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USING REPARAMETRIZATION INVARIANCE TO DEFINE VACUUM INFINITIES IN STRING PATH INTEGRALS*

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ABSTRACT

Path integrals for interacting world sheet sigma models play a key role in string theory. For open strings, the relevant path integral is one-dimensional and has direct physical interpretation as a source term for closed string fields. This means that the vacuum divergences (Möbius infinities) of the path integral must be renormalized correctly. In this paper we show that reparametrization invariance Ward identities, apart from specifying the equations of motion of spacetime background gauge fields, also serve to fix the renormalization scheme of the vacuum divergences.

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1. Introduction

In the Polyakov approach to string physics, open strings are associated with worldsheet boundaries and the effects of open string spacetime backgrounds are associated with boundary interactions, in the form of Wilson lines. In this connection, it is interesting to study the operator equivalent of inserting a single boundary, along with its Wilson line, into a closed string worldsheet. This object, called the boundary state and denoted by $|B\rangle$, summarizes the lowest-order effect of background open string matter on purely closed string physics (gravity, in particular). It can be thought of as a stringy generalization of the matter spacetime energy-momentum tensor.

It has been shown that the boundary state is equivalent to the vacuum amplitude of a one-dimensional field theory for D scalar fields (where D is the dimension of spacetime) [1]. This theory has an unusual dimension-one kinetic term, a dimension-one interaction term given by the Wilson line built out of the background gauge field and, finally, a set of linear source terms coupling the boundary fields to closed string creation operators. The source terms convert the vacuum amplitude of the one-dimensional field theory into a state in the closed string Hilbert space and the state so constructed is the boundary state [2,3]. For a constant, abelian background gauge field the one-dimensional action is quadratic and the path integral can be explicitly evaluated. For a general background, there are non-trivial interactions and the path integral defines a divergent but renormalizable perturbation theory.

In order to evaluate the boundary state (stringy energy-momentum tensor) associated with a given gauge field, it is necessary to specify a renormalization scheme for *all* the path integral divergences. These are of two kinds: logarithmic coupling constant divergences and linear vacuum divergences (referred to in other string contexts as Möbius infinities). It has recently been pointed out that both sorts of divergence can be absorbed by local Lagrangian counterterms [4] and, in principle, dealt with by standard renormalization methods. Several low-loop order calculations of the background field dependence of the open string path integral now exist [5]. A crucial feature of any renormalization scheme is a set of conditions to fix the finite parts of counterterms. In standard field theory applications, one is indifferent to vacuum divergences since they cancel out of S-matrix elements. Here, however, vacuum graphs provide a background-field-dependent normalization to the path integral which must be correctly defined if the path integral is to be used as a stringy energy-momentum tensor. An unconventional feature of this problem is therefore that we need renormalization conditions for vacuum divergences as well as coupling constant divergences. It seems to us that this problem has not been dealt with in a systematic way in the above-mentioned treatments of string path integral renormalization and it is the goal of this paper to derive, from basic string invariance principles, correct renormalization conditions for all the path integral divergences.

In standard field theory applications, renormalization conditions usually come from Ward identities for some underlying symmetry. In the end, that will turn out to be true here as well. Since local scale invariance (conformal invariance) picks out the two-dimensional sigma models compatible with string theory, it is plausible that we should require one-dimensional local scale invariance (reparametrization invariance) of the open string path integral. (The notion that reparametrization invariance is the fundamental dynamical principle of open string physics has been explored by Kleppe et al. [6], but the connection of their work to what we shall be doing here is not clear to us.) At the lowest level, this is achieved by choosing the coupling constants (background gauge fields) so that the theory sits at a renormalization group fixed point (zero of the beta functions). While this condition probably serves to specify the coupling constant renormalization conditions (and provides critical physical information in the form of a stringy generalization of Maxwell's equations), it doesn't say anything about the vacuum infinities and is not quite enough for our purposes. As mentioned earlier, the path integral is most properly thought of as generating a state $|B\rangle$ in the closed string Hilbert space which defines the coupling of the open string background to closed string physics. It can be demonstrated [3] that, for consistent coupling, $|B\rangle$ must be reparametrization invariant in the closed string sense. That is, it must be annihilated by the closed string Virasoro generators which correspond to boundary reparametrizations:

$$(L_n - \tilde{L}_{-n})\,|B\rangle = 0\,, \quad -\infty \leq n \leq \infty \tag{1.1}$$

Although the details are a bit subtle, this version of reparametrization invariance turns out to be just what we need: besides implying that the coupling constants sit at a zero of the beta functions, it provides a precise specification for the vacuum subtractions and gives precise meaning to the path integral. The bulk of this paper is devoted to working out the implications of the unconventional set of Ward identities implied by (1.1).

With luck and cleverness, we might be able to use this framework to study some more fundamental problems in string theory. In the two-dimensional conformal field theory approach one would like to classify all such field theories, to find theories which correspond to soliton solutions (topological sectors of string field theory?) and to learn how to sum over field theories in order to construct the string field theory path integral (or at least to quantize the collective coordinates of string solitons). These desires are currently frustrated by our lack of a sufficiently comprehensive understanding of conformal field theory. All of the above questions can be asked in the open string context and can be answered if we can characterize one-dimensional reparametrization-invariant field theory completely enough. Since one-dimensional field theory is a fairly simple system (closely related to multivariable quantum mechanics), one might hope to make serious progress along these lines. Conversely, one might hope to adapt the techniques described in this paper to the equally important and unsolved problem of fully defining the closed string path integral.

In this paper, we study the above-mentioned issues in the context of bosonic string theory. The extension to superstrings will be presented elsewhere. In Section 2 we introduce the path integral representation of the boundary state in a general background gauge field. In Section 3 we derive the Ward identity which follows from reparametrization invariance of the boundary state. In Section 4 we implement the Ward identity on the perturbation expansion of the one-dimensional field theory. In Section 5 we derive explicit one-loop order results and compare them to previous sigma model calculations [7]. Section 6 contains discussion and suggestions for future work.

2. Boundary States

The lowest order open string loop correction to a closed string process comes from adding a boundary to the world-sheet. The moduli of such loop amplitudes are taken into account by attaching the boundary to the old worldsheet through a cylinder, whose length is to be integrated over. In operator language a *closed* string state, $|B\rangle$, is created out of the vacuum at the boundary and the cylinder corresponds to a closed string propagator connecting the boundary state to a tree level closed string process [2, 8]. The form of the boundary state is determined by the open string boundary conditions and, in particular, may reflect the presence of a background gauge field in spacetime. The boundary state can thus be viewed as an open string source for closed string fields.

Let us first consider a free string in flat, empty spacetime. The boundary condition is simply

$$\partial_n X^\mu(\sigma,\tau)=0$$

where ∂_n denotes the normal derivative at the world-sheet boundary. We take the boundary to be at the end of a cylinder, at fixed world-sheet time τ, and we can choose $\tau=0$ for convenience. The mode expansion for a closed string is

$$X^\mu(\sigma,\tau)=q^\mu-2ip^\mu\tau+i\sum_{m\neq 0}\frac{1}{m}\left[\alpha_m^\mu e^{-m\tau-im\sigma}+\tilde{\alpha}_m^\mu e^{-m\tau+im\sigma}\right] \tag{2.1}$$

(Our conventions are detailed in the Appendix.) In terms of modes the above boundary condition reads

$$\begin{aligned} p^\mu &= 0\,, \\ \alpha^\mu_{-m} + \tilde{\alpha}^\mu_m &= 0\,. \end{aligned} \tag{2.2}$$

Upon quantization, the mode coefficients become operators satisfying the commutation relations

$$\begin{aligned} [\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] &= m\, \delta_{m+n,0}\, \delta^{\mu\nu} \quad , \\ [\alpha^\mu_m, \tilde{\alpha}^\nu_n] &= 0\,. \end{aligned} \tag{2.3}$$

As usual we take the α_n and $\tilde{\alpha}_n$ with $n > 0$ to be annihilation operators and those with $n < 0$ to be creation operators. The free boundary state must be annihilated by the combination of left- and right-moving creation and annihilation operators in (2.2). It is easy to see that the desired state is [2]

$$|B\rangle_{free} = \exp\{-\sum_{m=1}^{\infty} \frac{1}{m} \alpha_{-m} \cdot \tilde{\alpha}_{-m}\}\, |0\rangle \tag{2.4}$$

where $|0\rangle$ is the $SL(2,C)$ invariant vacuum of the closed string. If there is a gauge field in spacetime the boundary conditions are modified. In general they imply non-linear conditions for the closed string modes and the boundary state will no longer be so simple to obtain.

In [3] it is shown how the boundary state in an arbitrary spacetime gauge field is formally given in terms of a path integral for a certain one-dimensional field theory. The result, obtained by viewing the string as an infinite collection of oscillators and using some elementary properties of simple harmonic oscillator quantum mechanics, is

$$|B\rangle = \exp\Big(\sum_{m=1}^{\infty} \frac{1}{m} \alpha_{-m} \cdot \tilde{\alpha}_{-m}\Big) \int [\widehat{D\phi}]\, \mathrm{tr}\, P\mathrm{exp}\big[-S_0 - S_A - S_{ls}\big]\, |0\rangle\,, \tag{2.5}$$

where

$$
\begin{aligned}
S_0[\phi] &= -\frac{i}{4\pi}\oint ds\,\phi_+(s)\cdot\frac{d\phi_-(s)}{ds} \\
S_A[\phi] &= \frac{i}{4\pi}\oint ds\,A_\mu(\phi(s))\,\frac{d\phi^\mu(s)}{ds} \\
S_{ls}[\phi] &= \frac{1}{2\pi}\oint ds\,\alpha(s)\cdot\frac{d\phi(s)}{ds}\,.
\end{aligned}
\tag{2.6}
$$

The $\phi^\mu(s)$ are D scalar fields defined on S_1, parametrized by $s \in [0, 2\pi]$. $A_\mu(\phi)$ is the spacetime gauge potential and the trace and path ordering instructions are called for when dealing with non-abelian backgrounds. The linear source is a sum of closed string creation operators,

$$
\alpha^\mu(s) = \sum_{m=1}^{\infty}\frac{1}{m}\left(\tilde{\alpha}^\mu_{-m}\,e^{-ims} + \alpha^\mu_{-m}\,e^{ims}\right),
$$

so that the whole path integral can be regarded as a state in the closed string Hilbert space. A peculiar feature of the kinetic term is that it is written in terms of a decomposition of $\phi^\mu(s)$ into its positive-, negative- and zero-frequency Fourier modes:

$$
\phi^\mu(s) = \phi^\mu_0 + \phi^\mu_+(s) + \phi^\mu_-(s)\,,
$$

with

$$
\phi^\mu_+(s) = \sum_{m=1}^{\infty}\phi^\mu_m\,e^{-ims}, \qquad \phi^\mu_-(s) = \sum_{m=1}^{\infty}\phi^\mu_{-m}\,e^{ims}.
$$

The kinetic and interaction terms are linear in derivatives which has the consequence that the perturbation expansion for this path integral will turn out to be divergent (but power-counting renormalizable) rather than finite.

The kinetic term, S_0, is very peculiar indeed. For compactness of notation we have written it in a superficially local form, but this has required a breakup of ϕ^μ into positive and negative frequency parts with respect to a particular choice of parameter s on the worldsheet boundary. If we write S_0 in terms of the undecomposed field, we obtain instead

$$S_0[\phi] = \frac{1}{8\pi} \int_0^{2\pi} ds \int_{-\infty}^{\infty} ds' \, \frac{\left(\phi(s) - \phi(s')\right)^2}{(s - s')^2}.$$

This is both non-local and non-reparametrization invariant, but still well enough defined for a perturbation expansion of the path integral to exist. In view of the fundamental role of reparametrization invariance in this problem, and the fact that the interaction Lagrangian is manifestly reparametrization invariant, it is a bit surprising that the kinetic term should *not* be invariant to this symmetry! We will eventually see that this resolves itself in a natural way.

The zero mode, ϕ_0^μ, requires special treatment. It can be interpreted as the center of mass position of the closed string which is created at the boundary. As shown in [3], an important role of the boundary state is to provide source terms for the equations of motion of massless closed string fields (for example a gauge field energy momentum tensor term in the generalized Einstein equation). Since those equations are local in spacetime, it makes sense to regard the boundary state as a function of ϕ_0^μ. Indeed, $|B\rangle$ is always to be regarded as a state in the oscillator Fock space with projections onto individual Fock space states which are functions of the c-number zero mode coordinate ϕ_0^μ. Accordingly, the zero mode is *not* integrated over in the path integral and this is indicated by the hat over the measure $D\phi$ in (2.5). It does not appear in the kinetic term or the linear source term but in general the spacetime gauge potential, $A_\mu(\phi)$, which enters in the Wilson line interaction, is a function of ϕ_0^μ. The result of the path integral is therefore a functional of the closed string creation operators and the zero mode. When we study the response of this path integral to reparametrizations of the boundary coordinate s, this special treatment of the zero mode, as well as the fact that the kinetic term refers to a specific parametrization, will introduce unpleasant but, as far as we can see, unavoidable complications.

In the presence of a spacetime constant abelian gauge field strength, $F_{\mu\nu}$, the path integral is Gaussian and easily shown to take the value [3,1]

$$|B,F\rangle = \sqrt{\det(1+F)}\,\exp\Big(-\sum_{m=1}^{\infty}\frac{1}{m}\alpha^{\mu}_{-m}\big(\frac{1-F}{1+F}\big)_{\mu\nu}\tilde{\alpha}^{\nu}_{-m}\Big)\quad|0\rangle\,. \qquad (2.7)$$

(For an arbitrary gauge field, the boundary field theory is not free, and the path integral must be evaluated in a perturbation expansion.) That the dependence of this state on the oscillators is correct is verified by the fact that the state is annihilated by the appropriate quantized boundary conditions and that it reduces to the free boundary state when $F_{\mu\nu}=0$. The specific F-dependence of the determinant factor is also known, from a variety of consistency conditions, to be correct [9]. An important subtlety is hidden here. The normalization constant is actually the product of identical factors for each of the infinite number of string oscillators and is, strictly speaking, infinite. The correct finite result is obtained by an astute use of zeta-function regulation, while a more general view of the renormalization problem might lead one to conclude that the F-dependence of this factor is arbitrary! The resolution of this puzzle is that in order for the path integral to describe string theory, it must be renormalized so as to maintain reparametrization invariance. In the rest of this paper, we will show that implementation of certain reparametrization invariance Ward identities completely and correctly defines the renormalized path integral even for a general background gauge field.

3. Reparametrization Invariance

The free boundary state (2.4) is annihilated, for all n, by the combination $D_n = L_n - \tilde{L}_{-n}$ of closed string Virasoro generators. Since the D_n satisfy the Diff(S_1) algebra (with no central extension), this is evidently the condition that $|B\rangle_{free}$ be invariant to reparametrizations of the S_1 boundary of the worldsheet. In fact this is a completely general condition. It is shown in [3] that the leading effect of inserting an open string boundary into a closed string worldsheet is to cause a shift, $|\psi\rangle$, in the closed string vacuum satisfying

$$(Q + \tilde{Q})\,|\psi\rangle = |B\rangle\,. \tag{3.1}$$

Since the closed string BRST charge, $Q + \tilde{Q}$, is nilpotent this equation implies the consistency condition

$$(Q + \tilde{Q})\,|B\rangle = 0. \tag{3.2}$$

Upon stripping off the (trivial) ghost part of the boundary state [2,8], the consistency condition for the matter part of the boundary state is easily seen to be the reparametrization invariance condition

$$(L_n - \tilde{L}_{-n})\,|B\rangle = 0. \tag{3.3}$$

We take this as the fundamental string consistency condition on the boundary state. There is, of course, the question of which Virasoro generators to use in this equation. For the purposes of this paper, we take them to be the Virasoro generators of free closed string theory (i.e. the open strings are assumed to propagate in flat, empty spacetime). The spirit of this calculation is that we are looking for a solution of the open string matter equations in a given fixed gravitational background (implicitly specified by the choice of the L_n) and do not concern ourselves with the 'back-reaction' of the open string matter on the metric. There are obvious improvements one might make to this picture, but it is internally consistent as far as it goes. $|B\rangle$ is in general a complicated functional of the closed string creation operators and the zero mode, ϕ_0^μ, so (3.3) is a quite non-trivial condition on possible configurations of the spacetime gauge field. As explained in [3] the projection of (3.1) onto zero mass levels results in equations of motion for the dilaton and graviton with sources provided by open string condensates. Thus it is apparent that (3.1) should be regarded as a stringy generalization of Einstein's equations, with $|B\rangle$ as the energy-momentum tensor and that (3.2) should be regarded as the stringy generalization of the energy-momentum conservation law. This remark shows that the energy of an open string soliton, should we be able to construct one, can be read off from the associated boundary state thus giving us another reason for wanting to properly normalize $|B\rangle$.

Now we wish to recast (3.3) as a set of Ward identities for the correlation functions of the one-dimensional field theory which can in turn be used to constrain the renormalization procedure. Although (3.3) should be equivalent to invariance of the one-dimensional field theory under reparametrizations of S_1, on which the field theory lives, closer examination shows that the situation is not so simple. The infinitesimal variation of the field $\phi^\mu(s)$ under reparametrizations $s \to s + f(s)$ is

$$\delta_f \phi^\mu(s) = f(s)\,\frac{d\phi^\mu(s)}{ds} \tag{3.4}$$

(f is regarded as infinitesimal) and, although the interaction term in (2.6) is invariant under this transformation, the kinetic term is not! Specifically,

$$\begin{aligned}\delta_n S_0[\phi] &= \delta_n\Big(\frac{1}{2}\sum_{m=1}^{\infty} m\,\phi_m\cdot\phi_{-m}\Big)\\ &= \frac{1}{2}\sum_{m=1}^{n-1} m(n-m)\,\phi_m\cdot\phi_{n-m}\,.\end{aligned} \tag{3.5}$$

where we denote by δ_n the variation generated by D_n (corresponding to an infinitesimal reparametrization by $f(s) = ie^{ins}$). The failure of reparametrization invariance for the kinetic term stems from the fact that in writing it down we chose a specific parametrization of the S_1 in order to separate positive and negative Fourier modes. The Ward identities are significantly complicated by the fact that the underlying symmetry is *broken* reparametrization invariance.

The derivation of the Ward identities proceeds as follows. The path integral with the linear source term can be viewed as a generating functional for Feynman diagrams where the Wilson line supplies interaction vertices (details of the construction will be given shortly) and external lines terminate in α_{-n} and $\tilde{\alpha}_{-n}$ sources. In the remainder of this discussion the spacetime gauge field is taken to be abelian, or at any rate to be in an abelian subgroup of the full gauge group so that path ordering is not required in the path integral and the gauge group trace may be suppressed. The one-dimensional path integral

$$Z[\alpha, \tilde{\alpha}, \phi_0] = \int \widehat{D\phi}\ \exp\{-S[\phi] - i\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\phi_n + \tilde{\alpha}_{-n}\cdot\phi_{-n})\} \tag{3.6}$$

(where the one-dimensional action includes the kinetic and Wilson line terms, the linear source term has been expressed in terms of modes and the functional integration does not include the zero mode) can be regarded as the generating functional of all diagrams, both connected and disconnected. The generating functional, W, of connected diagrams is defined in the usual way by

$$Z[\alpha, \tilde{\alpha}, \phi_0] = e^{-W[\alpha,\tilde{\alpha},\phi_0]} \tag{3.7}$$

and a generating functional, Γ, of one-particle-irreducible diagrams (or effective action) can be obtained by Legendre transformation of W:

$$\begin{aligned}
\Gamma[\phi_{cl}] &= W[\alpha, \tilde{\alpha}, \phi_0] - i\sum_{m=1}^{\infty}(\alpha_{-m}\cdot\phi^{cl}_m + \tilde{\alpha}_{-m}\cdot\phi^{cl}_{-m}) \\
\phi^{cl}_m &= -i\frac{\partial W}{\partial\alpha_{-m}} \qquad \phi^{cl}_{-m} = -i\frac{\partial W}{\partial\tilde{\alpha}_{-m}} \\
\alpha_{-m} &= \ \ i\frac{\partial\Gamma}{\partial\phi^{cl}_m} \qquad \tilde{\alpha}_{-m} = \ \ i\frac{\partial\Gamma}{\partial\phi^{cl}_{-m}} \\
\phi^{cl}_0 &= \phi_0 \quad .
\end{aligned} \tag{3.8}$$

The boundary state itself, (2.5), has a simple expression in terms of the connected generating functional:

$$|B\rangle = \exp\Big(\sum_{m=1}^{\infty} \frac{1}{m}\alpha_{-m} \cdot \tilde{\alpha}_{-m} - W[\alpha, \tilde{\alpha}, \phi_0]\Big)\, |0\rangle\,. \tag{3.9}$$

Note that the boundary state is not quite identical to the sum of vacuum diagrams of the one-dimensional field theory. As is explained in detail in [3], the two objects differ by a factor which, although a simple Gaussian, plays an essential role in getting the right Ward identities.

Experience shows that the most convenient way to impose symmetry conditions such as (3.3) on a renormalization scheme is to convert the symmetry to a set of Ward identities for the one-particle-irreducible generating functional. To that end, we express the Virasoro generators in terms of mode operators:

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m\,, \tag{3.10}$$

represent the oscillator commutation relations (2.3) by replacing α_n^μ for $n > 0$ with $n\,\partial/\partial\alpha_{-n}^\mu$ and make the corresponding substitution for left movers. The zero modes satisfy $\alpha_0^\mu = \tilde{\alpha}_0^\mu = p^\mu$ and we represent spacetime momentum by $p^\mu = -i\,\partial/\partial\phi_0^\mu$. (this slightly unconventional normalization of α_0^μ is discussed in the Appendix). Using this representation for the Virasoro generators and (3.9) for the boundary state, the condition (3.3) can be rewritten for $n > 0$ as

$$\begin{aligned}\Big\{ \sum_{m=1}^{n-1} \Big[\frac{1}{2}m(n-m)\Big(\frac{\partial W}{\partial\alpha_{-m}} \cdot \frac{\partial W}{\partial\alpha_{m-n}} - \frac{\partial^2 W}{\partial\alpha_{-m} \cdot \partial\alpha_{m-n}}\Big) - (n-m)\tilde{\alpha}_{-m} \cdot \frac{\partial W}{\partial\alpha_{m-n}}\Big] \\ + \quad \Big[in\frac{\partial^2 W}{\partial\phi_0 \cdot \partial\alpha_{-n}} - in\frac{\partial W}{\partial\alpha_{-n}} \cdot \frac{\partial W}{\partial\phi_0}\Big] \\ + \sum_{m=n+1}^{\infty} \Big[-m\alpha_{n-m} \cdot \frac{\partial W}{\partial\alpha_{-m}} + (m-n)\tilde{\alpha}_{-m} \cdot \frac{\partial W}{\partial\tilde{\alpha}_{n-m}}\Big]\Big\}\ |B\rangle \ = \ 0\,.\end{aligned} \tag{3.11}$$

(A similar expression holds for $n < 0$.) In this equation there are only closed string creation operators, and no annihilation operators, acting on $|B\rangle$ so the expression in curly brackets must vanish by itself. By making use of the Legendre transformation and combining some terms, it can be cast in the simpler form

$$\sum_{m=-\infty}^{\infty}(m+n)\frac{\partial\Gamma}{\partial\phi_m^{cl}}\cdot\phi_{m+n}^{cl}=\sum_{m=1}^{n-1}\frac{1}{2}m(n-m)\,[\phi_m^{cl}\cdot\phi_{n-m}^{cl}+\frac{\partial^2 W}{\partial\alpha_{-m}\cdot\partial\alpha_{m-n}}] - in\frac{\partial^2 W}{\partial\phi_0\cdot\partial\alpha_{-n}}. \tag{3.12}$$

The left hand side of this equation is precisely $\delta_n\Gamma[\phi]$, the variation of the effective action under the reparametrization $s\rightarrow s+ie^{ins}$. Comparing with (3.5), we see that the first term on the r.h.s. is the variation under reparametrization of the classical action. Using the fact that the classical action is the tree-level effective action, and separating the tree and loop contributions to Γ, we cast the Ward identity into the following reasonably compact form:

$$\delta_n\Gamma_{loop}=\sum_{m=1}^{n-1}\frac{1}{2}m(n-m)\frac{\partial^2 W}{\partial\alpha_{-m}\cdot\partial\alpha_{m-n}}-in\frac{\partial^2 W}{\partial\phi_0\cdot\partial\alpha_{-n}}. \tag{3.13}$$

This Ward identity evidently expresses *broken* reparametrization invariance of the one-dimensional field theory. In fact, we have previously noted that reparametrization invariance seems to be broken by the kinetic term and by the special treatment of the zero mode in the path integral, and, indeed, the terms on the r.h.s. of (3.13) correspond to these two sources. It is perhaps disturbing that strict reparametrization invariance of the boundary state does not translate into strict reparametrization invariance of the underlying one-dimensional field theory. However, when we develop the renormalized perturbation expansion in the next section, we will find that the 'broken symmetry' Ward identity is easy to implement and contains all the right physics.

In fact we could have avoided having the last term on the right in (3.13) by considering the boundary state as a function on momentum space rather than spacetime itself. The Fourier transform to momentum space involves an integration over ϕ_0^μ (which removes the hat from the path integral measure in (2.5)) and adds $i\,p \cdot \phi_0$ to the linear source term in (2.6). The Ward identity is derived in the same manner as before but since the zero mode is no longer singled out of the path integral the last term in (3.13) does not appear. This simpler form of the Ward identity is convenient for formal discussions. However, the zero mode still has to be projected out when we define the perturbation theory, so we will be using (3.13) in the sequel.

4. Perturbation Theory

In this section we work out the perturbation theory rules for our one-dimensional field theory and carry out the renormalization program, including the implementation of the Ward identities, to one-loop order. In the process, we will rederive the known gauge field equations of motion and understand how the gauge-field-dependent normalization of the boundary state path integral is unambiguously determined. We use the background field expansion and proper time regulation, following quite closely methods recently used by Guadagnini [10] to study general coordinate invariance in the two-dimensional non-linear sigma model. His approach turns out to be very well-adapted to our problem. As in all other string theory applications of nonlinear sigma models, perturbation theory is an expansion in spacetime derivatives: The three point-coupling in the one-dimensional field theory turns out to be proportional to $\nabla_\lambda F_{\mu\nu}$, and we make the approximation of slowly-varying spacetime gauge field in order for perturbation theory to be valid. In the same spirit, we will neglect terms with more than one derivative of $F_{\mu\nu}$ wherever appropriate.

To carry out the background field expansion we shift the field in the path integral, $\phi^\mu(s) \to \phi^\mu(s) + \pi^\mu(s)$, taking $\pi^\mu(s)$ to be the new quantum field and $\phi^\mu(s)$ to be an arbitrary classical background. Then the action is expanded in powers of the π-fields to get a set of propagators and vertices which depend on the classical functions $\phi^\mu(s)$. By the usual rules, the effective action $\Gamma[\phi]$ is the sum of the one-particle-irreducible vacuum diagrams for the π-fields [11]. All the information we need is contained in the inverse propagator $H_{\mu\nu} = \delta^2 S[\phi]/\delta\phi^\mu\delta\phi^\nu$, which summarizes the terms quadratic in π in the background field expansion of the action. Formally, the one-loop effective action is

$$\Gamma_1 = \frac{1}{2}\log \operatorname{Det} H .$$

This expression needs regularization to be meaningful and it is convenient to use the proper time method to define

$$\Gamma_1^{reg} = -\frac{1}{2}\int_\epsilon^\infty \frac{d\tau}{\tau}\operatorname{Tr}[e^{-\tau H}]. \tag{4.1}$$

In this expression, the trace is both over states of the one-dimensional theory and spacetime indices of the matrix inverse propagator $H_{\mu\nu}$.

It is useful to introduce position and momentum eigenstates in the Hilbert space of the one-dimensional theory. Call the one-dimensional position operator q and write its eigenstates as $|s\rangle$. Define the momentum operator $p = -i\frac{d}{dq}$ and call its eigenstates $|m\rangle$. Summarize these definitions by the following relations:

$$\begin{aligned} q\,|s\rangle &= s\,|s\rangle & \qquad p\,|m\rangle &= m\,|m\rangle \\ e^{ips'}\,|s\rangle &= |s'+s\rangle & \qquad e^{inq}\,|m\rangle &= |m+n\rangle . \end{aligned}$$

The operator $H_{\mu\nu}$ has two pieces derived from S_0 and S_A respectively. Because of the non-local nature of the kinetic term we cannot write $H^0_{\mu\nu}$ explicitly in terms of q and p, but it acts in a simple manner on momentum eigenstates

$$H^0_{\mu\nu}\,|m\rangle = \frac{|m|}{2}\delta_{\mu\nu}\,|m\rangle\ , \qquad -\infty < m < \infty . \tag{4.2}$$

The part derived from S_A is

$$\begin{aligned} H^A_{\mu\nu} = &-\frac{1}{4}\Big[F_{\mu\nu}\big(\phi(q)\big)\,p + p\,F_{\mu\nu}\big(\phi(q)\big)\Big] \\ &-\frac{i}{4}\Big(\nabla_\mu F_{\nu\lambda}\big(\phi(q)\big) + \nabla_\nu F_{\mu\lambda}\big(\phi(q)\big)\Big)\phi'^\lambda(q) . \end{aligned} \tag{4.3}$$

The background field vertices, derived from the cubic and higher terms in the expansion of the action in powers of π, are all proportional to at least the first derivative of $F_{\mu\nu}$ and play no role in what follows.

We now want to express the Ward identity (3.13) as a condition on Γ_1. We first consider the response of Γ_1 to a reparametrization of the background field. Only H^A depends on $\phi^\mu(s)$ so

$$\delta_f \Gamma_1 = \frac{1}{2} \mathrm{Tr}\,[\delta_f H^A \frac{1}{H} e^{-\epsilon H}]\,. \tag{4.4}$$

The variation $\delta_f H^A$ has a simple operator expression

$$\delta_f H^A_{\mu\nu} = ipf(q) H^A_{\mu\nu} - i H^A_{\mu\nu} f(q) p\,. \tag{4.5}$$

which follows from reparametrization invariance of S_A, $\delta_f \langle \phi_1 | H^A_{\mu\nu} | \phi_2 \rangle = 0$, and the transformation rule of a scalar, $\delta_f |\phi\rangle = f(q) \frac{d}{dq} |\phi\rangle$.

Next we show how the first anomalous term on the right hand side of (3.13) can be cast in a form similar to the variation of Γ_1 in (4.4) and that the two combine to give a simple expression. Consider an infinitesimal reparametrization by $f_n(s) = ie^{ins}$ and construct an operator using $H^0_{\mu\nu}$ instead of $H^A_{\mu\nu}$ in the right hand side of (4.5). Let this operator act on a momentum eigenstate

$$\begin{aligned} \{ipf_n H^0_{\mu\nu} - iH^0_{\mu\nu} f_n p\} |m\rangle &= -\frac{1}{2}\delta_{\mu\nu}\left[(m+n)|m| - |m+n|m\right] |m+n\rangle \\ &= \begin{cases} \delta_{\mu\nu}(m+n)m\,|m+n\rangle\,, & \text{if } -n < m < 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Then note that second derivatives of the connected generating functional are precisely matrix elements of the propagator of the theory:

$$\frac{\partial^2 W}{\partial \alpha^\mu_{-m} \cdot \partial \alpha^\nu_{m-n}} = \langle -m | \frac{1}{H_{\mu\nu}} | n-m \rangle\,. \tag{4.6}$$

From these two observations it follows that

$$\frac{1}{2} \mathrm{Tr}\left[(ipf_n H^0 - iH^0 f_n p) \frac{1}{H} e^{-\epsilon H}\right] = -\sum_{m=1}^{n-1} \frac{1}{2} m(n-m) \frac{\partial^2 W}{\partial \alpha_{-m} \cdot \partial \alpha_{m-n}}$$

Assembling (4.4) - (4.6), we see that two terms in (3.13) combine very neatly to give

$$\delta_n \Gamma_1 - \sum_{m=1}^{n-1} \frac{1}{2} m(n-m) \frac{\partial^2 W}{\partial \alpha_{-m} \cdot \partial \alpha_{m-n}} = \frac{1}{2} \operatorname{Tr} [f_n' e^{-\epsilon H}] . \tag{4.7}$$

This is very similar to Guadagnini's result for the one-loop general covariance Ward identity [10] in the two-dimensional sigma model.

Finally, we must deal with the second anomalous term in (3.13), expressing the breaking of reparametrization invariance due to zero modes. According to our original definition of the path integral, we should not integrate over the zero modes of π and, since our expression (4.1) for the one-loop effective action has ignored that subtlety, the preceding calculations must be modified to take it into account. Define P_0 as the projection of any state onto the zero mode. Then the proper one-loop effective action is obtained by replacing $H_{\mu\nu}$ in (4.1) by its projection orthogonal to zero modes

$$H_{\mu\nu}^P = (1-P_0) H_{\mu\nu} (1-P_0). \tag{4.8}$$

The Ward identity involves the variation of the projected operator under a reparametrization. Since the projection P_0 does not depend on the background field ($\delta_f P_0 = 0$) we get

$$\begin{aligned} \delta_f H_A^P &= (1-P_0) \delta_f H_A (1-P_0) \\ &= (1-P_0)(ipfH_A - iH_A fp)(1-P_0) \\ &= ipfH_A^P - iH_A^P fp + f' P_0 H_A (1-P_0) + (1-P_0) H_A P_0 f' . \end{aligned} \tag{4.9}$$

(The last step uses the identity $pP_0 = P_0 p = 0$.) The first two terms fit into the arguments leading to (4.7) but the remaining two give an additional contribution

$$\operatorname{Tr}\big[f' P_0 H_A (1-P_0) \frac{1}{H^P}\big] \tag{4.10}$$

to $\delta_f \Gamma_1$. Once again take $f(s) = ie^{ins}$ to facilitate comparison with (3.13). The trace can be taken over momentum eigenstates and the P_0 projection eliminates all but one term to give the following extra contribution to (4.7) :

$$-\,n\,tr\,\langle 0|\,H_A(1-P_0)\frac{1}{H^P}\,|n\rangle\ .$$

We claim that this object is related to the generating functional of connected Green's functions by the following simple identity:

$$i\frac{\partial^2 W}{\partial\phi_0\cdot\partial\alpha_{-n}} = \mathrm{tr}\,\langle 0|\,H_A(1-P_0)\frac{1}{H^P}\,|n\rangle\,. \qquad (4.11)$$

To the order we are working (balancing one-loop anomalies in reparametrization invariance against explicit tree-level breaking) we only have to assert that this relation is true for tree graphs. The quantity $\partial W/\partial\alpha_{-n}$ is just ϕ_n^{cl} and is given by the sum of all connected trees with one distinguished external leg corresponding to ϕ_n^{cl}, all other external legs terminating on α sources and internal lines corresponding to propagating non-zero modes. The vertices correspond to expansion of the interaction Lagrangian in non-zero modes and, through their dependence on the spacetime background fields, are functions of the zero mode coordinate. Differentiation with respect to ϕ_0 acts in turn on all the vertices. A particular term in the graphical sum for $\partial^2 W/\partial\phi_0\cdot\partial\alpha_{-n}$ has a differentiated vertex connected to the distinguished ϕ_n^{cl} external leg by a diagram having the topology of a propagator. Summing over all diagrams will just convert such diagrams to the full background field propagator in the given ϕ^{cl} background. The same summation converts the differentiated interaction vertex to the matrix element of the background field quadratic action between one zero mode and one non-zero mode. This cumbersome argument is easier to see graphically than to state. The result is the claimed identity (4.11) as a consequence of which the last term in the Ward identity (3.13) serves to cancel the extra piece (4.10) of $\delta_f\Gamma_1$ which we found by treating the zero modes carefully. The net one-loop Ward identity,

$$\frac{1}{2}\mathrm{Tr}\big[f' e^{-\epsilon H^P}\big] = 0\,, \qquad (4.12)$$

is very simple indeed and identical to what we would have written down had we ignored the various broken reparametrization invariance subtleties we encountered!

In what follows, we will drop the superscript P from $H_{\mu\nu}$, but the correct treatment of the zero mode must be kept in mind.

5. Explicit Results

We now proceed to an explicit evaluation of the Ward identity. We will find that to satisfy (4.12) for arbitrary reparametrization f and background field ϕ, the spacetime gauge field must satisfy certain equations of motion and the counterterms needed to eliminate vacuum divergences must have specific finite parts. Taken together, these conditions will completely specify the value of the boundary state. The evaluation of the trace in (4.12) is complicated by the fact that the operator $H_{\mu\nu}$ depends both on p and q. We use a procedure employed by Guadagnini [10] which proves convenient here also. In a basis of position eigenstates the trace is

$$\frac{1}{2}\mathrm{Tr}[f' e^{-\epsilon H}] = \frac{1}{2}\oint \frac{ds}{2\pi} f'(s)\, \mathrm{tr}\, \langle s|\, e^{-\epsilon H(p,q)}\, |s\rangle\, . \tag{5.1}$$

We use the translation properties of the position eigenstates to write

$$\begin{aligned} \mathrm{tr}\, \langle s|\, e^{-\epsilon H(p,q)}\, |s\rangle &= \mathrm{tr}\, \langle 0|\, e^{-ips} e^{-\epsilon H(p,q)} e^{ips}\, |0\rangle \\ &= \mathrm{tr}\, \langle 0|\, e^{-\epsilon H(p,q+s)}\, |0\rangle \end{aligned} \tag{5.2}$$

Here q is an operator but s is a number. Furthermore q annihilates $|0\rangle$ so it is a good idea to Taylor expand $H_{\mu\nu}(p, q+s)$ in powers of q. That gives a series which can be arranged by powers of derivatives of $F_{\mu\nu}$

$$H_{\mu\nu}(p, q+s) = H^F_{\mu\nu}(p,s) + H^{\nabla F}_{\mu\nu}(p,s,q) + \dots$$

The two leading terms are

$$\begin{aligned} H^F_{\mu\nu}(p,s) =& \quad H^0_{\mu\nu}(p) - \frac{1}{2} F_{\mu\nu}(\phi(s))\, p\, , \\ H^{\nabla F}_{\mu\nu}(p,s,q) =& -\frac{1}{4}\nabla_\lambda F_{\mu\nu}(\phi(s))\phi'^\lambda(s)(qp+pq) \\ & -\frac{i}{4}\Big(\nabla_\mu F_{\nu\lambda}(\phi(s)) + \nabla_\nu F_{\mu\lambda}(\phi(s))\Big)\phi'^\lambda(q+s)\, . \end{aligned} \tag{5.3}$$

Note that H^F is a function of p and s and it is only the perturbation $H^{\nabla F}$ that depends on the position operator. We write out the exponential in (5.2) using an interaction picture trick:

$$e^{-\epsilon(H^F+H^{\nabla F})} = e^{-\epsilon H^F} - \epsilon\int_0^1 d\alpha\, e^{-(1-\alpha)\epsilon H^F} H^{\nabla F} e^{-\alpha\epsilon H^F}$$
$$+ \epsilon^2\int_0^1 \alpha d\alpha \int_0^1 d\beta\, e^{-(1-\alpha)\epsilon H^F} H^{\nabla F} e^{-\alpha(1-\beta)\epsilon H^F} H^{\nabla F} e^{-\alpha\beta\epsilon H^F} + \dots \tag{5.4}$$

Each of these terms is to be sandwiched between two zero position states and all q's are then commuted to the left or right until they annihilate on $|0\rangle$. The resulting expression involves only p operators and is easily evaluated by inserting a complete set of momentum states, $\sum_{m\neq 0} |m\rangle\langle m|$, with the zero momentum state left out because of the zero mode projection. The leading term in (5.4) gives

$$\begin{aligned}
\mathrm{tr}\,\langle 0|\, e^{-\epsilon H^F(p,s)}\,|0\rangle &= \sum_{m=1}^{\infty} \mathrm{tr}\big[e^{-\frac{m}{2}\epsilon(1-F)} + e^{-\frac{m}{2}\epsilon(1+F)}\big] \\
&= \mathrm{tr}\Big[\frac{1}{e^{\frac{1}{2}\epsilon(1-F)}-1} + \frac{1}{e^{\frac{1}{2}\epsilon(1+F)}-1}\Big] \\
&= \frac{4}{\epsilon}\mathrm{tr}\Big(\frac{1}{1-F^2}\Big) - D + O(\epsilon)
\end{aligned}$$

and the corresponding contribution to the reparametrization anomaly is

$$\frac{2}{\epsilon}\oint \frac{ds}{2\pi} f'(s)\,\mathrm{tr}\left(\frac{1}{1-F^2(\phi(s))}\right). \tag{5.5}$$

This linear divergence was to be expected since we are dealing with bosonic string theory. To eliminate it we simply include a counterterm interaction in our one-dimensional Lagrangian which describes a coupling to a background of open string tachyons. The general form of such a term is

$$S_T[\phi] = \oint \frac{ds}{2\pi} T\big(\phi(s)\big). \tag{5.6}$$

It is super-renormalizable but not reparametrization invariant. In fact its variation under a reparametrization is

$$\delta_f S_T[\phi] = -\oint \frac{ds}{2\pi} f'(s)\, T\big(\phi(s)\big)$$

so that the linear divergence in (5.5) is eliminated by choosing

$$T(\phi(s)) = \frac{2}{\epsilon}\,\mathrm{tr}\left(\frac{1}{1-F^2(\phi(s))}\right). \tag{5.7}$$

This counterterm also serves to eliminate the linear divergences of the theory and we might imagine that we could add to it a finite but background field dependent part. But the Ward identity restricts any such finite part to be at most a constant independent of $F_{\mu\nu}$ which leads to only a trivial overall constant ambiguity in the normalization of the path integral! We remark that the superstring has no tachyons and the corresponding boundary theory has a supersymmetry in one dimension which rids it of linear divergences of this kind.

The contribution of the second term in the expansion (5.4) is obtained in the same fashion. After a few lines of algebra we find that

$$\begin{aligned}\mathrm{tr}\ \langle 0|\,(-\epsilon)\int_0^1 d\alpha e^{-(1-\alpha)\epsilon H^F} H^{\nabla F} e^{-\alpha\epsilon H^F}\,|0\rangle \\ = 2i\Big(\frac{1}{1-F^2(\phi)}\Big)^{\mu\nu}\nabla_\mu F_{\nu\lambda}(\phi)\frac{d\phi^\lambda}{ds} + O(\epsilon)\end{aligned} \tag{5.8}$$

so that the finite $O(\nabla F)$ piece of the reparametrization anomaly is

$$i\oint \frac{ds}{2\pi} f'(s)\left(\frac{1}{1-F^2(\phi(s))}\right)^{\mu\nu}\nabla_\mu F_{\nu\lambda}\big(\phi(s)\big)\,\frac{d\phi^\lambda(s)}{ds}. \tag{5.9}$$

The boundary state is reparametrization invariant only if this vanishes for all $f(s)$ and arbitrary background $\phi^\mu(s)$, which requires the spacetime gauge field to satisfy the equation of motion

$$\left(\frac{1}{1-F^2}\right)^{\mu\nu} \nabla_\mu F_{\nu\lambda} = 0. \tag{5.10}$$

Our conventions (see Appendix) are such that $F_{\mu\nu}$ contains a factor of $2\pi\alpha'$. The equation of motion for the gauge field therefore contains terms to all orders in α' although it is only valid to leading order in derivatives of $F_{\mu\nu}$. At distances large compared to the string length scale it reduces to Maxwell's equation as it should. It is identical to the equation of motion previously derived in [7] by imposing conformal invariance on the two dimensional sigma model with a Wilson line interaction at a world-sheet boundary. This agreement provides a non-trivial check that reparametrization invariance is the dynamical principle which picks out acceptable boundary states in general.

We can now calculate the one-loop effective action itself, from which we can derive the value of the full boundary state path integral. Substituting the first term in (5.4) into (4.1) gives the leading contribution:

$$\begin{aligned}
\Gamma_1 &= -\frac{1}{2}\int_\epsilon^\infty \frac{d\tau}{\tau} \oint \frac{ds}{2\pi} \text{tr } \langle 0|\, e^{-\tau H^F(p,s)} \,|0\rangle \quad + \quad O(\nabla F) \\
&= -\frac{1}{2}\oint \frac{ds}{2\pi} \int_\epsilon^\infty \frac{d\tau}{\tau} \text{tr } \left[\frac{1}{e^{\frac{1}{2}\tau(1-F)}-1} + \frac{1}{e^{\frac{1}{2}\tau(1+F)}-1}\right] \\
&= -\frac{1}{2}\oint \frac{ds}{2\pi} \text{tr} \Big[\frac{4}{\epsilon}\big(\frac{1}{1-F^2}\big) + \log(1+F) + C_\epsilon + O(\epsilon)\Big].
\end{aligned} \tag{5.11}$$

The linearly divergent piece is canceled by the tachyon counterterm (5.7), which we found before. At this order the functional form of the normalization of the boundary state comes from the tr log $(1+F)$ term. In a constant gauge field that normalization is $\sqrt{\det(1+F)}$. This result has been around for some time and is usually arrived at using ζ-function techniques which dictate a specific subtraction procedure without any transparent physical motivation. We wish to emphasize that in the present approach there is no room for any ambiguity in the F dependence of the boundary state normalization. The reparametrization Ward identity does not allow a finite addition to the counterterm to depend on the spacetime gauge field. The constant of integration C_ϵ is logarithmically divergent, but it is simply some number independent of $F_{\mu\nu}$. This divergence simply reflects our freedom to choose the value of the open string loop coupling constant.

The $O(\nabla F)$ contribution to the one-loop effective action comes from the second term in the expansion (5.4) and is logarithmically divergent. One finds

$$\Gamma_1^{O(\nabla F)} = i \log \epsilon \oint \frac{ds}{2\pi} \Big(\frac{1}{1 - F^2(\phi)} \Big)^{\mu\nu} \nabla_\mu F_{\nu\lambda}(\phi) \frac{d\phi^\lambda}{ds} + \text{finite terms} \tag{5.12}$$

The functional form of the remaining finite expression is rather complicated and we do not write it down here. Evidently the one-loop beta function of the boundary field theory vanishes when the equation of motion (5.10) is satisfied. In fact, it is probably true to any loop order that the theory is required to sit at a renormalization group fixed point in order for the reparametrization anomaly to vanish.

6. Discussion

We have begun to explore the connection between string theory in a background of massless open string fields and a class of one-dimensional field theories. The field theory supplies a boundary state, in the closed string Hilbert space, created via open string physics at a world-sheet boundary. Reparametrization invariance is the key principle which allows one not only to determine acceptable open string backgrounds but also provides an unambiguous renormalization prescription for the couplings of the one-dimensional theory, which is vital when the boundary state is to be used to calculate the effect of open string matter on closed string fields. This field theory framework may lead to some insights about deeper issues in string theory. The two-dimensional conformal field theory program is in part aimed at finding interesting classical configurations of closed strings. If we can learn how to sum over such field theories, or at least to quantize some collective coordinates of string solitons, we would be a step closer to a quantum description of string field theory. Corresponding questions can be raised in open string theory and we believe that the way to get at the answers is through classifying reparametrization invariant boundary states. Since the field theories involved are only one-dimensional, one dares to hope that they can be characterized in some detail. In particular, it would be interesting to find, or prove the existence of, open string instantons or solitons (and calculate their energy-momentum).

Another reason to study boundary states is that they generalize the energy momentum tensor of matter in spacetime. In a string theory generalization of relativity open strings play the role of matter while the closed string sector corresponds to gravity itself. The boundary state is the open string source of closed string fields which enters in the BRST anomaly cancellation equations. Since these equations are the string theory generalization of Einstein's equation, knowledge about boundary states might shed some light on questions of energy positivity and occurrence of singularities in stringy relativity.

Reparametrization invariance of the boundary state manifests itself in the one-dimensional theory as a set of Ward identities. They are not as simple as one would have liked. Complications arise from the fact that the kinetic term explicitly refers to a specific parametrization of the one-dimensional manifold, and from the special treatment of the zero modes, whose identification again refers to a specific parametrization. Despite these nuisances, we found that the Ward identity at the one loop level can be stated in a very simple manner and leads to a simple equation of motion for the spacetime gauge field which has the same solutions as the variational equation of the non-linear Born-Infeld Lagrangian [7]. A straightforward extension of the present work would be to apply our Ward identities to higher loop calculations in the one-dimensional theory. It would also be desirable to drop the restriction to an abelian subgroup of the full gauge group. Steps have been taken in that direction in [3,5] but the problem is technically cumbersome because of path ordering.

Needless to say, the issue of defining the renormalization of vacuum divergences arises for the closed string path integral as well. Although it is not obvious how to proceed, we expect that a generalization of the Ward identity approach we have developed will be applicable in that context as well.

Our considerations are presumably connected with issues in statistical mechanics. The one-dimensional path integral is just the partition function of a one-dimensional statistical system and our equations of motion for the background gauge fields are just the condition that the system be at a critical point. One-dimensional systems with long-range interactions do indeed have critical points (the Ising model with r^{-2} interactions is the classic example) and our nonlocal kinetic term is of that type. It is possible that existing results on one-dimensional critical theories can be translated into useful information on open string theories. The work of Cardy [12] on conformal invariance and surface critical behavior is very interesting in this regard.

Although we have only been concerned with bosonic string theory in this paper, most of the ideas and techniques presented carry over to the more interesting case of superstrings. We will report on supersymmetric boundary states at a later date.

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APPENDIX

Here we state our conventions and normalizations. We use a Euclidean world-sheet metric and natural units $\hbar = c = 1$ throughout. A free closed string is parametrized by $\sigma \in [0, 2\pi]$ and its spacetime coordinates are periodic solutions of the two dimensional wave equation

$$X^\mu(\tau,\sigma) = q^\mu - \frac{i}{2} l^2 p^\mu \tau + \frac{i}{2} l \sum_{m\neq 0} \frac{1}{m} \left[\alpha_m^\mu e^{-m\tau - im\sigma} + \tilde{\alpha}_m^\mu e^{-m\tau + im\sigma}\right]. \qquad \text{(A.1)}$$

The length scale l is related to the string tension and the Regge slope parameter by

$$l = \frac{1}{\sqrt{\pi T}} = \sqrt{2\alpha'}. \tag{A.2}$$

The equal time commutation relations of $X^\mu(\tau,\sigma)$ and its conjugate momentum $P^\mu_\tau(\tau,\sigma) = T\partial_\tau X^\mu(\tau,\sigma)$ are

$$\left[X^\mu(\tau,\sigma), P^\nu_\tau(\tau,\sigma')\right] = \delta^{\mu\nu}\,\delta(\sigma-\sigma'). \tag{A.3}$$

Inserting the mode expansion one finds that $[q^\mu,p^\nu] = i\delta^{\mu\nu}$ and the commutation relations (2.3) for the mode operators.

Our conventions differ from those of Chapter 2 in [13] in that we use a Euclidean rather than Minkowski signature on the world-sheet, and we parametrize closed strings from 0 to 2π rather than 0 to π. We find it convenient to take the string length to be $l=2$ which corresponds to $\alpha'=2$ rather than $l=1$ and $\alpha'=\frac{1}{2}$ as in [13].

To figure out the normalization of α^μ_0 and $\tilde\alpha^\mu_0$ write X^μ as a sum of right and left moving parts

$$X^\mu(\tau,\sigma) = X^\mu_R(\tau+i\sigma) + X^\mu_L(\tau-i\sigma). \tag{A.4}$$

The time derivatives of X^μ_R and X^μ_L are

$$\begin{aligned}\dot X^\mu_R(\tau+i\sigma) &= -\frac{i}{2}l\sum_{m=-\infty}^{\infty}\alpha^\mu_{-m}e^{-m\tau-im\sigma},\\ \dot X^\mu_L(\tau-i\sigma) &= -\frac{i}{2}l\sum_{m=-\infty}^{\infty}\tilde\alpha^\mu_{-m}e^{-m\tau+im\sigma},\end{aligned} \tag{A.5}$$

Comparing with (A.1) one gets

$$\alpha^\mu_0 = \tilde\alpha^\mu_0 = \frac{1}{2}lp^\mu \tag{A.6}$$

which, on setting $l=2$, gives the momentum normalization used in Section 3.

The boundary path integral action, before α' is taken to equal 2, and with conventional normalization of the Wilson line is

$$S[\phi] = -\frac{i}{2\pi\alpha'}\oint ds\, \phi_+ \cdot \frac{d\phi_-}{ds} + i \oint ds\, A_\mu(\phi)\,\frac{d\phi^\mu}{ds}. \tag{A.7}$$

In (2.6) we have rescaled the gauge potential A_μ to include a factor of $2\pi\alpha'$. This makes the field strength $F_{\mu\nu}$ dimensionless and streamlines the notation.

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