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The Topology of Moduli Space and Quantum Field Theory*

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ABSTRACT

We show how an SO(2,1) gauge theory with a fermionic symmetry may be used to describe the topology of the moduli space of curves. The observables of the theory correspond to the generators of the cohomology of moduli space. This is an extension of the topological quantum field theory introduced by Witten to investigate the cohomology of Yang-Mills instanton moduli space. We explore the basic structure of topological quantum field theories, examine a toy U(1) model, and then realize a full theory of moduli space topology. We also discuss why a pure gravity theory, as attempted in previous work, could not succeed.

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1. Introduction

There is a widespread belief that the Lagrangian is the fundamental object for study in physics. The symmetries of nature are simply properties of the relevant Lagrangian. This philosophy is one of the remaining relics of classical physics where the Lagrangian is indeed fundamental. Recently, Witten has discovered a class of quantum field theories which have no classical analog.[1] These topological quantum field theories (TQFT) are, as their name implies, characterized by a Hilbert space of topological invariants. As has been recently shown, they can be constructed by a BRST gauge fixing of a local symmetry.[2][3] The classical Lagrangian is zero modulo a topological invariant. Thus, we see that the heart of the matter is in the symmetries one chooses to study. The Lagrangian is secondary, being completely determined by the symmetries. Topological theories have so far been constructed for investigating the instanton moduli spaces of Yang-Mills theories in 4-dimensions and nonlinear sigma models with target manifolds having an almost complex structure.[1][4][5] This was motivated by the works of Atiyah, Donaldson and Gromov.[6][7][8]

In this paper we follow up on previous work on two dimensional gravity.[5][9] It is well-known that pure gravity in two dimensions is a topological theory, since the classical action, $\frac{1}{4\pi}\int\sqrt{g}R$, is just the Euler characteristic. The relevant moduli space will be the familiar one of complex structures of Riemann surfaces of genus g, $\mathcal{M}_g$. The topology of this moduli space is of current interest in both mathematics and physics. String theory is believed to be fundamentally a theory defined on the moduli space of curves. In particular it is known that the superstring partition function is locally a total derivative on moduli space.[10] Thus, the partition function will only depend on the topology of moduli space. Mathematicians have also been interested in the topology of moduli space, since it is known to be highly nontrivial.[11] Unfortunately, there is still relatively little known about it. In this paper we will describe how elements of the cohomology of moduli space may be computed as observables of the theory.

We have found that the most fruitful approach to two dimensional gravity is to treat it as a topological gauge theory of SO(2,1). This is the isometry group of 2-dimensional gravity with a positive cosmological constant. We will show that by an appropriate gauge fixing this theory describes the topology of $\mathcal{M}_g$ for $g \geq 2$. When $g = 1$ the relevant group is ISO(1,1), and for $g = 0$ it is SO(1,2). This approach differs dramatically from the previous work[5] [9] on topological gravity where the emphasis was on the symmetry of pure gravity. It was not possible in a purely gravitational theory to investigate the topology of $\mathcal{M}_g$. Roughly speaking, one has to study the topological gauge theory whose moduli space is $\mathcal{M}_g$. We will discuss the reasons for this in section 5. This is analogous to the case of topological Yang-Mills, except that one must now deal with a noncompact group.

In this paper we will explicitly construct two dimensional gravity as the topological gauge theory of SO(2,1) for $g \geq 2$. The gauge choice is the vanishing of the field strength; this implies the standard conditions that the zweibein (a component of the gauge field) be covariantly conserved and the curvature of the metric constructed from this zweibein be a negative constant. Indeed, it is essential that the zweibein and spin connection be treated as independent fields which are related only by the gauge fixing condition that we have a flat connection. It is significant to observe that the SO(2,1) gauge theory on an arbitrary background Riemann surface of $g \geq 2$ is equivalent to two dimensional gravity with the same topology . In the case of $g = 1$ the gauge group is ISO(1,1), and the vanishing of the field strength implies that the curvature is zero. Similarly, for the sphere the gauge group is SO(1,2), and the gauge fixing constrains the curvature to be a positive constant.

In section 2, we will discuss the general relationship between moduli spaces and TQFT's. Section 3 will be devoted to the fundamental concepts necessary for constructing TQFT's emphasizing the fields and symmetries of topological gravity. In section 4, we will work out completely the toy example of a topological U(1) theory. Then we will review, in section 5, the previous work on topological gravity , its shortcomings and its relation to the current work. In section 6, we will discuss the degenerate case of $g = 1$ where the gauge group is ISO(1,1). The explicit construction of the Lagrangian of the SO(2,1) gauge theory will be given in section 7. In section 8 we will discuss its relation to moduli space and its invariant observables. Finally, we will summarize the work and comment on the open questions remaining.

2. Moduli Space and TQFT

In gauge theories, moduli spaces are the spaces of solutions of the classical equations modulo gauge equivalence. They are finite dimensional, since they correspond to solutions of non-linear partial differential equations. A proper treatment of a quantum field theory with a local gauge symmetry requires an integration over the relevant moduli . In Yang-Mills theories on 4-dimensional Euclidean space, instantons provide an important tool for computing nonperturbative effects. The moduli spaces must then be understood in order to complete the final integration.

Knowledge about the topology of these spaces has recently been dramatically increased by the work of Donaldson.[7] Witten's topological Yang-Mills theories (TYM) have provided a quantum field theoretic realization of Donaldson's work.[1] In string theory where the gauge symmetry is that of world-sheet diffeomorphisms one has an analogous structure. The integration over the gauge configurations (i.e. the metric) simplifies into an integration over the moduli space of complex structures of Riemann surfaces, $\mathcal{M}_g$. Indeed, string theory measures are expected to be sections of line bundles on moduli space as advocated by Belavin and Knizhnik.[12]

Knowledge about the topology of moduli space is then important for gaining a better understanding of string theory. Witten has shown that topological quantum field theories may provide a fruitful new tool for studying moduli space.

Mathematicians investigate the topology of moduli space using the theory of principal G-bundles and classifying spaces. Topological quantum field theories can also be used to study the topology of moduli space. In this section, we will clarify the relation between these two approaches. Readers who are only interested in the field theory should proceed to section 3. In section 8, we will show precisely how the observables of TQFT's probe the topology of the universal bundle. In the following we outline the relevant mathematical terminology needed to describe the structure of TQFT's.

A principal G-bundle is a fiber bundle with fiber given by the structure group G; it contains a mapping $\pi : P \to B$ and a mapping $g : P \times G \to P$. P is called the *total space*; B is the *base space*, and π is the projection. P is a manifold on which the group G acts freely. The coset space $B := P/G$ can also be made into a manifold such that the projection $\pi : P \to B$ is smooth. If P is contractible , then the principal bundle is called *universal*, and $B = P/G$ is called a *classifying space* for G and often denoted as BG (base space of G). A good deal is known about the cohomology and homotopy of classifying spaces which is closely related to that of the moduli space. The moduli space is the finite dimensional subspace where only the classical gauge configurations are allowed. The classifying spaces are then useful in studying the topology of moduli space. In particular we can choose a principal G-bundle where $P = \mathcal{A} = \{$connections on a fixed G-bundle over some manifold $M\}$, then $BG = \mathcal{A}/\mathcal{G}$ with $\mathcal{G}$ being the group of gauge transformations . Indeed, BG is the space one is usually concerned with in quantum field theory, since the space of gauge fields in the path integral is defined to be $\mathcal{A}/\mathcal{G}$. For the case of instantons the subspace of $\mathcal{A}/\mathcal{G}$ with self-dual field strengths is the moduli space.

The observables of TQFT's are the generators of $H^*(\mathcal{M})$, the cohomology ring of instanton moduli space. These are also elements of $H^*(\mathcal{A}/\mathcal{G})$. There is a whole lore on these observables which mathematicians call characteristic classes. They are used to classify the topology of G-bundles; they generate the cohomology of the classifying space of a Lie group, G. In general, for a simply connected, finite dimensional Lie group G, we have the following generators for the cohomology rings:

$$H^*(BG) \simeq \mathbb{R}[y_1, ..., y_k], \text{ degree } y_i = d_i + 1, \\ H^*(G) \simeq E[x_1, ..., x_k], \text{ degree } x_i = d_i, \tag{2.1}$$

where $\mathbb{R}[...]$ is a polynomial algebra over the reals and $E[...]$ is an exterior algebra. The generators, x_i, y_i, are forms, and by *degree* we mean the degree of the form. Recall that $H^*(BG) \simeq H^*(\mathcal{A}/\mathcal{G})$. Thus, the observables of TYM will be a polynomial algebra over the generators, y_i. These are referred to as the Donaldson polynomials. Thus, TYM theories provide quantum field theoretic techniques for the explicit construction of these polynomials. The details are discussed in ref.[1].

For the moduli space of curves there is an analogous structure. The classification is, however, much more difficult because the gauge group is noncompact. We must then be content with a weaker result. Miller proved that there is an injective mapping from a polynomial algebra, $\mathbb{Q}$, over the rationals to the cohomology ring of the moduli space of the oriented diffeomorphisms of genus g surfaces, Λ_g:[13]

$$\mathbb{Q}[\kappa_1, ..., \kappa_{g-2}] \rightarrow H^*(B\Lambda_g; \mathbb{Q}) \tag{2.2}$$

where $\kappa_i \in H^{2i}(B\Lambda_g)$.

Recall that $\mathcal{M}_g$ is the space of metrics for genus g Riemann surfaces modulo diffeomorphisms. As in Yang-Mills theory, $H_*(B\Lambda_g)$ and $H_*(\mathcal{M}_g)$ are intimately related.[11] Indeed, their rational cohomologies are isomorphic:

$$H_*(B\Lambda_g; \mathbb{Q}) \simeq H_*(\mathcal{M}_g; \mathbb{Q}). \tag{2.3}$$

Mumford gave a construction for the classes, κ_i.[14] We will refer to them as Mumford invariants. They are elements of the stable cohomology of $\mathcal{M}_g$. Stability is the statement that for $g \gg i$, κ_i is independent of g. In fact, for $i \geq g - 1$, κ_i

is a polynomial in $\kappa_1, ..., \kappa_{g-2}$. We will have more to say about these classes in section 8 where the observables of the TQFT will be discussed.

For the quantum field theory we will construct in this paper, we expect that our topologically invariant observables will include the Mumford classes. Since the Mumford classes do not necessarily generate the entire $H^*(\mathcal{M}_g)$, we also might find other observables. Indeed, we hope that our quantum field theoretic construction of the generators of $H^*(\mathcal{M}_g)$ will increase our understanding of $H^*(\mathcal{M}_g)$.

We will have more to say about the invariant observables in section 8. In the next section, we will outline basic issues for the assembly of TQFT's, in particular those related to gravity and the groups ISO(1,1) and SO(2,1).

3. Symmetries, Fields and Actions in Topological Gravity

Our objective is to obtain a topological theory for the moduli space of curves. Therefore, a natural candidate for such a TQFT is obviously topological gravity (TG). Topological quantum field theories are characterized by an invariance under a local symmetry which guarantees that all the observables of the theory will be metric independent. In TYM, for example, the symmetry is $\delta A_\mu = \theta_\mu(x)$ and the observables are the so called Donaldson polynomials. In the following we will discuss the symmetries and fields relevant for describing topological theories of gravity.

The fields usually used to describe two dimensional gravity are the metric $g_{\alpha\beta}$, zweibein and spin-connection. The zweibein $e_{\alpha a}$, defined by $g_{\alpha\beta} = e^a_\alpha e_{\beta a}$, is used to transform a world vector V^α into a tangent space vector V_a via $V_a = e_{\alpha a} V^\alpha$. The action of the spin connection ω_α is defined in the covariant derivative by $D_\alpha V^a = \partial_\alpha V^a + \omega_\alpha \epsilon^{ab} V_b$. It can be expressed in terms of $e_{\alpha a}$ by using the requirement of no torsion, namely $D_\beta e_{\alpha a} = 0$. The symmetries of TG should include diffeomorphisms ($\delta_D(v)$), Weyl rescaling ($\delta_W(\rho)$), local Lorentz transformations ($\delta_L(\lambda)$), and the additional "topological symmetry" ($\delta_{TG}(\theta)$) which allows the metric to be gauged away locally. Under these transformations, the fields $g_{\alpha\beta}$, $e_{\alpha a}$ and ω_α transform as follows:

$$\begin{aligned}
\delta_D g_{\alpha\beta} &= D_\alpha v_\beta + D_\beta v_\alpha, & \delta_L g_{\alpha\beta} &= 0,\\
\delta_D e^a_\alpha &= e^{\beta a} D_\alpha v_\beta, & \delta_L e^a_\alpha &= \Lambda \epsilon^{ab} e_{\alpha b},\\
\delta_D \omega_\alpha &= -\epsilon^{ab} D_a D_\alpha v_b, & \delta_L \omega_\alpha &= -\partial_\alpha \Lambda,\\
\\
\delta_W g_{\alpha\beta} &= \rho g_{\alpha\beta}, & \delta_T g_{\alpha\beta} &= \theta_{\alpha\beta} + \theta_{\beta\alpha},\\
\delta_W e^a_\alpha &= \frac{1}{2}\rho e^a_\alpha, & \delta_T e^a_\alpha &= \theta^a_\alpha,\\
\delta_W \omega_\alpha &= -\frac{1}{2}\epsilon_{\alpha\beta}\partial^\beta \rho, & \delta_T \omega_\alpha &= -\epsilon^{ab}\partial_a \theta_{\alpha b}.
\end{aligned} \tag{3.1}$$

A different approach to the symmetries of the theory of gravity was presented by Witten for the case of three dimensions.[15] He showed that the Einstein-Hilbert action was equal to a Chern-Simons action for the group ISO(2,1). In analogy, we want to analyse the ISO(1,1) theory. First, we want to compare the gauge transformations of $e_{\alpha a}$ and ω_α , the gauge fields of the ISO(1,1), to those given in (3.1). We denote the ISO(1,1) gauge fields by

$$A_\alpha = e^a_\alpha P_a + \omega_\alpha \frac{1}{2}\epsilon^{ab} J_{ab} \tag{3.2}$$

where P_a and $J = \frac{1}{2}\epsilon^{ab} J_{ab}$ are the generators of the two translations and the Lorentz transformation. They obey the following algebra:

$$[P_a, P_b] = [J, J] = 0, \qquad [J, P_a] = \epsilon_{ab} P^b. \tag{3.3}$$

Using the usual transformation for gauge fields, $\delta A_\alpha = -D_\alpha u$ where $u = v^a P_a + \Lambda J$, one gets

$$\begin{aligned}
\delta_{(1,1)} e^a_\alpha &= -\partial_\alpha v^a + \Lambda \epsilon^{ab} e_{\alpha b} - \epsilon^{ab}\omega_\alpha v_b\\
\delta_{(1,1)} \omega_\alpha &= -\partial_\alpha \Lambda.
\end{aligned} \tag{3.4}$$

The Λ transformation is identical to the Lorentz transformation given in (3.1).

If one uses $D_\beta e^a_\alpha = 0$ when $\Lambda = 0$, then the difference between (3.4) and (3.1) is only a Lorentz transformation with a parameter $\tilde{\Lambda} = V^\alpha \omega_\alpha$. It is thus possible to use the ISO(1,1) gauge transformations instead of the diffeomorphism and local Lorentz transformations when applied to $e_{\alpha a}$, ω_α. The covariant derivative is as usual $D_\alpha = \partial_\alpha + [A_\alpha, \;\;]$ and the field strength is

$$F_{\alpha\beta} = [D_\alpha, D_\beta] = D_{[\alpha} e^a_{\beta]} P_a + \partial_{[\alpha}\omega_{\beta]} J \tag{3.5}$$

where $_{[\,]}$ denote anti-symmetrization. Under the gauge transformation given in (3.4) $F_{\alpha\beta}$ transforms as follows:

$$\delta_{(1,1)} F_{\alpha\beta} = (\Lambda \epsilon^{ab} D_{[\alpha} e_{\beta]b} - \epsilon^{ab} \partial_{[\alpha}\omega_{\beta]} v_b) P_a \tag{3.6}$$

Note, however, that the transformations (3.4) do not include the Weyl rescaling.

The ISO(1,1) group which is the isometry group of a flat Minkowski space-time can be generalized to maximally symmetric spaces with positive and negative curvatures, the de-Sitter and anti-de-Sitter spaces, respectively. This is achieved by introducing a cosmological constant, λ. For the de-Sitter space, $\lambda > 0$, the isometry group is SO(2,1) and for $\lambda < 0$ it is SO(1,2). The gauge field is still given by (3.2), but the algebra is now:

$$[P_a, P_b] = \lambda \epsilon_{ab} J \qquad [J, J] = \ 0, \qquad [J, P_a] = \epsilon_{ab} P^b. \tag{3.7}$$

The invariant quadratic form which is consistent with a non-degenerate Casimir operator is

$$< J, J >= 1, \qquad\qquad < P_a, P_b >= \lambda \delta_{ab}. \tag{3.8}$$

For positive λ we can rescale $P_a \to \frac{P_a}{\sqrt{\lambda}}$ and just take $\lambda = 1$. For negative λ one can rescale with $\sqrt{|\lambda|}$ and set $\lambda = -1$. As will be clarified later, we will be interested only in $\lambda = 1$; so from here on we will discuss only this case. The transformation laws of the gauge fields are

$$\begin{aligned} \delta_{(2,1)} e^a_\alpha &= -\partial_\alpha v^a + \Lambda \epsilon^{ab} e_{\alpha b} - \epsilon^{ab} \omega_\alpha v_b, \\ \delta_{(2,1)} \omega_\alpha &= -\partial_\alpha \Lambda - \epsilon^{ab} e_{\alpha a} v_b. \end{aligned} \tag{3.9}$$

The field-strength, which is now

$$F_{\alpha\beta} = [D_\alpha, D_\beta] = D_{[\alpha} e^a_{\beta]} P_a + (\partial_{[\alpha} \omega_{\beta]} + \epsilon^{ab} e_{\alpha a} e_{\beta b}) J, \tag{3.10}$$

transforms under SO(2,1) as follows:

$$\delta_{(2,1)} F_{\alpha\beta} = (\Lambda \epsilon^{ab} D_{[\alpha} e_{\beta]b} - \epsilon^{ab} \partial_{[\alpha} \omega_{\beta]} v_b) P_a + \epsilon^{ab} D_\alpha e_{\beta a} v_b J. \tag{3.11}$$

The "topological" ISO(1,1) or SO(2,1) transformations are now different from the "topological" gravitational transformations given in (3.1). The former are simply

$$\delta A_\alpha = \theta_\alpha \Longrightarrow \quad \delta e^a_\alpha = \theta^a_\alpha, \qquad \delta \omega_\alpha = \tilde{\theta}_\alpha. \tag{3.12}$$

The difference is obviously due to the fact that in the ISO(1,1) and SO(2,1) approach we, apriori, treat $e_{\alpha a}$ and ω_α as independent gauge fields. We expect that the relation $\omega_\alpha(e_{\alpha a})$ will emerge from the equations of motion. In the standard geometric picture of space-time the condition for having no torsion relates ω_α and $e_{\alpha a}$.

The next stage in defining TG after specifying the fields and the symmetries, is choosing the action. We advocate the Lagrangian $\mathcal{L} = 0$ modulo topological invariants and the elimination of auxiliary fields. However, we believe that unless there is a topological invariant expressed in terms of the given fields a non-trivial TQFT cannot be constructed. There must be some topological quantity left invariant which constrains the global properties of the ghost fields, for otherwise the gauge field could be completely transformed away. For example, in TYM the instanton number must be left invariant, and in two dimensional gravity the Euler number must be left invariant. We, thus, prefer to construct a topological invariant as our original Lagrangian. In two dimensional gravity the natural invariant is the Einstein-Hilbert action:

$$I_0 = \int d^2x \sqrt{g} R. \tag{3.13}$$

This can be re-expressed in terms of $e_{\alpha a}$ and ω_α as follows:

$$\begin{aligned} \mathcal{L}_0 = \sqrt{g}R &= det(e) e^{\alpha a} e^{\beta}_{a} R^{\gamma}_{\alpha\gamma\beta} = det(e) e^{\alpha a} e^{\beta}_{a} R_{\alpha\gamma}{}^{c}{}_{d} e_{\beta c} e^{\gamma d}, \\ &= det(e) e^{\alpha a} e^{\gamma d} \epsilon_{ad} \partial_{[\alpha}\omega_{\gamma]}, \\ &= \frac{1}{2} det(e) det(e^{-1}) \epsilon^{\alpha\beta} \partial_{[\alpha}\omega_{\beta]} = \epsilon^{\alpha\beta}\partial_\alpha \omega_\beta, \end{aligned} \tag{3.14}$$

where we used $R^{ab}_{\alpha\beta} = \epsilon^{ab}\partial_{[\alpha}\omega_{\beta]}$ for the Riemann two form assuming $D_\alpha e^a_\beta = 0$.

The last expression raises two issues: (i) Since we get a total derivative, it looks as if the action is zero for two dimensional manifolds without boundaries ; whereas, on the other hand, it is well known that the Einstein action is in fact the Euler number $8\pi(1-g)$ where g is the genus of the manifold. This is resolved by noting that $\sqrt{g}R$ is only locally a total derivative. In fact it is an element of the second cohomology group of the manifold and measures our inability to construct globally flat coordinates. (ii) The action (3.13) is just a topological abelian action (i.e. the first Chern number):

$$I_0 = \int d^2x \epsilon^{\alpha\beta}\partial_\alpha\omega_\beta = \frac{1}{2}\int d^2x \epsilon^{\alpha\beta}F_{\alpha\beta} = \frac{1}{2}\int F. \tag{3.15}$$

So it may look as if TG is equivalent to the topological Maxwell theory. This statement is incorrect, since the action (3.15) is independent of $e_{\alpha a}$; thus, the relation $D_\alpha e^a_\beta = 0$ cannot emerge as an equation of motion. Hence, TG is not equivalent to the topological Maxwell theory. Nevertheless, we now want to analyze the topological U(1) theory, and later in section 5 we will come back to TG.

4. Topological Abelian Gauge Theory

Leaving aside momentarily TG theory, we now proceed to analyze the action given in (3.15). The field ω_α is now an abelian gauge field which is not related to the two-dimensional metric. This will be a toy model useful for understanding the formalism of topological gauge theories though there is no interesting topology for this case. However, as we have seen it is closely related to gravity, since the two-dimensinal Lorentz group is just U(1). Our initial topological invariant is the first Chern number $c_1 = \frac{1}{2\pi}\int F$. Considering a non-compact Euclidean space (as is often done for instanton applications) one has,

$$\int F = \int d^2x\epsilon^{\alpha\beta}F_{\alpha\beta} = \int_{\delta\Sigma} \omega_\alpha = \oint_{|x|=\infty} \omega_\alpha dx^\alpha, \tag{4.1}$$

where the boundary $\delta\Sigma$ is a circle at infinity. Non-vanishing results for c_1 (i.e. QED instantons) are known to exist for scalar QED with spontaneous symmetry breaking.[16] There are, however, no pure 2-dimensional QED instantons. For our case, even though a priori there is no equation of motion for ω_α, as will be clarified below, the Maxwell equation will emerge as the equation of motion once the topological symmetry is gauge-fixed. Thus, these instanton configurations are not relevant to us. Moreover, we are interested only in compact Riemann surfaces without boundaries. For these manifolds $c_1 \neq 0$ only if the second cohomology, $H^2(\Sigma)$, is nontrivial; i.e. $dF = 0$ (Maxwell's equation) but with $F \neq d\omega$ globally. The relevant configurations are thus non-trivial vector bundle over Σ. In holomorphic coordinates these are the meromorphic vector fields. The c_1 which is also the Euler number measures our inability to construct global holomorphic vector fields and is given by the number of the poles.

We now gauge fix the topological symmetry, $\delta\omega_\alpha = \theta_\alpha$, while maintaining the ordinary U(1) symmetry. This is done following the procedure we introduced for other topological quantum field theories.[2]

$$\begin{aligned}\mathcal{L}^{(1)} = \mathcal{L}^{(1)}_{(GF+FP)} =& \hat{\delta}_{T1}[i\tilde{\chi}(\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta - C + i\tilde{B})], \\ =& i\tilde{B}(\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta - C) - i\tilde{\chi}\epsilon^{\alpha\beta}\partial_\alpha\tilde{\psi}_\beta + \tilde{B}^2,\end{aligned} \tag{4.2}$$

where $(GF + FP)$ stands for gauge fixing and Faddeev-Popov, and the BRST transformation is denoted by $\delta_{T1} = i\epsilon\hat{\delta}_{T1}$ with ϵ a constant anti-commuting parameter. The commuting constant C has the same sign as the Chern number. Under this transformation the gauge field, ω_α, the ghost field, $\tilde{\psi}_\alpha$, the anti-ghost, $\tilde{\chi}$, and the auxiliary field $\tilde{B}$ transform as follows:

$$\begin{aligned}\hat{\delta}_{T1}\omega_\alpha &= \tilde{\psi}, & \hat{\delta}_{T1}\tilde{\psi} &= 0, \\ \hat{\delta}_{T1}\tilde{\chi} &= \tilde{B}, & \hat{\delta}_{T1}\tilde{B} &= 0.\end{aligned} \tag{4.3}$$

Since we have gauge-fixed only one degree of freedom, we expect an additional local symmetry. It is straightforward to check that (4.2) is invariant under the ghost symmetry: $\hat{\delta}_{T1}\tilde{\psi}_\alpha = i\partial_\alpha\tilde{\phi}$ which is the U(1) transformation of a gauge field. We fix this additional symmetry by:

$$\mathcal{L}^{(2)} = \mathcal{L}^{(2)}_{GF+FP} = \hat{\delta}_{T1}[\frac{-i}{2}\tilde{\lambda}\partial_\alpha\tilde{\psi}^\alpha] = \frac{1}{2}\tilde{\lambda}\partial_\alpha\partial^\alpha\tilde{\phi} - i\tilde{\eta}\partial_\alpha\tilde{\psi}^\alpha. \tag{4.4}$$

The BRST transformations of the ghosts $\tilde{\lambda}$, $\tilde{\eta}$ and $\tilde{\phi}$ are:

$$\hat{\delta}_{T1}\tilde{\lambda} = 2\tilde{\eta} \qquad \hat{\delta}_{T1}\tilde{\eta} = 0 \qquad \hat{\delta}_{T1}\tilde{\phi} = 0. \tag{4.5}$$

Note that while $\mathcal{L}^{(1)}$ was expressed in terms of forms, in $\mathcal{L}^{(2)}$ we had to introduce a metric. Therefore, each term of the Lagrangian which does not include $\epsilon^{\alpha\beta}$ is in fact multiplied by $\sqrt{g}$, and the derivatives have to be covariant with respect to the two dimensional diffeomorphisms. The BRST algebra is closed up to a U(1) gauge transformation; namely, $\hat{\delta}^2_{T1}\omega_\alpha = \partial_\alpha\tilde{\phi}$, and $\hat{\delta}^2_{T1} = 0$ on the rest of the fields since they are neutral. Altogether, after eliminating the auxiliary fields, we are left with the following U(1) gauge invariant Lagrangian:

$$
\begin{aligned}
\mathcal{L} = \mathcal{L}^0 + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} = & \frac{1}{2}\epsilon^{\alpha\beta}F_{\alpha\beta} + \frac{1}{4}(\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta - C)^2 - i\tilde{\chi}\epsilon^{\alpha\beta}\partial_\alpha\tilde{\psi}_\beta \\
& - i\tilde{\eta}\partial_\alpha\tilde{\psi}^\alpha + \frac{1}{2}\tilde{\lambda}\partial_\alpha\partial^\alpha\tilde{\phi}.
\end{aligned}
\tag{4.6}
$$

The anti-commuting part of the Lagrangian (on a flat background) can be rewritten as :

$$
\tilde{\chi}\epsilon^{\alpha\beta}\partial_\alpha\tilde{\psi}_\beta + \tilde{\eta}\partial_\alpha\tilde{\psi}^\alpha \equiv \tilde{\chi}^T \not{\partial}\tilde{\psi} \tag{4.7}
$$

where we have denoted the vector $(\tilde{\psi}_o, \tilde{\psi}_1)$ by $\tilde{\psi}$ and the two scalars $(\tilde{\eta}, \tilde{\chi})$ by $\tilde{\chi}$. The operator $\not{\partial}$ is given by,

$$
\not{\partial} = \gamma^\alpha\partial_\alpha
$$

where the gamma matrices are:

$$
\gamma^0 = \sigma_1 \text{ and } \gamma^1 = -\sigma_3, \text{ with } \{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta} \tag{4.8}
$$

We then see that $\not{\partial}$ is just the usual 2-dimensional Dirac operator. Writing eq. (4.7) in holomorphic coordinates we get,

$$
\tilde{\chi}^T \not{\partial}\tilde{\psi} = \tilde{\chi}_+\partial_{\bar{z}}\tilde{\psi}_+ + \tilde{\chi}_-\partial_z\tilde{\psi}_- \tag{4.9}
$$

where $\partial_z = \partial_1 + i\partial_0$, $\tilde{\psi}_\pm = \tilde{\psi}_1 \pm i\tilde{\psi}_0$, $\tilde{\chi}_\pm = \tilde{\chi}_1 \pm i\tilde{\chi}_0$. The anti-commuting zero modes will then be the $2g$ (anti)holomorphic differentials for $\tilde{\psi}$ and the two constant zero modes for the scalars, $\tilde{\chi}$. Interestingly, we then have that the Index$(\not{\partial}) = 2 - 2g$ which is just the Euler number. We will come back to this later when considering the SO(2,1) theory.

For the calculation of the invariants of the theory, we obviously have to gauge-fix the abelian gauge symmetry. To maintain the closure of the BRST algebra , $\hat{\delta}_{T1}^2 = 0$, on each of the fields we have to use a covariant gauge fixing by introducing the familiar $c, \bar{c}, b$ ghosts. After eliminating the auxiliary fields, the Lagrangian (4.6) takes the form:

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \bar{c}\partial_\alpha \partial^\alpha c + (\partial_\alpha \omega^\alpha)^2, \tag{4.10}$$

and we have to modify and add new transformations to (4.3) and (4.5) as follows:

$$\begin{aligned} \hat{\delta}_{T1}\omega_\alpha &= \tilde{\psi}_\alpha + \partial_\alpha c \qquad \hat{\delta}_{T1} c = -\tilde{\phi} \\ \hat{\delta}_{T1}\bar{c} &= b = \partial_\alpha \omega^\alpha. \end{aligned} \tag{4.11}$$

We will now discuss the conserved currents for the action in (4.6) without the topological term $\mathcal{L}^{(0)}$. The BRST Noether current which follows from the latter action is given by:

$$J_\alpha = F_{\alpha\beta}\tilde{\psi}^\beta - \tilde{\eta}\partial_\alpha \tilde{\phi} - \epsilon_{\alpha\beta}\tilde{\chi}\partial^\beta \tilde{\phi}. \tag{4.12}$$

The energy-momentum tensor $T_{\alpha\beta}$ now takes the form,

$$\begin{aligned} T_{\alpha\beta} =& \frac{1}{2}[(F_{\alpha\gamma}F_\beta^\gamma - \frac{1}{4}g_{\alpha\beta}(F_{\gamma\delta}F^{\gamma\delta} + 2C^2)] \\ & - (\partial_\alpha\tilde{\phi}\partial_\beta\tilde{\lambda} + \partial_\alpha\tilde{\lambda}\partial_\beta\tilde{\phi} - g_{\alpha\beta}\partial_\gamma\tilde{\lambda}\partial^\gamma\tilde{\phi}) \\ & + i(\partial_\alpha\tilde{\eta}\tilde{\psi}_\beta + \partial_\beta\tilde{\eta}\tilde{\psi}_\alpha - g_{\alpha\beta}\partial_\gamma\tilde{\eta}\tilde{\psi}^\gamma) \\ =& \{Q, \lambda_{\alpha\beta}\} \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} \lambda_{\alpha\beta} =& i\sqrt{g}[\frac{1}{2}[(F_{\alpha\sigma} + \sqrt{g}\epsilon_{\alpha\sigma}C)\epsilon_\beta^\sigma - \frac{1}{4g}g_{\alpha\beta}(F_{\gamma\delta}\epsilon^{\gamma\delta} + 2\sqrt{g}C)]\tilde{\chi} \\ & + (\tilde{\psi}_\alpha\partial_\beta\tilde{\lambda} + \tilde{\psi}_\beta\partial_\alpha\tilde{\lambda} - g_{\alpha\beta}\tilde{\psi}_\gamma\partial^\gamma\tilde{\lambda}). \end{aligned}$$

Thus $T_{\alpha\beta}$ is a BRST commutator. It is straightforward to check that $D_\alpha T^{\alpha\beta} = 0$.

To check for scale transformation we find that

$$T^{\alpha}_{\alpha} = \frac{1}{4}(F_{\alpha\beta}F^{\alpha\beta} - C^2) \tag{4.14}$$

Therefore, the action is not invariant under local scale transformations. The four-dimensional topological theories are invariant under global scale transformations. This is not the case here, since T^{α}_{α} can not be written as a total derivative, $T^{\alpha}_{\alpha} \neq D_{\alpha}R^{\alpha}$. Thus the action is not invariant under global scaling. Nevertheless, there is a further U(1) ghost number symmetry which will play an important role in the construction of the invariants of the theory. Under this symmetry the fields $(\omega_\alpha, \tilde{\psi}_\alpha, \tilde{\chi}, \tilde{\eta}, \tilde{\phi}, \tilde{\lambda})$ carry the charges (0,1,-1,-1,2,-2).

In constructing the observables we follow the procedure outlined for the Donaldson polynomials in the TYM theory.[1] We first search for a BRST invariant operator O which is not a BRST commutator of another operator O'; namely: $\hat{\delta}_{T1}O = 0$, but $O \neq \hat{\delta}_{T1}O'$. This operator must also be metric independent. Obviously, $\tilde{\phi}$ fulfills this condition. Thus, we take for the zero homology cycle $W_0^{(2n)} = \tilde{\phi}^n$ where the superscript of W indicates the ghost number . The associated observable is $I_0 =< W_0^{(2n)} >$. We can now create the chain of higher homology invariants as follows:

$$\begin{aligned} 0 &= i\{Q, W_0^{(2n)}\}, & W_0^{(2n)} &= \tilde{\phi}^n, \\ dW_0^{(2n)} &= i\{Q, W_1^{(2n-1)}\}, & W_1^{(2n-1)} &= n\tilde{\phi}^{n-1}\tilde{\psi}, \\ dW_1^{(2n-1)} &= i\{Q, W_2^{(2n-2)}\}, & W_2^{(2n-2)} &= n\tilde{\phi}^{n-1}F + n(n-1)\tilde{\phi}^{n-2}\tilde{\psi}\wedge\tilde{\psi}, \\ dW_2^{(2n-2)} &= 0. \end{aligned} \tag{4.15}$$

The metric independent observables then take the form $I_k^{(2n)} = \int_{\gamma_k} W_k^{(2n)}$ where γ_k is a k-dimensional homology cycle. For the simplest case ($n = 1$) we get $I_0^{(2)} =< \tilde{\phi} >$, $I_1^{(1)} = \oint \tilde{\psi}$ and $I_2 = \int_\Sigma F = 2\pi c_1$. Contrary to TYM, in the abelian theory the $(\tilde{\phi}, \tilde{\lambda})$ system does not have a potential and thus has constant zero modes. We, therefore, believe that all the invariants, $< W_0^{(2n)} >$, vanish.

By noting that the $(\bar{c}, c)$ system is the same as the $(\tilde{\phi}, \tilde{\lambda})$ system apart from the different statistics, we see that in the partition function the zero modes of the two systems will cancel. However, for $< W_0 >$ the $(\bar{c}, c)$ zero modes will lead to a vanishing result. Just as in the case of the Donaldson invariants, non-zero observables are those expectation values of operators which can absorb the zero modes of the anti-commuting ghost, $\tilde{\psi}$. The only operator obeying this condition is $\prod_{i=1}^{g}(\int_\Sigma \tilde{\psi} \wedge \tilde{\psi})$ where g is the genus of the Riemann surface. This operator cancels the $2g$ zero modes of $\tilde{\psi}$. Its expectation value can be computed explicitly. It is easily seen to be independent of the metric:

$$\langle \prod_{i=1}^{g} \int_\Sigma \tilde{\psi} \wedge \tilde{\psi} \rangle = det[\int \tilde{\psi}_i^{(0)} \wedge \tilde{\psi}_j^{(0)}] \frac{det' \not{\partial}}{det'^{1/2} \partial^2}, \tag{4.16}$$

where the $\tilde{\psi}_i^{(0)}$ are the zero modes of $\tilde{\psi}$. We choose the basis $\tilde{\psi}^{(0)} = \sum_i [\theta^i \omega_i(z) + \bar{\theta}^i \bar{\omega}_i(\bar{z})]$ where the $\omega_i(z)$ are abelian differentials and the θ_i are anticommuting parameters. We normalize the differentials by choosing the canonical $H_1(\Sigma)$ basis, a_i, b_i, such that

$$\oint_{a_i} \omega_j = \delta_{ij}. \tag{4.17}$$

Then, the period matrix of the Riemann surface, Σ is given by:

$$\oint_{b_i} \omega_j = \tau_{ij}. \tag{4.18}$$

Using the Riemann bilinear identity for closed 1-forms, ρ_1, ρ_2,

$$\int_\Sigma \rho_1 \wedge \rho_2 = \sum_{i=1}^{g} [\oint_{a_i} \rho_1 \oint_{b_i} \rho_2 - \oint_{b_i} \rho_1 \oint_{a_i} \rho_2], \tag{4.19}$$

it is simple to show that,

$$det[\int_\Sigma \tilde{\psi}_i^{(0)} \wedge \tilde{\psi}_j^{(0)}] = det[\int_\Sigma \omega_i \wedge \bar{\omega}_j] \tag{4.20}$$

where,

$$\int_{\Sigma} \omega_i \wedge \bar{\omega}_j = Im(\tau_{ij}). \tag{4.21}$$

Hence, we have that,

$$\langle \prod_1^g \int_{\Sigma} \tilde{\psi} \wedge \tilde{\psi} \rangle = det'^{1/2}(Im\tau_{ij}) \frac{det' \not\partial}{det'^{1/2}\partial^2}. \tag{4.22}$$

$Det' \not\partial$ is just the determinant of the Dirac operator with all periodic boundary conditions. Since the theory is topological and thus does not depend on the complex structure, we have that,

$$\frac{det' \not\partial}{det'^{1/2}\partial^2} \sim [det(Im\ \tau)]^{-1/2}. \tag{4.23}$$

For $g = 1$, we know that the above determinants are given by:

$$\begin{aligned} det' \not\partial =& (\frac{\tau_2}{2\pi})^{1/2} \left| \frac{\vartheta_1'(0;\tau)}{\eta(\tau)} \right| = \sqrt{\tau}_2 |\eta(\tau)|^2, \\ det'^{1/2}\partial^2 =& \tau_2 |\eta(\tau)|^2, \end{aligned} \tag{4.24}$$

which verifies the general relation given by eq. (4.23). Indeed, $(detIm\tau)^{-1/2}$ is just the zero mode normalization factor. We could have normalized our $\psi_i^{(0)}$ so the their determinant would be one; then the ratio of determinants in (4.23) would also be unity. We thus see that there are no interesting observables in this theory. In fact, the only observables are the identity and the Chern number.

5. The Problems of a Pure Gravity Theory

Since the topological U(1) cannot provide quantum field theoretic expressions for the desired invariants on moduli space, we also want to briefly summarize the difficulties with topological gravity in two dimensions.

We begin with the same topological action as in the U(1) theory. However, the gauge field is now identified with ω_α, the spin connection. We are, therefore, forced to add to the Lagrangian (4.6) terms which emerge from the BRST variation of the $\sqrt{g}$ and the Christoffel symbols. For example, in the first stage of gauge fixing we have to add $[\hat{\delta}_{T1}\sqrt{g}]\tilde{\chi}\tilde{B}$, and in the second stage $\sqrt{g}\tilde{\lambda}[\hat{\delta}_{T1}\Gamma^{\alpha}_{\alpha\beta}]\tilde{\psi}^{\beta}$ where $\Gamma^{\gamma}_{\alpha\beta}$ is the Christoffel connection. The problem is that we cannot invert the relation $\omega_\alpha(e_{\alpha a})$ and express the transformation $\hat{\delta}_{T1}e_{\alpha a}$ in terms of $\tilde{\psi}$. A natural way to resolve this difficulty is to express the topological transformation of ω_α in terms of $\hat{\delta}_{T1}e^a_\alpha = \psi^a_\alpha$. As was given in (3.1), $\hat{\delta}_{Tg}\omega_\alpha = \epsilon^{ab}D_b\psi_{\alpha a} = det(e^{-1})\epsilon^{\beta\gamma}D_\gamma\psi_{\alpha\beta}$. Gauge-fixing to a flat connection, $\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta = 0$, we take,

$$\mathcal{L}^1 = \hat{\delta}_{Tg}[\tilde{\chi}\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta] = \tilde{B}\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta - det(e^{-1})\tilde{\chi}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}D_\alpha D_\delta\psi_{\beta\gamma}. \tag{5.1}$$

Using $\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} = (detg)(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma})$, we get,

$$\begin{aligned}\mathcal{L}^1 &= \tilde{B}\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta - det(e)\tilde{\chi}(D_\alpha D_\beta\psi^{\alpha\beta} - D_\alpha D^\alpha\psi^\beta_\beta),\\ &= \sqrt{g}[(\frac{1}{2}\tilde{B}R - \tilde{\chi}(D_\alpha D_\beta\psi^{\alpha\beta} - D_\alpha D^\alpha\psi^\beta_\beta)],\end{aligned} \tag{5.2}$$

which is exactly the same gauge fixed action for pure gravity that we derived in the past.[5] The gauge fixing of the additional ghost symmetry will also be the same. The BRST algebra is closed up to a diffeomorphism and Lorentz transformation, provided the "ghost for ghost" ϕ^a ($\hat{\delta}_{Tg}\psi^a_\alpha = D_\alpha\phi^a$) is not a BRST scalar, $\hat{\delta}_{Tg}\phi^a = e^\alpha_b\psi^a_\alpha\phi^b$. This, as was explained in the introduction, is the source of the inability to derive nontrivial topological invariants. Different gauge-fixing procedures such as $\hat{\delta}_{Tg}[\epsilon^{\alpha\beta}\tilde{\chi}_\beta\omega_\alpha]$ or those presented in reference [9], cannot escape $\hat{\delta}_{Tg}\tilde{\phi}_\alpha \neq 0$.

At this stage it appears that we have reconfirmed the statements made in [9] that TG does not contain a quantum field theoretic realization of the Mumford classes and that the only topological invariant operator is the identity. This means that TG (in both two and four dimensions) is an empty quantum theory. What is the reason for this situation? Technically, it is due to the fact that while in TQFT the zero homology invariant is a scalar, here it is a vector. This is obvious because the BRST algebra is closed up to a diffeomorphism whose parameter is a vector.

Since both the topological U(1) and the TG theories cannot provide invariants on Riemann moduli space, we must look elsewhere. We showed earlier that there is an equivalence between the ISO(1,1) (or SO(2,1) for $g \geq 2$) gauge transformations and the diffeomorphism and Lorentz transformations of ω_α and e^a_α. It is, therefore, natural to ask whether the topological ISO(1,1) and SO(2,1) non-abelian theories can save the day. The analysis of the topological $ISO(1,1)$ will be the subject of next section.

6. Topological ISO(1,1) and g=1 Riemann Surfaces

This section will be devoted to the topological ISO(1,1) theory. This will provide a description of the topology of the genus one moduli space ($\mathcal{M}_1$), since ISO(1,1) is the isometry group of flat two-dimensional Minkowski space. We know from mathematicians that the only homology groups of $\mathcal{M}_1$ are torsion groups.[17]⋆ We do not expect a quantum field theory to probe torsion (unless it is through global anomalies). Thus, interesting observables are not expected for the ISO(1,1) theory. Nevertheless, let us see what we get from the quantum field theory approach; this will be useful for comparing with the richer theory of SO(2,1) for higher genus.

⋆ A torsion group is an abelian group in which every element has a finite order; e.g. Z_N.

We begin with the $ISO(1,1)$ gauge field, $A_\alpha = e^a_\alpha P_a + \omega_\alpha J$, which was discussed in section 3. Recall, that $\mathcal{L}^{(0)}$, the first Chern number, is invariant under the topological transformation of the gauge field, A_α,

$$\hat{\delta}_{T(1,1)} A_\alpha = \Psi_\alpha \qquad \Longrightarrow \hat{\delta}_{T(1,1)} e^a_\alpha = \psi^a_\alpha, \qquad \hat{\delta}_{T(1,1)} \omega_\alpha = \tilde{\psi}_\alpha. \tag{6.1}$$

The strategy for eliminating the redundent (local) degrees of freedom is the usual one for topological theories, namely gauge fixing the topological symmetry while maintaining the ISO(1,1) local gauge symmetry. Since we are to describe $\mathcal{M}_g$, we gauge fix to configurations having a flat connection: $F_{\alpha\beta} = 0$. This implies that $\epsilon^{\alpha\beta}\partial_\alpha \omega_\beta = 0$, and $\epsilon^{\alpha\beta} D_\alpha e^a_\beta = 0$. This configuration of gauge fields can be simply used to define a covariantly conserved metric with zero curvature. Since we are dealing with compact surfaces, this describes a torus ($g = 1$). Let us now write the topological gauge-fixed Lagrangian,

$$\begin{aligned} \mathcal{L}^{(1)} = \frac{1}{2}[\epsilon^{\alpha\beta}\chi F_{\alpha\beta}] =& \hat{\delta}_{T(1,1)}[i\tilde{\chi}(\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta) + \epsilon^{\alpha\beta}\chi_a D_\alpha e^a_\beta] \\ =& i[(\tilde{B}\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta + \epsilon^{\alpha\beta}B_a D_\alpha e^a_\beta) \\ & - i(\tilde{\chi}\epsilon^{\alpha\beta}\partial_\alpha\tilde{\psi}_\beta + \chi_a D_\alpha \psi^a_\beta + \epsilon^{ab}\chi_a\tilde{\psi}_\alpha e_{\beta b}) \end{aligned} \tag{6.2}$$

where $\delta_{T(1,1)} = i\epsilon\hat{\delta}_{T(1,1)}$ denotes the topological variation. From here on expressions written in group multiplet will always be traced. The anti-ghost $\chi = \chi^a P_a + \tilde{\chi} J$ transforms as follows:

$$\hat{\delta}_{T(1,1)}\chi = B \Longrightarrow \begin{cases} \hat{\delta}_{T(1,1)}\chi_a = & B_a, \\ \hat{\delta}_{T(1,1)}\tilde{\chi} = & \tilde{B}. \end{cases} \tag{6.3}$$

Note that we took the quadratic form to be $< J, J >= 1$, $< P_a, P_b >= \delta_{ab}$. The corresponding "Casimir" does not commute with P_a. We know that the quadratic Casimir of ISO(1,1) is degenerate. It is just $P_a P^a$. At any rate, we have no choice but to use the above quadratic form. The Lagrangian will still have an ISO(1,1) gauge symmetry, but χ will transform by a nonstandard gauge transformation. The gauge transformations of the fields are given by eq. (6.9). These complications will not appear in the higher genus SO(2,1) theory which does have a non-degenerate quadratic Casimir.

Since the topological symmetry was only partially fixed, we expect to have an additional ghostly symmetry. Because the BRST algebra will close up to an ISO(1,1) gauge transformation we anticipate that the transformation of Ψ under this symmetry will be like that of a gauge field under ISO(1,1), namely,

$$\hat{\delta}_{T(1,1)}\Psi_\alpha = iD_\alpha\Phi \Longrightarrow \begin{cases} \hat{\delta}_{T(1,1)}\tilde{\psi}_\alpha = & i\partial_\alpha\tilde{\phi}, \\ \hat{\delta}_{T(1,1)}\psi^a_\alpha = & i(D_\alpha\phi^a + \epsilon^{ab}\tilde{\phi}e_{\alpha b}). \end{cases} \tag{6.4}$$

By transforming the Lagrangian (6.2) by (6.4), it is straightfrward to verify that it is invariant, provided we simultaneously transform B as follows:

$$\hat{\delta}_{T(1,1)}\tilde{B} = \epsilon^{ab}\phi_a\chi_b, \qquad \hat{\delta}_{T(1,1)}B^a = \epsilon^{ab}\tilde{\phi}\chi_b. \tag{6.5}$$

This again could have been anticipated by the closure of the BRST algebra.

So far we have not introduced a world-sheet metric, since our gauge-fixing action was constructed only from forms. In the second stage of gauge fixing, the introduction of a metric is unavoidable. We have to decide whether our gauge fields, ω_α and e^a_α, will be identified with the world-sheet metric, or simply with ISO(1,1) gauge fields (completely unrelated to the two-dimensional metric). Following the first alternative will mean that the ISO(1,1) gauge symmetry cannot be maintained, since the additional functionals of the metric, $\sqrt{g}$ and the Christoffel symbols, cannot be expressed in an ISO(1,1) covariant way. From the bundle structure we would like the theory to possess (as discussed in section 2) and the above considerations, we emphasize that the gauge fields of ISO(1,1) are not directly related to the world-sheet metric. As in TYM, we simply treat the world-sheet metric as a background field. The second stage of gauge fixing thus takes the form:

$$\mathcal{L}^{(2)} = \mathcal{L}^{(2)}_{GF+FP} = \sqrt{g}\hat{\delta}_{T(1,1)}[\frac{-i}{2}\lambda D_\alpha \Psi^\alpha + \chi B]$$
$$= \frac{1}{2}\lambda D_\alpha D^\alpha \Phi - i\eta D_\alpha \Psi^\alpha + \lambda[\Psi_\alpha\ ,\Psi^\alpha] + B^2 + \chi\hat{\delta}_{T(1,1)}B] \tag{6.6}$$

In terms of the "component" fields the Lagrangian is:

$$\begin{aligned}\mathcal{L}^{(2)} =& \frac{1}{2}\sqrt{g}[\tilde{\lambda}\partial_\alpha\partial^\alpha\tilde{\phi} + \lambda_a D_\alpha D^\alpha \phi^a \\ &+ \sqrt{g}[i\tilde{\eta}\partial_\alpha\tilde{\psi}^\alpha + i\eta^a(D_\alpha\psi^a_\alpha - \epsilon_{ab}e^b_\alpha\tilde{\psi}^\alpha] \\ &+ \sqrt{g}[-i\epsilon^{ab}\lambda_a\psi_{\alpha b}\tilde{\psi}^\alpha + \tilde{B}^2 + B_a B^a + i\epsilon^{ab}\phi_a\chi_b\tilde{\chi}],\end{aligned} \tag{6.7}$$

where we used $\hat{\delta}_{T(1,1)}\lambda = 2\eta, \Longrightarrow \hat{\delta}_{T(1,1)}\tilde{\lambda} = \tilde{\eta}, \qquad \hat{\delta}_{T(1,1)}\lambda_a = \eta_a$. Following our work on TYM, we can insert in equation (6.6) a term $\lambda(D_\alpha\Psi^\alpha - \frac{1}{2}\mathcal{B})$ instead of $\lambda D_\alpha\Psi^\alpha$.[2] In this case in order to conserve the U(1) global ghost number we have to take $\mathcal{B} = [\Phi, \eta]$ and $\hat{\delta}_{T(1,1)}\eta = \frac{i}{2}[\phi, \lambda] \Longrightarrow \hat{\delta}_{T(1,1)}\tilde{\eta} = 0 \qquad \hat{\delta}_{T(1,1)}\eta_a = \frac{i}{2}\epsilon_{ab}(\tilde{\phi}\lambda^b - \tilde{\lambda}\phi^b)$.

This will lead to an additional term in the Lagrangian,

$$\mathcal{L}^{(3)} = \frac{i}{4}\sqrt{g}\hat{\delta}_{T(1,1)}[\epsilon^{ab}\lambda_a(\tilde{\phi}\eta_b - \tilde{\eta}\phi_b)] = \frac{i}{4}(\lambda_a\tilde{\lambda}\phi_b - \epsilon^{ab}\tilde{\eta}\eta_a\phi_b). \tag{6.8}$$

Let us now verify that the BRST algebra is closed up to an ISO(1,1) gauge transformation with a transformation parameter $\Phi = \tilde{\phi}J + \phi^a P_a$. Explicitly, this can be realized directly from the transformations:

$$\begin{aligned}&\hat{\delta}^2_{T(1,1)}A_\alpha = D_\alpha\Phi &&\Longrightarrow \hat{\delta}^2_{T(1,1)}e^a_\alpha = D_\alpha\phi^a - \epsilon^{ab}e_{\alpha b}\tilde{\phi}, && \hat{\delta}^2_{T(1,1)}\omega_\alpha = \partial_\alpha\tilde{\phi}, \\ &\hat{\delta}^2_{T(1,1)}\Psi_\alpha = [\Phi, \Psi] &&\Longrightarrow \hat{\delta}^2_{T(1,1)}\psi^a_\alpha = \epsilon^{ab}(\tilde{\phi}\psi_{\alpha b} - \phi_b\tilde{\psi}_\alpha), && \hat{\delta}^2_{T(1,1)}\tilde{\psi} = 0, \\ & && \hat{\delta}_{T(1,1)}\tilde{\chi} = \epsilon^{ab}\chi_a\phi_b, && \hat{\delta}^2_{T(1,1)}\chi^a = \epsilon^{ab}\tilde{\phi}\chi_b.\end{aligned} \tag{6.9}$$

One may suspect that the third line in (6.9) does not describe a gauge transformation of χ; but, in fact, it does as can be verified from the gauge invariance of the first expression in (6.2).

The invariants of the theory will be discussed in section 8. We now proceed to explore the more interesting case of the higher genus Riemann surfaces.

7. Higher genus Riemann surfaces and the topological SO(2,1) theory

Riemann surfaces of $g \geq 2$ can be described by a constant negative curvature metric (hyperbolic metric). As will soon become obvious, this corresponds to a positive ($\lambda = +1$) cosmological constant. The corresponding symmetry group is that of SO(2,1). Following our discussion in section 3, we proceed to construct the topological SO(2,1) gauge theory. We follow the same procedure that we used in the last section.

First, we fix $F_{\alpha\beta}$ to zero so as to project onto configurations with a flat connection. For SO(2,1) this implies the conditions, $\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta + det(e) \;= 0$ and $\epsilon^{\alpha\beta}D_\alpha e^a_\beta = 0$. Since $\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta = -det(e)R$, we, in fact, gauge fix the curvature to a negative constant ($R = -1$). The resulting Lagrangian is

$$\begin{aligned}\mathcal{L}^{(1)} = \frac{1}{2}\hat{\delta}_{T(2,1)}[\epsilon^{\alpha\beta}\chi F_{\alpha\beta}] =&\hat{\delta}_{T(2,1)}[i\tilde{\chi}(\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta + det\ e) + \epsilon^{\alpha\beta}\chi_a D_\alpha e^a_\beta],\\ =&i\epsilon^{\alpha\beta}[(\tilde{B}(\partial_\alpha\omega_\beta + \frac{1}{2}\epsilon^{ab}e_{\alpha a}e_{\beta b}) + B_a D_\alpha e^a_\beta)\\ &- i[\tilde{\chi}(\epsilon^{\alpha\beta}\partial_\alpha\tilde{\psi}_\beta + \epsilon^{ab}\epsilon^{\alpha\beta}\psi_{\alpha a}e_{\beta b})\\ &+ \chi_a(D_\alpha\psi^a_\beta + \epsilon^{ab}\tilde{\psi}_\alpha e_{\beta b})],\end{aligned} \tag{7.1}$$

where $\delta_{T(2,1)} = i\epsilon\hat{\delta}_{T(2,1)}$ denotes the SO(2,1) topological variation, and the ghost fields $\tilde{\chi}, \chi_a, \tilde{B}$ and B_a are defined in the same way as in section 6. Under the ghost symmetry, Ψ transforms again like the gauge field under SO(2,1) (as in eq 3.11), namely,

$$\hat{\delta}_{T(2,1)}\Psi_\alpha = D_\alpha\Phi \implies \begin{cases}\hat{\delta}_{T(2,1)}\tilde{\psi}_\alpha = & \partial_\alpha\tilde{\phi} + \epsilon^{ab}e_{\alpha a}\phi_b,\\ \hat{\delta}_{T(2,1)}\psi^a_\alpha = & D_\alpha\phi^a + \epsilon^{ab}\tilde{\phi}e_{\alpha b},\end{cases} \tag{7.2}$$

and the auxiliary field B transforms as,

$$\hat{\delta}_{T(2,1)}B = -i[\Phi,\chi] \Longrightarrow \begin{cases} \hat{\delta}_{T(2,1)}\tilde{B} = & \epsilon^{ab}\phi_a\chi_b, \\ \hat{\delta}_{T(2,1)}B^a = & \epsilon^{ab}(\tilde{\phi}\chi_b - \tilde{\chi}\phi_b). \end{cases} \tag{7.3}$$

Similarly, the second stage of gauge fixing is:

$$\begin{aligned} \mathcal{L}^{(2)} = \mathcal{L}^{(2)}_{GF+FP} &= \sqrt{g}\hat{\delta}_{T(2,1)}[\frac{-i}{2}\lambda D_\alpha\Psi^\alpha + \chi B], \\ &= \frac{1}{2}\lambda D_\alpha D^\alpha\Phi - i\eta D_\alpha\Psi^\alpha - \frac{i}{2}\lambda[\Psi_\alpha\ ,\Psi^\alpha] + B^2 - i\chi[\Phi,\chi]. \end{aligned} \tag{7.4}$$

In terms of the "component" fields the Lagrangian is:

$$\begin{aligned} \mathcal{L}^{(2)} =& \frac{1}{2}\sqrt{g}[\tilde{\lambda}\partial_\alpha\partial^\alpha\tilde{\phi} + \lambda_a D_\alpha D^\alpha\phi^a], \\ &+ \sqrt{g}[i\tilde{\eta}(\partial_\alpha\tilde{\psi}^\alpha + \epsilon^{ab}\epsilon^{\alpha\beta}e_{\alpha a}\psi_{\beta b}) + i\eta^a(D_\alpha\psi^a_\alpha - \epsilon_{ab}e^b_\alpha\tilde{\psi}^\alpha)], \\ &+ \sqrt{g}[-i\epsilon^{ab}(\lambda_a\psi_{\alpha b}\tilde{\psi}^\alpha + \frac{1}{2}\tilde{\lambda}\psi^a_\alpha\psi^\alpha_b) + \tilde{B}^2 + B_aB^a \\ &- i\epsilon^{ab}(2\phi_a\chi_b\tilde{\chi} + \tilde{\phi}\chi_a\chi_b)], \end{aligned} \tag{7.5}$$

where we used $\hat{\delta}_{T(2,1)}\lambda = 2\eta$, $\hat{\delta}_{T(2,1)}\tilde{\lambda} = \tilde{\eta}$, $\hat{\delta}_{T(2,1)}\lambda_a = \eta_a$. The additional term in the Lagrangian which was introduced in the last section for ISO(1,1), now takes the form

$$\begin{aligned} \mathcal{L}^{(3)} &= \frac{i}{4}\sqrt{g}\hat{\delta}_{T(2,1)}(\lambda[\Phi,\eta]) = \frac{i}{4}\sqrt{g}(2\eta[\Phi,\eta] - \frac{i}{2}\lambda[\Phi,[\Phi,\lambda]]), \\ &= \frac{i}{4}\sqrt{g}\epsilon^{ab}(\tilde{\phi}\tilde{\eta}_a\tilde{\eta}_b + 2\tilde{\eta}\eta_a\phi_b) + \frac{\sqrt{g}}{8}[(\tilde{\phi}^2\lambda_a\lambda^a - \tilde{\lambda}^2\phi_a\phi^a - (\epsilon^{ab}\phi_a\lambda_b)^2], \end{aligned} \tag{7.6}$$

using that $\hat{\delta}_{T(2,1)}\eta = [\Phi,\lambda]$.

We can combine these expressions to write the total Lagrangian, $\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)}$, and collect the kinetic terms:

$$\begin{aligned} \mathcal{L}_{kin} =& \sqrt{g}[(\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta)^2 + (\partial_{[\alpha}e^a_{\beta]})^2] \\ &+ i\sqrt{g}[-(\tilde{\chi}\epsilon^{\alpha\beta}\partial_\alpha\tilde{\psi}_\beta + \chi_a\partial_\alpha\psi^a_\beta) + (\tilde{\eta}\partial_\alpha\tilde{\psi}^\alpha + \eta^a\partial_\alpha\psi^a_\alpha)] \\ &+ \frac{1}{2}\sqrt{g}[\tilde{\lambda}\partial_\alpha\partial^\alpha\tilde{\phi} + \lambda_a\partial_\alpha\partial^\alpha\phi^a]. \end{aligned} \tag{7.7}$$

The gauge interaction terms can be written as $\mathcal{L}_{gauge} = \sqrt{g}(J^{\alpha}_{(A)} A_\alpha)$ where

$$
\begin{aligned}
J^{\alpha}_{(A)} =& [A^\beta, ([A^\alpha, A_\beta] + 2\partial^{[\alpha} A_{\beta]})] \\
& - i([\eta, \psi^\alpha] + \epsilon^{\alpha\beta}[\chi, \psi_\beta]) + \frac{1}{2}([\Phi, \partial^\alpha \lambda] + [\partial^\alpha \Phi, \lambda] + [[A^\alpha, \Phi], \lambda]) \\
=& \{\epsilon_{ab} e^{\beta a}[2\partial^{[\alpha} e^b_{\beta]} + \epsilon^{cb}(\omega^\alpha e_{\beta c} - e^\alpha_c \omega_\beta) \\
& - i\epsilon^{ab}(\eta_a \psi^\alpha_b + \epsilon^{\alpha\beta}\chi_a \psi_{\beta b} + [\partial^\alpha(\phi_a \lambda_b) + (\omega^\alpha \phi_a - e^\alpha_a \tilde{\phi})\lambda_b])\} J \\
& \{+\epsilon^{ab}[\omega^\beta(\omega^\alpha e^c_\beta - e^{\alpha c}\omega_\beta)\epsilon_{ca} - e^{\beta a}\epsilon^{cd} e^\alpha_c e_{\beta d} + 2(\omega^\beta \partial^{[\alpha} e^a_{\beta]} - e^{\beta a}\partial^{[\alpha}\omega_{\beta]}] \\
& + \epsilon^{ab}[-(\tilde{\eta}\psi^\alpha_a - \tilde{\psi}^\alpha \eta_a) + \epsilon^{\alpha\beta}(\tilde{\chi}\psi_{\beta a} - \chi_a \tilde{\psi}_\beta) \\
& + \partial^\alpha(\tilde{\phi}\lambda_a) - \partial^\alpha(\tilde{\lambda}\phi_a) + \epsilon^{bd} e^\alpha_d \phi_c \lambda_a - \tilde{\lambda}(\omega^\alpha \phi_a - e^\alpha_a \tilde{\phi})]\} P_b
\end{aligned}
\tag{7.8}
$$

and the other interactions are,

$$
\begin{aligned}
\mathcal{L}_{int} =& \sqrt{g}[-i\epsilon^{ab}(\lambda_a \psi_{\alpha b}\tilde{\psi}^\alpha + \frac{1}{2}\tilde{\lambda}\psi^a_\alpha \psi^\alpha_b) - i\epsilon^{ab}(2\phi_a \chi_b \tilde{\chi} + \tilde{\phi}\chi_a \chi_b)], \\
=& \frac{i}{4}\sqrt{g}\epsilon^{ab}(\tilde{\phi}\tilde{\eta}_a \tilde{\eta}_b + 2\tilde{\eta}\eta_a \phi_b) + \frac{\sqrt{g}}{8}[\tilde{\phi}^2 \lambda_a \lambda^a - \tilde{\lambda}^2 \phi_a \phi^a - (\epsilon^{ab}\phi_a \lambda_b)^2],
\end{aligned}
\tag{7.9}
$$

The derivation of the Noether current J^{BRST}_α and the energy-momentum tensor is similar to the derivation for the topological abelian theory (4.12-4.14) and therefore the same conclusions hold here also. The BRST algebra is closed up to an SO(2,1) gauge transformation with a parameter $\Phi = \tilde{\phi} J + \phi^a P_a$. This can be realized directly from the transformations:

$$
\begin{aligned}
\hat{\delta}^2_{T(2,1)} A_\alpha =& D_\alpha \Phi \Longrightarrow \begin{cases} \hat{\delta}^2_{T(2,1)} e^a_\alpha = & D_\alpha \phi^a - \epsilon^{ab} e_{\alpha b}\tilde{\phi}, \\ \hat{\delta}^2_{T(2,1)} \omega_\alpha = & \partial_\alpha \tilde{\phi} + \epsilon^{ab} e^a_\alpha \phi_b, \end{cases} \\
\hat{\delta}^2_{T(2,1)} \Psi_\alpha =& [\Phi, \Psi_\alpha] \Longrightarrow \begin{cases} \hat{\delta}^2_{T(2,1)} \psi^a_\alpha = & \epsilon^{ab}(\tilde{\phi}\psi_{\alpha b} - \phi_b \tilde{\psi}_\alpha), \\ \hat{\delta}^2_{T(2,1)} \tilde{\psi} = & \epsilon^{ab}\phi_a \psi_{\alpha b}, \end{cases} \\
\hat{\delta}^2_{T(2,1)} \chi =& [\Phi, \chi] \Longrightarrow \begin{cases} \hat{\delta}^2_{T(2,1)} \tilde{\chi} = & \epsilon^{ab}\chi_a \phi_b, \\ \hat{\delta}^2_{T(2,1)} \chi^a = & \epsilon^{ab}\tilde{\phi}\chi_b. \end{cases}
\end{aligned}
\tag{7.10}
$$

In general the BRST algebra on any operator is closed as follows,

$$
[\delta_{T(2,1)\epsilon_1}, \delta_{T(2,1)\epsilon_2}] O = -2\epsilon_1 \epsilon_2 [\Phi, O]. \tag{7.11}
$$

Any explicit calculation will require a further gauge fixing of the SO(2,1) gauge symmetry. This is achieved by the standard Fadeev-Popov procedure. In the next section we will elaborate on the topological observables of the theory.

8. Topological Invariants of Moduli Space

We will now discuss the quantum field theoretic approach to the topological invariants of moduli space. Before proceeding into an explicit construction of the observables, let us discuss why, in fact, our theory provides a description of the topology of the moduli space of curves. In this context, moduli space is equivalent to the space of Riemann metrics modulo diffeomorphisms. We have chosen a gauge group whose action on the metric is just that of diffeomorphisms. For $g \geq 2$ the group is SO(2,1); for $g = 1$ it is ISO(1,1), and for $g = 0$ it is SO(1,2). Since ISO(1,1) does not have a nondegenerate quadratic Casimir and consequently $\mathcal{M}_1$ has no interesting topology, save for a torsion group of order 12 [17], we will from now on only consider $g \geq 2$.

The structure of our theory is that of a principal SO(2,1) bundle consisting of a projection $\pi : E \to B$. The fiber is the gauge group SO(2,1). From our gauge fixing condition we see that:

$$E = \{\text{flat } \mathrm{SO}(2,1) \text{ connections over } F_g\}$$

where F_g is an oriented surface of genus g. We will now also show that E is equivalent to the space of metrics on F_g. Thus, $B = E/SO(2,1)$ would be homeomorphic to the moduli space. Recall that the gauge fixing condition (i.e. vanishing field strength) on the generators of SO(2,1) was such that:

$$\begin{aligned} D_{[\alpha}e^a_{\beta]} &= \partial_{[\alpha}e^a_{\beta]} + \omega_{[\alpha}e^a_{\beta]} = 0, \\ \partial_{[\alpha}\omega_{\beta]} &= - \, det(e), \end{aligned} \tag{8.1}$$

We can define a covariantly conserved metric by $g_{\alpha\beta} = e^a_\alpha e_{\beta a}$.* From the first

* It should be noted that this is to be distiguished from the world-sheet metric in the quantum field theory.

relation above we can determine ω_α as a function of e^a_α . Then, substituting the expression for $\omega_\alpha(e^a_\beta)$ into the second relation above will give us, as shown earlier,

$$\sqrt{g}R = -\sqrt{g} \implies R = -1. \tag{8.2}$$

This is the condition that we have constant negative curvature. But we know that all Riemann surfaces of $g \geq 2$ can be described by constant negative curvature metrics (i.e. hyperbolic metrics). For $g = 0$, the gauge group is SO(1,2) and we have positive curvature. This then completes the verification that our theory provides for a description of the moduli space.

Let us now compute the dimension of this space using the index of the fermionic operator in our Lagrangian. This is simply: {number of Ψ_α zero modes} – {number of (χ, η) zero modes}. Since Ψ_α is a vector and (χ, η) are scalars, we have that the dimension of moduli space is:

$\mathrm{Dim}\{SO(2,1)\} \times (2g-2) = 6g-6$. This is, of course, a well-known result. There is one unfortunate fact that we must note. Because we always have (χ, η) zero modes, the Ψ_α zero modes do not form a good coordinate basis for moduli space. In TYM Witten was able to construct an explicit mapping of the Donaldson polynomials into differential forms on the instanton moduli space after an integration of the nonzero mode fields. This was possible because his analogous Ψ zero modes (instanton deformations) could be used to parametrize moduli space, since he considered the generic case in which there are no (χ, η) zero modes. In general, the χ zero modes are the obstructions to choosing a Ψ which coordinatizes moduli space.

For the moduli of Riemann surfaces we know that the quadratic differentials are good coordinates. Since we must be content with our Ψ_α and χ, we will glean whatever information we can even though the explicit mapping onto the differential forms on moduli space will not be so easy.

Let us now proceed with the construction of our observables, or topologically invariant correlation functions. This will be very much along the lines of ref.

[1]. Witten showed that the observables will be a ring of differential forms on the background manifold. These will be a sequence of forms of increasing degree beginning with the zero forms constructed from the invariant polynomials of the BRST scalars:

$$
\begin{aligned}
W_0^{(4n)} =& Tr(\Phi^2)^n, \\
\int_\gamma W_1^{(4n-1)} =& Tr(2n \int_\gamma \Psi_\alpha \Phi^{2n-1} dx^\alpha), \\
\int_\Sigma W_2^{(4n-2)} =& Tr(2n \int_\Sigma F_{\alpha\beta} \Phi^{2n-1} dx^\alpha \wedge dx^\beta \\
& + 2n(2n-1) \int_\Sigma \phi^{2n-2} \Psi_\alpha \Psi_\beta dx^\alpha \wedge dx^\beta),
\end{aligned}
\tag{8.3}
$$

where we have $dW_N = \{Q_{BRST}, W_{N+1}\}$, $\{Q, W_0\} = 0$, and $W_0 \neq \{Q, \Lambda\}$. This is then the structure of the generators for the topological invariant polynomials. Mathematicians refer to these as characteristic classes. We see that they are of the form $\int_{\gamma_k} W_k$ where $\gamma_k \in H_k(\Sigma)$ and $W_k \in H^k(\Sigma)$. In the quantum field theory we must choose a combination of these classes such that the fermionic zero modes are absorbed. This involves choosing a product of classes with ghost number $6g-6$. The ghost number of a differential form on the Riemann surface will correspond to the degree of the differential form on moduli space. The observables can be considered as mappings from forms on a Riemann surface to forms on the corresponding moduli space. The Mumford invariants, as mentioned previously, are elements of the even cohomology groups of $\mathcal{M}_g$. They will then be identified with the even ghost number observables . For example, $\kappa_1 \in H^2(\mathcal{M}_g)$ will be identified with $W_2^{(2)}$. It is known that $\kappa_1^{3g-3} \neq 0$; this corresponds to:

$$
\kappa_1^{3g-3} \to \langle \mathrm{Tr} \prod_{i=1}^{3g-3} \int_\Sigma \Psi \wedge \Psi \rangle. \tag{8.4}
$$

The expression above can be computed in the weak coupling limit where it is identical to the computation in the abelian theory (done in section 4). This is an exact result because the theory is independent of the coupling constant. It verifies that $\kappa_1^{3g-3} \neq 0$. The Euler character of moduli space was computed by Harer and Zagier.[18] They found it to be:

$$\chi(\mathcal{M}_g) = -\frac{\varsigma(1-2g)}{2(g-1)} \quad (g > 1) \tag{8.5}$$

where ς denotes the Riemann zeta function. This should now also be computable using quantum field theory. Since the quantum field theory is two dimensional and a weak coupling expansion can be used, the computation should not be too difficult. It appears to be related to a zero point energy. We leave this for future work.

In summary our observables describe the following mappings between cohomology elements on the Riemann surface and those on moduli space:

$$\begin{aligned} W_0^{(2n)} &: H^0(\Sigma) \rightarrow H^{2n}(\mathcal{M}_g), \\ W_1^{(2n-1)} &: H^1(\Sigma) \rightarrow H^{2n-1}(\mathcal{M}_g), \\ W_2^{(2n-2)} &: H^2(\Sigma) \rightarrow H^{2n-2}(\mathcal{M}_g), \end{aligned} \tag{8.6}$$

9. Conclusion

In this paper we have constructed a quantum field theoretic description of the topology of moduli space. The guiding principle has been the study of moduli space topology by using topological gauge theories. These are gauge invariant theories whose Lagrangians are assembled by gauge fixing onto certain classical configurations. They probe the topology of the manifold of classical configurations modulo gauge equivalence (i.e. moduli space).

For the study of the moduli space of curves we chose the gauge group to be the symmetry group of the metric of a Riemann surface of a given genus. This is SO(2,1) for $g \geq 2$, ISO(1,1) for $g = 1$, and SO(1,2) for $g = 0$. Then by gauge-fixing the connection to be flat, we obtained a field theoretic description of $\mathcal{M}_g$.

The previous attempts at obtaining a topological theory for $\mathcal{M}_g$ were based on developing a topological theory of pure gravity.[9] We have seen in this paper that moduli space is described naturally by an SO(2,1) gauge theory on a curved two-dimensional background.

There remain several open problems. Since so little is known about moduli space topology, it would be fruitful to explicitly compute some of the invariants. This should not be too difficult, since we are dealing with a two dimensional field theory about which much is known. There is also a question as to whether there are any global anomalies. We don't know what effect torsion groups in moduli space will have on the quantum field theory. It would also be interesting to extend this work to obtain a description of super-moduli space. We would naively expect that a supersymmetric version of a topological SO(2,1) gauge theory will do the job. This should be a straightforward extension of the current work. It may lead to interesting consequences.

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