16]

First, let us examine the super-Poincaré algebra, $ISO(2,1|2)$, in 2+1 dimensions. The generators of this super-Lie algebra are the Lorentz generator, J_{ab}, and the super-translation generators P_a and Q_α. Defining the dual $J_a \equiv \frac{1}{2}\epsilon_{abc}J^{bc}$, the graded-algebra reads

$$
\begin{aligned}
[J_a, J_b\} &= \epsilon_{abc} J^c \quad , \\
[J_a, P_b\} &= \epsilon_{abc} P^c \quad , \\
[P_a, P_b\} &= 0 \quad , \\
[Q_\alpha, Q_\beta\} &= 2(\gamma^a)_{\alpha\beta} P_a \quad , \\
[Q_\alpha, J_b\} &= i\frac{1}{2}(\gamma_b)_\alpha{}^\beta Q_\beta \quad , \\
[Q_\alpha, P_b\} &= 0 \quad ,
\end{aligned}
\tag{3.1}
$$

The P's and Q's form the supersymmetry algebra.

Now the Casimir element of the $ISO(2,1)$ group is [7] $C_0 = \frac{1}{2}\epsilon_{abc} P^a J^{bc} = P \cdot J$. It is straightforward to show that

$$
C \;=\; P \cdot J \;-\; i\frac{1}{8} Q^\alpha Q_\alpha \quad , \tag{3.2}
$$

is a Casimir element of the $ISO(2,1|2)$ super-algebra given above. This element is defined in terms of a non-singular, graded-symmetric, bilinear form, $\mathcal{G}^{AB}$, as $C \equiv \mathcal{G}^{AB} X_A X_B$ where $X_A \in \{J_a, P_a, Q_\alpha\}$. Hence inner products in the super-Lie algebra are given by

$$
\begin{aligned}
\langle P^a, J^b \rangle &= \eta^{ab} \quad , \\
\langle Q^\alpha, Q^\beta \rangle &= i\frac{1}{8} C^{\alpha\beta} \quad ,
\end{aligned}
\tag{3.3}
$$

with the other bilinear forms vanishing. The anti-symmetric symbol, $C^{\alpha\beta}$, is the spinor metric.

The fact that eqn. (3.2) is the Casimir of $ISO(2,1|2)$ is interesting. The Casimir of $ISO(3,1)$ is the Pauli-Lubanski vector: $W_a = \frac{1}{2}\epsilon_{abcd} P^b J^{cd}$. Its supersymmetric generalization is relatively complicated in that it is a product of quartic and quadratic forms in P, Q and J. Now C_0 and W_0 are clearly of the same structure. This is not the case with C and its $ISO(3,1|4)$ counterpart.

There is another interesting consequence of the structure of the Casimir element. Inner products defined by "super-metrics" which are block diagonal (as in eqn. (3.3)) imply that the supersymmetric action will be composed of two pieces. The first will be a simple supersymmetric generalization of the purely bosonic result. The second will be a pure supersymmetric artifact as it arises only from the translation of the spinor coordinates.

With the results of the previous section and eqns. (3.1-3.3), the construction of the ECSSG action proceeds as follows. Work in a super-coordinate basis and define the super-covariant derivative $\nabla_M \equiv \partial_M - i\Gamma_M$ where

$$\Gamma_M \equiv E_M{}^a P_a + E_M{}^\alpha Q_\alpha + \Omega_M{}^a J_a \ , \tag{3.4}$$

is the gauge superpotential expanded over the generators. The superfields $E_m{}^a$, $E_m{}^\alpha$ and $\Omega_m{}^a$ respectively contain the dreibein, $e_m{}^a$, gravitino, $\psi_m{}^\alpha$ and the spin-connection, $\omega_m{}^a$ as their lowest components. Traditional SYM generators may also be added to the right-hand-side of eqn. (3.4). The super-curvature (superfield strength) is then

$$\begin{aligned}
\mathcal{F}_{MN} \ =& \ \left[\partial_{[M}E_{N)}{}^c \ - \ iE_{[M}{}^a\Omega_{N)}{}^b\epsilon_{ab}{}^c \ - \ i2(-)^N(\gamma^c)_{\alpha\beta}E_M{}^\alpha E_N{}^\beta\right]P_c \\
&+ \ \left[\partial_{[M}\Omega_{N)}{}^c \ - \ i\Omega_M{}^a\Omega_N{}^b\epsilon_{ab}{}^c\right]J_c \\
&+ \ \left[\partial_{[M}E_{N)}{}^\gamma \ - \ \frac{1}{2}(\gamma_a)_\beta{}^\gamma\Omega_{[M}{}^aE_{N)}{}^\beta\right]Q_\gamma \ .
\end{aligned} \tag{3.5}$$

As constraints led to the solution of the classical SYM theory in terms of certain prepotentials, the constraints will allow for the solution of certain supergravity superfields in terms of others. The superfield-strength $\mathcal{F}_{MN}$ is a super 2-form. This means that $\mathcal{F}_{AB} = (-)^{M(N+B)}E_A{}^M E_B{}^N \mathcal{F}_{MN}$. If we impose the constraint $\mathcal{F}_{\alpha\beta} = 0$, the relations

$$\begin{aligned}
E_a{}^P \ =& \ i\frac{1}{2}(\gamma_a)^{\alpha\beta}[E_\alpha{}^M\partial_M E_\beta{}^P \ - \ E_\alpha{}^M\partial_M E_\beta{}^N E_N{}^\gamma E_\gamma{}^P] \ , \\
\Omega_b{}^c \ =& \ i\frac{1}{2}(\gamma_b)^{\alpha\beta}E_\alpha{}^M\partial_M\Omega_\beta{}^c \ - \ i\frac{1}{4}\Omega_{\alpha b}\Omega^{\alpha c} \\
&+ \ \frac{1}{4}(\gamma^d)^{\alpha\beta}\epsilon_{abd}\Omega_\alpha{}^a\Omega_\beta{}^c \\
&- \ \frac{1}{4}(\gamma_b)^{\alpha\beta}\Omega_\alpha{}^a\Omega_\beta{}^d\epsilon_{ad}{}^c \ , \\
\Omega_\alpha{}^a \ =& \ (\gamma^a)_\alpha{}^\gamma E_{(\gamma}{}^M\partial_M E_{\beta)}{}^N E_N{}^\beta \ - \ (\gamma^a)^{\gamma\beta}E_\gamma{}^M\partial_M E_\beta{}^N E_{N\alpha} \ ,
\end{aligned} \tag{3.6}$$

are found. Thus the conventional constraint results in expressions for the spin-connection in terms of the superfield $E_\alpha{}^M$. In the $ISO(2,1|2)$ super-gauge theory, Ω and E are independent gauge superfields. This means that we must relax the conventional constraint. However, we will still impose the constraint on $\mathcal{F}_{\alpha b}$.

The superspace action for the $ISO(2,1|1)$ gauge theory is

$$S \; = \; \int d^3x d^2\theta \; \frac{1}{6}(\gamma^m)^{\mu\nu} Tr\big[i\Gamma_{(\mu}\mathcal{F}_{\nu)m} \; + \; i\Gamma_m \mathcal{F}_{\mu\nu} \; + \; \Gamma_\mu [\Gamma_m, \Gamma_\nu\}\big] \quad . \tag{3.7}$$

Constructed as a gauge theory, the action is invariant under the $ISO(2,1|2)$ super-gauge transformations

$$\begin{aligned}
\delta E_M{}^\alpha \; &= \; - \partial_M K^\alpha \; + \; i\frac{1}{2}(\gamma_a)_\beta{}^\alpha(\omega_M{}^a K^\beta \; - \; E_M{}^\beta L^a) \quad , \\
\delta E_M{}^c \; &= \; - \partial_M K^c \; + \; E_M{}^a L^b \epsilon_{ab}{}^c \; - \; \omega_M{}^a \rho^b \epsilon_{ab}{}^c \; + \; 2E_M{}^\alpha K^\beta (\gamma^c)_{\alpha\beta} \quad , \\
\delta \Omega_M{}^c \; &= \; - \partial_M L^c \; - \; \Omega_M{}^a L^b \epsilon_{ab}{}^c \quad .
\end{aligned} \tag{3.8}$$

The superfield parameter, K, of the gauge group is expanded over the generators as $K \equiv K^a P_a + L^a J_a + K^\alpha Q_\alpha$. As the gauge group includes bosonic and fermionic translations along with Lorentz transformations, these appear with independent parameters in eqn. (3.8). The bosonic component parameters of ref. [7] are given by $\rho^a = K^a|$ and $\tau^a = L^a|$. The spinor super-parameter contains the local supersymmetry parameter via $\epsilon^\alpha = K^\alpha|$. The W-Z gauge $\chi_\mu = \Gamma_\mu| \equiv 0$ corresponds to choosing the $\mathcal{O}(\theta)$ components of K^A and L so that the lowest components of $E_\mu{}^A$ and $\Omega_\mu{}^a$ are zero.

Three dimesnional N=1 supergravity is a SCS gauge theory on the coset space $ISO(2,1|2)/SO(2,1)$. So to recover the supersymmetric Einstein-Hilbert action, the spin-connection must no longer be independent. It must be given in terms of the dreibein and gravitino, as usual. As we saw in eqn. (3.6), the conventional constraint gives us exactly this. A solution for the Lorentz parameter in terms of the diffeomorphism parameter may be found by identifying $\delta_{DIFF} E_M{}^A \equiv \delta E_M{}^A$. This identification is as given in ref. [7].

Computing the component $ISO(2,1|2)$ action may be done in two ways. Equation (3.7) may be directly projected onto components before using the inner product of eqn. (3.3) or we may use the component results of section II. Using the latter method and eqn. (2.14), in particular, one finds

$$
\begin{aligned}
S_{ISO} = \int d^3x\{\epsilon^{mnp}[& 2e_m{}^a(\partial_{[n}\omega_{p]a} + \omega_n{}^c\omega_p{}^b\epsilon_{abc}) \\
& + i\frac{1}{4}\psi_{m\beta}\partial_n\psi_p{}^\beta - \frac{4}{3}(\gamma_a)_{\alpha\beta}\omega_m{}^a\psi_n{}^\alpha\psi_p{}^\beta \\
& - \frac{4}{3}v_m{}^a(\partial_{[n}\phi_{p]a}) - v_m{}^a\phi_n{}^c\phi_p{}^b\epsilon_{abc} - \phi_m{}^a e_n{}^b\phi_p{}^c\epsilon_{abc} \\
& + 2(\gamma_a)_{\alpha\beta}\phi_m{}^a\psi_n{}^\alpha\rho_p{}^\beta + \frac{1}{16}(\gamma_a)_\alpha{}^\gamma\phi_m{}^a\psi_{n\gamma}\psi_p{}^\alpha \\
& - i\frac{1}{6}\rho_{m\beta}\partial_n\rho_p{}^\beta - \frac{1}{16}(\gamma_a)_\alpha{}^\gamma\psi_{m\gamma}\omega_n{}^a\rho_p{}^\alpha]\} \ .
\end{aligned}
\tag{3.9}
$$

In addition to this, the following constraints must be imposed:

$$
\begin{aligned}
& (e_n{}^a\phi^{nb} + v_n{}^a\omega^{nb})\epsilon_{abc} - 2(\gamma_c)_{\alpha\beta}\psi_n{}^\alpha\rho^{n\beta} = 0 \ , \\
& (\gamma_a)_{\alpha\gamma}(\psi_n{}^\alpha\phi^{na} - \omega_n{}^a\rho^{n\alpha}) = 0 \\
& \omega_n{}^a\phi^{nb}\epsilon_{abc} = 0 \ .
\end{aligned}
\tag{3.10}
$$

These arise from the auxiliary B field's equation of motion (see eqn. (2.14)). A second set of constraints is given by the ψ equation: $\lambda_\mu = 0$. This only serves to define the $\mathcal{O}(\theta)$ components of the superfield. As $\lambda_\mu \equiv W_\mu|$, the field-strenght $\mathcal{F}_{\nu m}|$, vanishes. This gives

$$
\begin{aligned}
\nabla_\nu E_m{}^c| &= \partial_m e_\nu{}^c + ie_\nu{}^a\omega_m{}^b e_{ab}{}^c - ie_m{}^a\omega_\nu{}^b\epsilon_{ab}{}^c + i2(\gamma^c)_{\alpha\beta}e_\nu{}^\alpha\psi_m{}^\beta \ , \\
\nabla_\nu\Omega_m{}^c| &= \partial_m\omega_n{}^c + i\omega_\nu{}^a\omega_m{}^b e_{ab}{}^c \ , \\
\nabla_\nu E_m{}^\gamma| &= \partial_m e_\nu{}^\gamma + \frac{1}{2}(\gamma_a)_\beta{}^\gamma[\omega_\nu{}^a\psi_m{}^\beta - \omega_m{}^a e_\nu{}^\beta] \ .
\end{aligned}
\tag{3.11}
$$

These results were computed in the W-Z gauge. Consequently, the right-hand sides of the expressions in eqn. (3.11) are identically zero. The component fields are defined by

$$
\begin{aligned}
& e_M{}^a \equiv E_M{}^a| \ , \qquad \psi_M{}^\alpha \equiv E_M{}^\alpha| \ , \qquad \omega_M{}^a \equiv \Omega_M{}^a| \ , \\
& V_m \equiv v_m{}^a P_a + \phi_m{}^a J_a + \rho_m{}^\alpha Q_\alpha \ .
\end{aligned}
\tag{3.12}
$$

Although the procedure of varying the minimal action with respect to the spin-connection gives the correct expression for ω, one must in fact impose the latter relation as a constraint. This is the "torsion-free" constraint, $\mathcal{D}_{[m}e_{n]}{}^{a} = 0$. With this constraint, the partition function is defined as a integral over the vielbein, only. The same process holds in superfield theories. Here the constraints on the torsions and curvatures (or field-strenghts in this case), restrict the number of "fundamental" fields in the theory. Consequently, the measure of the partition function is defined only over a subset of the original superfields. For the ISO gauge theories, ω, e and ψ are all independent fields. So they must each be integrated over. Had we imposed the conventional constraint, only e and ψ would be fundamental [3] fields. Hence the constraint was removed. However, in doing so extra fields were introduced. These fields were not auxiliary and thus they showed up in the component action. Their role is to maintain the supersymmetry transformation laws when ω is not a functional of e and ψ.

IV. Supersymmetric D=3 Topological Quantum Field Theory

Topological quantum field theories (TQFT) have been constructed in four [1], three [10] and two [1] dimensions. The genesis of these theories is an almost all encompasing symmetry of the form $\delta A_a = \mathcal{A}_a$. For a manifold with boundary, this symmetry is imposed everywhere except at the boundary. In this section, an extension of such a symmetry to a supersymmetric version in 2+1 dimensions, will be given. Note that this symmetry does not include (super) Yang-Mills transformations. An important use of these theories

[3] In the langauge of superspace, there would have been one such superfield, $H_\alpha{}^a$, through which the remaining superfields are defined. The latter superfield is the prepotential of the D=3, N=1 superfield supergravity theory [15,17].

is in the construction of the cohomology of moduli spaces of various (super) Yang-Mills configurations.

In this section, we will only work with minimal super Yang-Mills theory. Then the fundamental superfield is Γ_α. So we would like to gauge fix the symmetry

$$\delta\Gamma_\alpha = \mathcal{G}_\alpha \quad . \tag{4.1}$$

The first curiosity is whether or not there exist Lagrangians which are invariant under the above topological transformations. There are currently two philosophies, the first is that $\mathcal{L}_0 = 0$ and the second is that $\mathcal{L}_0$ is a surface term. Notice that neither theory exists classically. They may only exist quantum mechanically. Topological surface terms are easily constructed as expresssions which involve a Bianchi Identity (BI). For example, the D=4 YM BI, $\epsilon^{abcd}\mathcal{D}_b F_{cd} = 0$, leads to the Pontryagin density, $Tr(F^\star F)$. Similarly, in D=3 we have the BI $\epsilon^{abc}\mathcal{D}_a F_{bc} = 0$; which leads to $Tr(F \wedge \mathcal{D}\phi)$ as a surface term. In two dimensions, there is the pullback of a symplectic form, J, on some target manifold [1,18]. Its form is that of the WZW Lagrangian for the non-linear sigma model of the string. Unfortunately, because one has to impose $dJ = 0$, it is not quite the latter as in the string's spectrum $dB = G \neq 0$, where G is the 3-form of D=10, N=1 supergravity.

Returning to the three dimensional supersymmetric theory, we find that there is a surface term for the vector superfield. It is

$$\mathcal{L}_0 \;=\; -Tr[W^\alpha \nabla_\alpha \Phi] \quad , \tag{4.2}$$

where Φ is a scalar superfield. The BI is $\nabla^\alpha W_\alpha = 0$ as is given in section II. Projecting this to components is easy:

$$S_0 \;=\; \int d^3x \; Tr[F \wedge \mathcal{D}\phi] \quad . \tag{4.3}$$

There are no supersymmetric completion terms. It is simple to see that this Lagrangian is invariant under $\delta A_a = \mathcal{A}_a$ and $\delta\phi = \mathcal{A}$, because of the BI.

Gauge fixing the symmetry in eqn. (4.1) is done in a BRST invariant fashion. Although a BRST superspace could be introduced here, it would only complicate the issue. However, for the $Sp(2)$ invariant BRST and anti-BRST gauge fixing however, such a construction would prove to be very useful. A gauge slice must be chosen. Among the wide class of slices, there is one which is of topological interest: ${}^{*}F = \mathcal{D}\phi$. This is the Bogomol'nyi equation [11]. This slice was used in ref. [10] to construct monople invariants. It is the dimensional reduction of the four-dimensional instanton solution. In gauge fixing the dual field-strength, we will literally be creating Lagrangians from nothing. What's more, the energy-momentum tensors will be the BRST (Q) transforms of something: $T_{ab} = [Q, \lambda_{ab}]$. This means that the Hamiltonians will also be a BRST transform: $\mathcal{H} = [Q, \lambda_{00}]$. These are all properties of TQFT's.

Introduce the BRST operator, $\hat{\delta}$, by $\delta = i\epsilon\hat{\delta}$, where ϵ is a real, constant, anti-commuting parameter. If we choose the gauge slice

$$W^{\alpha} \equiv \nabla^{\alpha}\Phi \ , \tag{4.4}$$

we see that this would imply the component projections:

$$\begin{gathered} W_{\alpha}| = \lambda_{\alpha} \ , \qquad \nabla_{\gamma}W_{\alpha}| = i\frac{1}{2}\epsilon^{abc}(\gamma_c)_{\alpha\gamma}F_{ab} \ , \qquad \nabla^2 W_{\alpha} = i(\gamma^a)_{\alpha}{}^{\beta}\mathcal{D}_a\lambda_{\beta} \ , \\ \Phi| \equiv \phi \ , \qquad \nabla_{\alpha}\Phi \equiv \beta_{\alpha} \ , \qquad \nabla^2\Phi| \equiv A \ , \\ \nabla_{\gamma}\nabla_{\alpha}\Phi| = i(\gamma^{a})_{\alpha\gamma}\mathcal{D}_a\phi + C_{\alpha\gamma}A \ , \end{gathered} \tag{4.5}$$

which give ${}^{*}F = \mathcal{D}\phi$. Equation (4.4) has the correct dimension as $[(\nabla^{\alpha}, \Phi, W^{\alpha})] = (\frac{1}{2}, 1, \frac{3}{2})$. Other component expressions resulting from it may be read off from eqn. (4.5). The spin-0 component field, A, is auxiliary. Equation (4.4) is then the supersymmetric analog of the Bogomol'nyi equation. So we write[4] , in space-time superspace,

$$\mathcal{L}^{(P)}_{(GF+FP)} = \hat{\delta}_P[\Sigma^{\alpha}(W_{\alpha} + \nabla_{\alpha}\Phi + \frac{1}{2}c_0\mathcal{B}_{\alpha})]$$

[4] Traces are implied in all Lagrangians to follow.

$$
\begin{aligned}
&= \mathcal{B}^{\alpha}(W_{\alpha} + \nabla_{\alpha}\Phi + \frac{1}{2}c_0\mathcal{B}_{\alpha}) \\
&\quad + \frac{1}{2}\Sigma^{\alpha}[i(\gamma^{a})_{\alpha}{}^{\beta}\nabla_{a}\Psi_{\beta} + \nabla^2\Psi_{\alpha}] \\
&\quad - \Sigma^{\alpha}\big[\nabla_{\alpha}\Psi + i[\Psi_{\alpha},\Phi\}\big] \ .
\end{aligned} \tag{4.6}
$$

Here Σ^{α} is the anti-ghost superfield, Ψ_{α} is the ghost superfield and $\mathcal{B}_{\alpha}$ is the auxiliary ghost superfield. The first two superfields are commuting spinors while $\mathcal{B}_{\alpha}$ is an anti-commuting spinor. The BRST transformation laws are

$$
\begin{aligned}
\hat{\delta}_P\Gamma_{\alpha} &= \Psi_{\alpha} \ , \qquad \hat{\delta}_P\Phi = \Psi \ , \\
\hat{\delta}_P\Gamma_{a} &= -i\frac{1}{2}(\gamma_{a})^{\alpha\beta}\nabla_{\alpha}\Psi_{\beta} \ , \\
\hat{\delta}_P\Psi_{\alpha} &= \hat{\delta}_P\Psi = 0 \ , \\
\hat{\delta}_P\Sigma^{\alpha} &= \mathcal{B}^{\alpha} \ , \\
\hat{\delta}_P\mathcal{B}^{\alpha} &= 0 \ .
\end{aligned} \tag{4.7}
$$

The operator $\hat{\delta}_P$ is nilpotent on all superfields including Γ_a. The label 'P' indicates "primary" as will soon become obvious.

Now go to the gauge $c_0 = 0$. Then there is the secondary ghost symmetry

$$
\begin{aligned}
\hat{\delta}_S\Psi_{\alpha} &= \nabla_{\alpha}X \ , \\
\hat{\delta}_S\mathcal{B}_{\alpha} &= -i[\Sigma_{\alpha}, X\} \ , \\
\hat{\delta}_S\Psi &= -i[\Phi, X\} \ ,
\end{aligned} \tag{4.8}
$$

where X is the superfield parameter of the secondary symmetry (S-symmetry). Once again there is a reversal of statistics so X is commuting. This ghost-for-ghosts structure is another feature of TQFT's. It is just the statement that not all of the original symmetry (4.1) was gauge fixed in eqn. (4.7). The dimensions of the superfields introduced thus far are $[(\Gamma_{\alpha}, \Gamma_{a}, \Phi, W_{\alpha}, \Psi_{\alpha}, \Psi_{a}, \Psi, \Sigma^{\alpha}, \mathcal{B}^{\alpha})] = (\frac{1}{2}, 1, 1, \frac{3}{2}, \frac{1}{2}, 1, 1, \frac{3}{2}, \frac{3}{2})$. The natural choice for the gauge fixing of this second symmetry is to demand that $\hat{\delta}_P\Gamma$ is orthogonal to the super

Yang-Mills transformations. Thus the condition

$$\nabla^2 \nabla^\alpha \Psi_\alpha \; + \; i[\Phi, \Psi\} \; = \; 0 \;\; , \tag{4.9}$$

is imposed. The left-hand-side is of dimension 2. When reduced to components, this gauge fixing condition yields

$$\mathcal{D}^a \psi_a \; + \; i[\phi, \psi\} \; - \; i2[\lambda^\alpha, \psi_\alpha\} \; = \; 0 \;\; , \tag{4.10}$$

as its lowest component. Here the component fields are defined as $\psi_A \equiv \Psi_A|$, $\phi \equiv \Phi|$, $\psi \equiv \Psi|$ and $\lambda_\alpha \equiv W_\alpha$. Following the four-dimensional work of ref. [4], we write

$$\mathcal{L}^{(S)}_{(GF+FP)} \; = \; \hat{\delta}_T[\frac{1}{2}\Lambda(\nabla^2\nabla^\alpha\Psi_\alpha \; + \; i[\Phi,\Psi\} \; + \; s[\nabla^2 N, X\}) \; - \; 4\Sigma^\alpha \mathcal{B}_\alpha] \;\; , \tag{4.11}$$

where $\hat{\delta}_T \equiv \hat{\delta}_P + \hat{\delta}_S$ and s is a real constant. The complete set of BRST transformations read:

$$\begin{aligned}
\hat{\delta}_T \Gamma_\alpha \; &= \Psi_\alpha \;\; , \qquad \hat{\delta}_T \Phi \; = \; \Psi \;\; , \\
\hat{\delta}_T \Gamma_a \; &= \; - i\frac{1}{2}(\gamma_a)^{\alpha\beta} \nabla_\alpha \Psi_\beta \; \equiv \; \Psi_a \;\; , \\
\hat{\delta}_T \Psi_\alpha \; &= \nabla_\alpha X \;\; , \qquad \hat{\delta}_T \Psi \; = \; -i[\Phi, X\} \;\; , \\
\hat{\delta}_T \Sigma^\alpha \; &= \mathcal{B}^\alpha \;\; , \qquad \hat{\delta}_T \mathcal{B}^\alpha \; = \; -i[\Sigma^\alpha, X\} \;\; , \\
\hat{\delta}_T \Lambda \; &= 2N \;\; , \qquad \hat{\delta}_T N \; = \; i\frac{1}{2}[X, \Lambda\} \;\; , \\
\hat{\delta}_T X \; &= 0 \;\; .
\end{aligned} \tag{4.12}$$

These have the property that the commutator $[\delta_T(\epsilon_1), \delta_T(\epsilon_2)]$ acting on any of the superfields (with the exception of X) is a gauge transformation with gauge parameter $2\epsilon_1\epsilon_2 X$. In this sense, $\hat{\delta}_T$ is not a traditional BRST operator which is nilpotent. The operators δ_P and δ_S do not commute.

Putting these last two sets of equations together with eqn. (4.6), gives the total gauge fixing plus Faddeev-Poppov ghost Lagrangian as $\mathcal{L}_T = \mathcal{L}^{(P)} + \mathcal{L}^{(S)}$:

$$
\begin{aligned}
\mathcal{L}_T = & \frac{1}{8}(W^2 + W^\alpha \nabla_\alpha \Phi - \Phi \nabla^2 \Phi) \\
& + \frac{1}{2}\Sigma^\alpha \left[i(\gamma^a)_\alpha{}^\beta \nabla_a \Psi_\beta + \nabla^2 \Psi_\alpha \right] - \Sigma^\alpha \nabla_\alpha \Psi + 2N \nabla^a \Psi_a + \Lambda \nabla^a \nabla_a X \\
& - i2N[W^\alpha, \Psi_\alpha\} + i\Lambda[W^\alpha, \nabla_\alpha X\} - i\Lambda[\Psi^a, \Psi_a\} - i\frac{1}{2}\Lambda[\nabla^2 \Psi^\alpha, \Psi_\alpha\} \\
& + iN[\Phi, \Psi\} + i\frac{1}{2}\Lambda[\Psi, \Psi\} + \frac{1}{2}[\Phi, X\}[\Lambda, \Phi\} + \frac{1}{2}(\gamma^a)^{\alpha\beta}\Lambda[\nabla_a \Psi_\beta, \Psi_\alpha\} \\
& + i4X[\Sigma^\alpha, \Sigma_\alpha\} - i\Sigma^\alpha[\Psi_\alpha, \Phi\} \\
& + s\big[X[N, \nabla^2 N\} + i\frac{1}{4}[X, \nabla^2 \Lambda\}[X, \Lambda\} + i\frac{1}{4}[\nabla^2 X, \Lambda\}[X, \Lambda\} \\
& - i\frac{1}{2}[\Psi^\gamma, \nabla_\gamma N\}[X, \Lambda\} + i\frac{1}{4}[\nabla^\gamma \Psi_\gamma, N\}[X, \Lambda\} + i\frac{1}{2}[\nabla^\gamma X, \nabla_\gamma \Lambda\}[X, \Lambda\}\big] \ .
\end{aligned}
\tag{4.13}
$$

Notice that the second generation component anti-ghosts appear as the highest components of the N and Λ superfields. On the other hand, the second generation component ghost is the lowest ($\theta = 0$) component of the X superfield. The dimensions and ghost numbers of all the superfields are listed in Table I.

Given the lagrangian (4.8), it is appropriate to ask what the observables are. As mentioned in section III, observables will be required to be gauge invariant and generally covariant (independent of the metric). As usual, they are defined as the path integral of the associated operator, $\mathcal{O}$. Such an operator must satisfy $[Q, \mathcal{O}\} = 0$ but $\mathcal{O} \neq [Q, \Delta\}$ for some operator Δ [1].

An operator which satisfies these BRST properties is X. But this is not YM-gauge invariant. However, the polynomials $Tr(X^{2n})$ ($n = 1, \ldots,$ rank of the group) are gauge invariant [1]. Furthermore, these polynomials are metric independent. The cohomology is constructed by first defining the invariant (for a rank 1 group, for example)

$$
\Omega_0 \equiv \frac{1}{2} Tr(X^2) \ . \tag{4.14}
$$

Next use the fact that $[Q, \Gamma_A\} = \Psi_A$ with $[Q, \Psi_A\} = \nabla_A X$, to find the sequence

$$
\begin{aligned}
0 &= [Q, \Omega_0\} \ , & & \\
d\Omega_0 &= [Q, \Omega_1\} \ , & \Omega_1 &\equiv Tr(X \wedge \Psi) \ , \\
d\Omega_1 &= [Q, \Omega_2\} \ , & \Omega_2 &\equiv Tr(\frac{1}{2}\Psi \wedge \Psi \ + \ X \wedge \mathcal{F}) \ , \\
d\Omega_2 &= [Q, \Omega_3\}, \ , & \Omega_3 &\equiv Tr(\Psi \wedge \mathcal{F}) \ , \\
d\Omega_3 &= [Q, \Omega_4\} \ , & \Omega_4 &\equiv Tr(\frac{1}{2}\mathcal{F} \wedge \mathcal{F}) \ , \\
d\Omega_4 &= 0 \ .
\end{aligned}
\tag{4.15}
$$

The Ω_q ($q = 0, \ldots, 4$) are super q-forms on the space-time supermanifold. All the expressions are in terms of superfields with d being the exterior super-derivative [15]. The super 1-form, Ψ_A, which appears in eqn. (4.15) as Ψ, should not be confused with the super 0-form, $[Q, \Phi\} = \Psi$, which does not appear in eqn. (4.15). Whereas a 4-form does not exist in three dimensions, a super 4-form (such a Ω_4) does exists in the "five-dimensional" superspace we are working in. For example, $\Omega_{\alpha bcd} = -\frac{1}{2}(\gamma_{[b|})_\alpha{}^\epsilon \epsilon_{|cd]e}(\gamma^e)^{\gamma\delta} Tr(W_\epsilon \nabla_\gamma W_\delta)$, is one element in Ω_4. So the structure of the supersymmetric sequence differs from that of the non-supersymmetric sequence in that the 3-form there is closed. Eqn. (4.15) is identical, in form, to the four dimensional result [1]. In fact, the ghost number of Ω_q is $4 - q$.

The D=3, N=1 superspace analogs of the Donaldson invariants [2] are constructed [1] form the integrals over the (super)cycles on the (super)manifold: $I(\mathcal{C}) = \int_{\mathcal{C}} \Omega_q$, where $\mathcal{C}$ is a super q-cycle. As shown in ref. [1], they are BRST invariant since $[Q, \Omega_q\} = d\Omega_{q-1}$. Hence up to a BRST commutator, these integrals depend only on the homology class of $\mathcal{C}$. The supersymmetric Donaldson invariants are the correlations functions of these integrals. More importantly, the correlation functions are independent of the topological structure on the super-manifold, since $\langle [Q, \mathcal{O}\} \rangle = 0$ for any $\mathcal{O}$. Thus they are topological invariants. An explicit computation of these objects will be left for the future. Note that supersymmetry is manifest because the q-forms were all computed in superspace.

Earlier in this section, it was mentioned that the Hamiltonians of TQFT's are the BRST transforms of some operator: $\mathcal{H} = [Q, \lambda_{00}\}$. This means that the vacuum energy is zero: $\langle \mathcal{H} \rangle = 0$. Consequently, as long as the Lagrangian is the BRST transform of something, supersymmetry is unbroken. Furthermore, the supersymmetry algebra closes onto the BRST commutator of something: $[Q_\alpha, Q_\beta\} = 2(\gamma^a)_{\alpha\beta}[Q, \lambda_{0a}\}$, where the Q_α's are the supercharges.

It may be useful to study the theory given by eqn. (4.13) in the context of the space of solutions to the Bogomol'nyi equation. Much work has been done on the monopole moduli space. An action for the geodesic motion in this space has been given in ref. [20]. The metric (kinetic part) is the integral of the square of the "electric" field. The potential is the integral of the kinetic terms of the gauge and Higgs fields. This would be given by

$$U = \frac{1}{8} \int d^3x d^2\theta \; Tr(W^2 - \Phi \nabla^2 \Phi) \quad , \tag{4.16}$$

in the context of eqn. (4.13). It has been shown [21] that the monopole solutions may be constructed from algebraic curves (so called spectral curves) defined by a certain polynomial equation whose degree is given by the monopole's charge. Finding the number of Ψ_α, Ψ, N and Σ^α zero-modes may provide a convenient way of obtaining these results.

V. Conclusion

Non-minimal D=3, N=1 super Yang-mills gauge theory was developed and used to construct the super-Poincaré gauge theory. The additional supermultiplet allows for a theory in which the spin-connection, dreibein and gravitino are functionally independent. The equations of motion for the auxiliary fields which appear in these super-multiplets place constraints on the new set of fields.

The topological super Yang-Mills field theory of three dimensions was constructed in a BRST invariant manner. Its topological invariants are polynomials in the secondary ghost superfield. The associated moduli space is that of monopole configurations modulo super Yang-Mills transformations. The generally covariant observable of the $U(1)$ Chern-Simons theory is the Wilson Loop. Supersymmetrization does not alter its value. This is due to the fact that in the supersymmetric theory, the fermions are either pure gauge degrees of freedom or are auxiliary. In a generally covariant theory, there can be no supercharge.

The results obtained here open interesting avenues for further research. These include the possibility of using the gauge theory of the supersymmetry algebra to study supersymmetry breaking. Also, if the gauge group of the topological super Yang-Mills field theory is chosen to be $ISO(2,1|2)$, the moduli space of gauge superfields modulo local supersymmetry transformations may be investigated. It is straightforward to extend section III to a gauge theory of the superconformal algebra. A topological sigma model in three dimensions may be constructed in terms of the pull-back of a closed 3-form of some target (super)manifold. This may be of some use in (super)membrane theory.

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Appendix: Observables, Phase Factors and Super-Curves

In topological quantum field theories, the observables must be gauge invariant and generally covariant. For Chern-Simons gauge theory, a class of observables is given by the Wilson Loops [12]. One would like to compute the expectation value of these loops.

The expectation value of the Wilson Loop gives the Polyakov Phase Factor [14]. This phase plays an important role is the theory of fractional spin fields in 2+1 dimensions. In particular, the bosons of the CP^1 sigma model were shown to transmute into fermions as a result of this phase. There the group is $U(1)$ and the manifold is S^3.

Studied in either context, the *vev* of the Wilson Loop is then an important construct in the Chern-Simons gauge theory. However, an ill-defined integral appears and it must be carefully regularized. Poyakov did this by regularizing the delta function which appears as the integrand of the phase. For generally covariant theories, one would like to regularize in a manner which maintains general covariance. Unfortunately, the supersymmetric Chern-Simons action is not generally covariant outside of the W-Z gauge. This is directly because of the manner in which the fermions enter the lagrangian (2.7). So whether or not the superspace version of the phase factor needs to be regularized is irrelevant from the point of view of general covariance. In either case, one looses manifest general covariance. Nevetheless, the superspace theory is interesting in its own right since it allows for knots in superspace. Let us look at the superspace integrand for the self-linking number. First, a quick (albeit incomplete) review of the self-linking integrand is in order.

For purposes of the linkage with fraction spin, let us introduce a coefficient, $\frac{k}{8\pi}$, in front of the $U(1)$ super Chern-Simons action (2.6). It was shown in ref. [14] that the *vev* of the Wilson Loop,[2] $\langle W \rangle$, is

$$\langle W \rangle \;=\; \langle \exp\,[i \oint_C dx^a A_a] \rangle \;=\; \exp\,[-\frac{1}{2} \oint_C dx^a \oint_C dy^b \; \langle A_a(x) A_b(y) \rangle] \quad . \qquad (A.1)$$

[2] By the Wilson Loop, we will mean the operator, not its *vev*.

It is then simple to show that, for the Chern-Simons theory, this is

$$\begin{aligned}\langle W\rangle &= \exp\,[i\frac{\pi}{k}\Phi(C)] \quad , \\ \Phi(C) &= \frac{1}{4\pi}\oint_C dx^a \oint_C dy^b \epsilon_{abc}\frac{(x - y)^c}{|x - y|^3} \quad .\end{aligned} \tag{A.2}$$

The phase factor Φ is known in the mathematics literature as the self-linking number (SL) obtained when the Gauss linking number, $\Phi(C_i, C_j)$, is calculated for identified paths, *i.e.* $\Phi(C) = \Phi(C, C)$. It is known that SL is an integer [13]. For the physicist, this is not so obvious as the integrand of $\Phi(C)$ is ill-defined near $x = y$.

Witten [12] used a mechanism, called framing, for regularizing $\Phi(C)$ which is essentially point-splitting. A new curve, let us call it C', is defined by displacing C in the direction of a normal along C. Consequently, one has two closed curves which are everywhere infinitesimally separated from each other. The original curve, C, becomes a thin ribbon. SL is then calculated in the limit where the curves approach each other. In fact, a detailed discussion of this framing or ribboning of the $U(1)$ Wilson Loop was previously given, in the physics literature, in ref. [19a] (see ref. [19b] also).

Let us now turn to the supersymmetric theory. We will work with the minimal formulation of sub-section II.1. For our purposes, a super-curve, $\mathcal{C}$, will be a mapping from a connected, open interval of R to a super-manifold. It will be parameterized as $z^A(\tau) \equiv (x^a(\tau), \theta^\alpha(\tau))$. The supersymmetric Wilson Loop then has as its exponent: $i\oint_{\mathcal{C}} dz^A\Gamma_A$. Note that this is generally super-covariant. The $U(1)$ super Chern-Simons action is

$$S_{SCS} = \frac{k}{8\pi}\int d^3x d^2\theta\; \Gamma_\alpha D^\beta D^\alpha \Gamma_\beta \quad . \tag{A.3}$$

By analogy with eqn. (A.2), we can then write the Wilson Loop's *vev* as

$$\langle W\rangle = \exp\,[-\frac{1}{2}\oint_{\mathcal{C}} dz^A \oint_{\mathcal{C}} dw^B \;\langle\Gamma_A(z)\Gamma_B(w)\rangle] \quad . \tag{A.4}$$

Given eqns. (A.3) and (2.2), it is straightforward to compute the super-propagators:

$$\begin{aligned}\langle\Gamma_\alpha(x,\theta)\Gamma_\beta(x',\theta')\rangle &= -\frac{4\pi}{k}\frac{D_\alpha D_\beta}{\Box}\delta^2(\theta-\theta')\delta^3(x-x') \ , \\ \langle\Gamma_\alpha(x,\theta)\Gamma_a(x',\theta')\rangle &= i\frac{4\pi}{k}\epsilon_{abc}(\gamma^b)_\alpha{}^\beta\frac{\partial^c D_\beta}{\Box}\delta^2(\theta-\theta')\delta^3(x-x') \ , \\ \langle\Gamma_a(x,\theta)\Gamma_b(x',\theta')\rangle &= \frac{4\pi}{k}\epsilon_{abc}\frac{\partial^c}{\Box}D^2\delta^2(\theta-\theta')\delta^3(x-x') \ .\end{aligned} \tag{A.5}$$

Let us look at the vector-vector super-propagator. Then we have integrals of the form

$$-i\frac{1}{2k}\oint_C dx^a\oint_C dy^b\epsilon_{abc}\frac{(x^c - y^c - i\theta_y^\alpha(\gamma^c)_{\alpha\beta}\theta_x^\beta)}{|x - y - i\theta_y\theta_x|^3} \ . \tag{A.6}$$

Since the integrals in eqn. (A.2) are over the same curves, there is a potential singularity at a crossing of the curve. Now, in eqn. (A.6) such a problem does not only occur at crossings $((x,\theta_x)=(y,\theta_y))$, but may also occur when $(x,\theta_x)\neq(y,\theta_y)$. There are also all the other points on the cycle where $x^a - y^a = i\theta_y^\alpha(\gamma^a)_{\alpha\beta}\theta^\beta$. These may be away from crossings. So it would appear that the supersymmetric case is worse than the non-supersymmetric theory.

However, there are no fermions in the spectrum of the pure super Chern-Simons theory. Consequently, $\langle W\rangle$ is identical to that of the pure bosonic theory. This means that the θ's contribute nothing to the integral. To see this, recall that supersymmetric invariance translates into supertranslation invariance, in the language of superspace. Accordingly, simultaneous shifts in the coordinates of the form $x^a \rightarrow x^{a\prime} = x^a - i\epsilon_x^\alpha(\gamma^a)_{\alpha\beta}\theta^\beta$ and $\theta_x^\alpha \rightarrow \theta_x^{\alpha\prime} = \theta_x^\alpha + \epsilon_x^\alpha$, leave the *vev*, $\langle W\rangle$, invariant. Choose $\epsilon_x \propto \theta_x$ and $\epsilon_y \propto \theta_y$; then it may be shown that the coefficients may be fixed so that eqn. (A.4) reduces to eqn. (A.2). So supersymmetry does not improve the integrand.

TABLE I

Superfield	Dimension	Ghost Number	Grassmann Parity
Γ_α	$\frac{1}{2}$	0	odd
Γ_a	1	0	even
Φ	1	0	even
W_α	$\frac{3}{2}$	0	odd
Ψ_α	$\frac{1}{2}$	1	even
Ψ_a	1	1	odd
Ψ	1	1	odd
$\Sigma^{\dot\alpha}$	$\frac{3}{2}$	-1	even
$\mathcal{B}^\alpha$	$\frac{3}{2}$	0	odd
X	0	2	even
Λ	1	-2	even
N	1	-1	even

Dimensions, Ghost Numbers and Grassmann Parities of the superfields in the topological super Yang-Mills theory of section IV.

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