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CONSTRAINTS FOR CONFORMAL FIELD THEORIES ON THE PLANE: REVIVING THE CONFORMAL BOOTSTRAP*

DAVID C. LEWELLEN

Stanford Linear Accelerator Center,
Stanford University, Stanford, CA 94309

ABSTRACT

We obtain a number of results by reexamining conformal field theory on the plane, using only elementary mathematics in the process. In particular we rederive the constraints on conformal dimensions found by Vafa, using the crossing symmetry of four-point amplitudes; find stronger, inequality versions of these conditions which constrain the magnitude of the conformal dimensions of primary fields which can appear as intermediate states in a given amplitude; and study the conformal bootstrap perturbatively using the power series expansions for conformal block functions. This latter approach gives constraints of a quite different type, in particular tree level constraints on the central charge.

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1. INTRODUCTION

The study of two-dimensional conformal field theories (CFT's) has flourished in recent years. CFT's have proved useful for the study of infinite dimensional algebras in mathematics, represent many two-dimensional statistical systems at their critical points, and provide the building blocks for the classical solutions of string theories. Not surprisingly, then, considerable attention has focused on the problem of classifying all CFT's. In the seminal work in this field, Belavin, Polyakov and Zamolodchikov (BPZ) gave an essentially complete formulation of the parameters which characterize a CFT, the constraints that these must satisfy, and how these constraints might be solved in principle (the "conformal bootstrap") [1].

A CFT is completely fixed in principle, once we have specified its central charge, the conformal dimensions of its primary fields and the structure constants defining the three-point couplings. The sole nontrivial constraints that these parameters must satisfy is the crossing symmetry of the four-point amplitudes on the plane. These amplitudes are built from functions whose coordinate dependence is determined entirely by the conformal symmetry, the so-called conformal blocks.

Unfortunately, in practice we only know how to calculate the conformal blocks laboriously, order by order in a power series expansion, which in general is not a suitable form for imposing the crossing symmetry constraints. For this reason, interest in the conformal bootstrap has faded and a number of reformulations of the classification problem have been proposed. Probably the most popular of these focuses entirely on the analytic and modular properties of the correlation functions of a particular subset of CFT's, the so-called rational conformal field theories (RCFT) [2]-[10]. Using this approach, Verlinde [3], Vafa [4] and others [5-9] have derived constraints on the central charge and conformal dimensions which may appear in a RCFT as a function of the fusion rule algebra of its primary fields (that is, the knowledge of which primary fields couple to which) and, more recently, Moore and Seiberg have suggested an explicit realization of this "modular geometry" approach in terms of the duality matrices relating different sets of conformal blocks [5].

Despite such advances there has been no definitive progress made along these lines in the general problem of classifying conformal field theories. Indeed some of the simplicity, power and generality of the original conformal bootstrap has been lost along the way. In emphasizing modular geometry we have sacrificed our detailed knowledge that conformal symmetry in fact completely fixes the conformal blocks. In restricting ourselves to RCFT's we are hoping that this subset is in some sense a good approximation to the space of all CFT's; but even if RCFT's are dense in the space of CFT's (as has been suggested [2]) they may not provide any natural or useful parameterization of it, much as an infinitely branched tree can be made dense in the plane but provides a hopelessly awkward description of it.

For these (and other) reasons we believe that the conformal bootstrap still represents the most promising approach to classifying CFTs, and so deserves renewed attention. The chief utility of RCFTs, we believe, lies in providing simpler constraints which might help in constructing and testing new CFT's. In this spirit we present three results in this paper. First, we will rederive the constraints on conformal dimensions which seem to us to be of the most practical use, namely those found by Vafa [4]. We will present this entirely in the language of BPZ, showing how these conditions follow directly and naturally from the crossing symmetry constraint of the conformal bootstrap. Second, using a trick inspired by work of Mathur, Mukhi and Sen [6], we will strengthen Vafa's constraints, turning them into inequalities which, in essence, tell us that the dimensions of the primary fields appearing as intermediate states in any given amplitude must not be "too" big relative to the dimensions of the external fields. For example, if two self-conjugate primary fields, ϕ_1 and ϕ_2 (of the Virasoro algebra or some extension), fuse only to a third primary field, ϕ_3 (and its descendents), then the conformal dimensions must satisfy $\Delta_3 + k/2 = \Delta_1 + \Delta_2$ with k a *non-negative* integer or the CFT will be inconsistent.

Finally, we will show that in some cases the conformal bootstrap can be solved "perturbatively," using the power series expansion of the conformal blocks. The trick is to consider particular amplitudes, such as those involving only a single intermediate state primary field, which are fixed up to a finite number of parameters by their known analytic structure. The unknown parameters are fixed using the first few terms in the relevant conformal block expansions; the crossing symmetry constraints may then be imposed order by order in the expansions. This method gives constraints of a rather different sort than those found by "modular geometry" methods, including tree-level constraints on the central charge. For example: any CFT which contains a dimension 1/2 primary field, ψ, such that the $\psi\psi$ operator product expansion contains only a single Virasoro primary field (and its descendents) must have central charge $c = 1/2$, regardless of the remaining operator content of the theory, or else the $\langle\psi\psi\psi\psi\rangle$ amplitude will not be crossing symmetric.

In section 2 and the beginning of section 3 of this paper we will review in some detail the conformal bootstrap of BPZ and introduce rational conformal field theories and some of their most basic properties. The latter part of section 3 contains the derivation of Vafa's conditions. The new inequality constraints are presented in section 4 and the perturbative treatment of the bootstrap comprises section 5. All of these results follow just from studying the four-point amplitudes of primary fields on the plane. This allows us to avoid the use of sophisticated mathematics which is often required to discuss general N point amplitudes on arbitrary genus Riemann surfaces and characterizes most of the recent literature on this subject. Thus our discussions may also serve as a pedestrian introduction to these more abstract formulations. Indeed, the four-point functions on the plane often represent the simplest nontrivial examples of the general structures they consider and so it is instructive to see constraints arise at this level. We will make comments to this effect throughout, as well as argue why higher genus considerations are of secondary importance in the problem of classifying CFTs. In section 6 we conclude with some discussion and speculation on the classification of CFTs and how this issue impacts on string theory.

2. CONFORMAL FIELD THEORY ON THE PLANE

The bulk of our knowledge of conformal field theory is contained in the classic paper of Belavin, Polyakov and Zamolodchikov (BPZ) [1]. We assume a basic knowledge of this work (see e.g. [11] for a good introduction) and begin by collecting together those formulae and results which we will need for the subsequent discussions.

The operators in a conformal field theory fall into representations of two independent Virasoro algebras L and $\bar{L}$; these algebras are completely specified by the central charges c and $\bar{c}$. Specifically,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{2.1}$$

and the analogous relation for $\bar{L}$. The irreducible representations consist of primary fields, $\phi_i(z, \bar{z})$ with holomorphic and anti-holomorphic conformal dimensions Δ_i and $\bar{\Delta}_i$, satisfying,

$$[L_m, \phi_i(z, \bar{z})] = z^{m+1}\frac{\partial}{\partial z}\phi_i(z, \bar{z}) + \Delta_i(m+1)z^m\phi_i(z, \bar{z}) \tag{2.2a}$$

$$[\bar{L}_m, \phi_i(z, \bar{z})] = \bar{z}^{m+1}\frac{\partial}{\partial \bar{z}}\phi_i(z, \bar{z}) + \bar{\Delta}_i(m+1)\bar{z}^m\phi_i(z, \bar{z}) \tag{2.2b}$$

and descendents of primaries obtained by applying products of $L_m, \bar{L}_n$ operators $(m, n < 0)$ on the primary fields. Following BPZ, z and $\bar{z}$ are treated throughout as independent complex variables; one is restricted to be the complex conjugate of the other only at the very end of a computation. The Virasoro generators are Fourier components of the stress energy tensors $T(z)$ and $\bar{T}(\bar{z})$, e.g.,

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \quad . \tag{2.3}$$

These are not primary fields, but descendents of the identity operator.

The form of the two- and three-point functions of primary fields in a conformal field theory is fixed by demanding invariance under projective transformations, that is the subgroup of the conformal group generated by $\{L_0, L_1, L_{-1}\}$ which smoothly maps the complex plane onto itself. Specifically,

$$\langle\phi_n(z_1,\bar{z}_1)\phi_m(z_2,\bar{z}_2)\rangle = \delta_{nm}(z_1-z_2)^{-2\Delta_n}(\bar{z}_1-\bar{z}_2)^{-2\bar{\Delta}_n} \tag{2.4}$$

$$\begin{aligned}\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle = C_{123}& \\ (z_1-z_2)^{\Delta_3-\Delta_1-\Delta_2}(z_1-z_3)^{\Delta_2-\Delta_1-\Delta_3}(z_2-z_3)^{\Delta_1-\Delta_2-\Delta_3}& \\ (\bar{z}_1-\bar{z}_2)^{\bar{\Delta}_3-\bar{\Delta}_1-\bar{\Delta}_2}(\bar{z}_1-\bar{z}_3)^{\bar{\Delta}_2-\bar{\Delta}_1-\bar{\Delta}_3}(\bar{z}_2-\bar{z}_3)^{\bar{\Delta}_1-\bar{\Delta}_2-\bar{\Delta}_3}& \quad .\end{aligned} \tag{2.5}$$

In writing (2.4) we have normalized the primary fields and defined them such that they are self-conjugate (which we can always do) in order to keep our notation as uncluttered as possible; in particular, with this choice the structure constants C_{ijk} are completely symmetric in their indices. This definition is not always the most natural or convenient for a given conformal field theory so we will indicate, when appropriate, how our final results are modified when fields are not self-conjugate.

It is often convenient to calculate correlation functions within an operator framework with Hamiltonian $L_0+\bar{L}_0$, and "in" and "out" vacuum and primary states defined by

$$L_n|0\rangle = \bar{L}_n|0\rangle = 0 \quad n \geq -1 \tag{2.6a}$$

$$\langle 0|L_n = \langle 0|\bar{L}_n = 0 \quad n \leq 1 \tag{2.6b}$$

$$|i\rangle = \phi_i(0)|0\rangle \tag{2.7a}$$

$$\langle i| = \lim_{z,\bar{z}\to\infty} \langle 0|\phi_i(z,\bar{z})z^{2L_0}\bar{z}^{2\bar{L}_0} \quad . \tag{2.7b}$$

Then (2.4) and (2.5) become

$$\langle n|m\rangle = \delta_{nm} \tag{2.8}$$

$$\langle \ell|\phi_m(z,\bar{z})|n\rangle = C_{\ell mn} z^{\Delta_\ell - \Delta_m - \Delta_n} \bar{z}^{\bar{\Delta}_\ell - \bar{\Delta}_m - \bar{\Delta}_n} \quad , \tag{2.9}$$

and the states of the form

$$L_{-k_1} L_{-k_2} \ldots L_{-k_N} \bar{L}_{-\bar{k}_1} \ldots \bar{L}_{-\bar{k}_M} |i\rangle \quad ; \quad \begin{array}{c} k_1 \leq \ldots \leq k_N \\ \bar{k}_1 \leq \ldots \leq \bar{k}_M \end{array} \tag{2.10}$$

form a basis for the space of descendents of the primary state $|i\rangle$. Note that $\{L_0, L_1, L_{-1}\}$ is the largest subgroup of the Virasoro algebra which leaves the vacuum invariant.

As BPZ showed, a conformal field theory is completely specified once the central charges, $c, \bar{c}$, dimensions of the primary fields, $\Delta_i, \bar{\Delta}_i$, and structure constants for the primary fields, C_{ijk}, are given. Any correlation function involving descendent states [or equivalently powers of $T(z)$ or $\bar{T}(\bar{z})$] can be reduced to one involving primary fields alone using (2.1)–(2.3) and (2.6). Correlation functions of more than three primary fields can be reduced to sums (generally infinite) of products of three-point functions using the operator product expansions of primary fields which, apart from the C_{ijk}, are completely fixed by conformal invariance. In practice, it is the inherent difficulties involved in this procedure which prevents the complete solution of generic conformal field theories. We will examine this in some detail for the four-point function; this will be the chief object of our attention throughout the present work.

Invariance of the vacuum under projective transformations allows us to write an arbitrary correlation function of four primary fields in the form

$$\begin{aligned}&\langle\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\phi_3(z_3,\bar z_3)\phi_4(z_4,\bar z_4)\rangle\\&=\prod_{i<j}(z_i-z_j)^{\gamma/3-\Delta_i-\Delta_j}(\bar z_i-\bar z_j)^{\bar\gamma/3-\bar\Delta_i-\bar\Delta_j}Y(x,\bar x)\quad ,\end{aligned}\tag{2.11}$$

with $\gamma\equiv\sum_{i=1}^4\Delta_i$ and Y a function of the anharmonic ratios

$$x\equiv\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}\qquad\bar x\equiv\frac{(\bar z_1-\bar z_2)(\bar z_3-\bar z_4)}{(\bar z_1-\bar z_3)(\bar z_2-\bar z_4)}\quad .\tag{2.12}$$

The factors in front of Y in (2.11) are fixed only up to multiplication by factors of x or $\bar x$; We have chosen the specific form to be completely symmetric in the four coordinates $(z_i,\bar z_i)$. In the operator formulation,

$$\begin{aligned}Y(x,\bar x)&\doteq x^{\Delta_3+\Delta_4-\gamma/3}(1-x)^{\Delta_2+\Delta_3-\gamma/3}\bar x^{\bar\Delta_3+\bar\Delta_4-\bar\gamma/3}(1-\bar x)^{\bar\Delta_2+\bar\Delta_3-\bar\gamma/3}\\&\langle1|\phi_2(1,1)\phi_3(x,\bar x)|4\rangle\quad .\end{aligned}\tag{2.13}$$

We can re-express Y as an infinite sum of products of three-point functions by inserting the identity operator in (2.13) in the form $I=\sum\mid s\rangle\langle s\mid$ where the sum runs over all states (properly normalized). In conformal field theory all states are primaries or their descendents given by (2.10), so (2.13) becomes

$$\begin{aligned}Y(x,\bar x)=A(x,\bar x)&\sum_{\substack{\text{primaries}\\ \text{P}}}\sum_{\{k_i\}\{k'_j\}\{\bar k_i\}\{\bar k'_j\}}\\&\langle1|\phi_2(1,1)\mathcal{L}_{-\{k_i\}}\bar{\mathcal{L}}_{-\{\bar k_i\}}|p\rangle\langle p|\mathcal{L}_{\{k'_j\}}\bar{\mathcal{L}}_{\{\bar k'_j\}}\phi_3(x,\bar x)|4\rangle\\&(\mathcal{M}^p)^{-1}_{\{k_i\}\{k'_j\}}(\mathcal{M}^p)^{-1}_{\{\bar k_i\}\{\bar k'_j\}}\quad ,\end{aligned}\tag{2.14}$$

where A is the prefactor in (2.13) and

$$\begin{aligned}\mathcal{L}_{-\{k_i\}}&\equiv L_{-k_1}L_{-k_2}\cdots L_{-k_N}\quad 0<k_1\le k_2\le\cdots\le k_N\\ \mathcal{L}_{\{k_j\}}&\equiv L_{k_M}\cdots L_{k_2}L_{k_1}\quad 0<k_1\le k_2\le\cdots\le k_M\end{aligned}\tag{2.15}$$

$$\mathcal{M}^p_{\{k_i\}\{k_j'\}} \equiv \langle p|\mathcal{L}_{\{k_j'\}}\mathcal{L}_{-\{k_i\}}|p\rangle \tag{2.16}$$

and the analogous definitions for the anti-holomorphic quantities labeled by $\bar{k}$. The three-point functions appearing in (2.14) can be explicitly evaluated using (2.2), (2.6) and (2.9). The difficult part of the calculation is computing and inverting the matrix $\mathcal{M}$ in (2.16) which arises because the basis of descendent states (2.10) is not orthonormal. Note that

$$\mathcal{M}^p_{\{k_i\}\{k_j'\}} = 0 \quad \text{unless} \sum_{i=1}^{N} k_i = \sum_{j=1}^{M} k_j' \quad , \tag{2.17}$$

so that $\mathcal{M}$ is block diagonal and may be inverted "level by level" with the level given by $\sum k_i$. This corresponds to a computation of (2.14) order by order in powers of x. Also observe that for fixed p, (2.14) is proportional to the structure constants C_{12p}, C_{34p} and is a product of holomorphic and anti-holomorphic terms. Thus,

$$Y(x,\bar{x}) = \sum_p C_{12p}\ C_{34p}\ f_{12,34}(p|x)\bar{f}_{12,34}(p|\bar{x}) \quad , \tag{2.18}$$

with the "conformal block" $f_{12,34}$ given explicitly by

$$\begin{aligned} f_{12,34}(p|x) \equiv (C_{12p}\ C_{34p})^{-1} x^{\Delta_3+\Delta_4-\gamma/3}(1-x)^{\Delta_2+\Delta_3-\gamma/3} \sum_{\{k_i\}\{k_j'\}} \\ \langle 1|\phi_2(1,1)\mathcal{L}_{-\{k_i\}}|p\rangle (\mathcal{M}^p)^{-1}_{\{k_i\}\{k_j'\}} \langle p|\mathcal{L}_{\{k_j'\}}\phi_3(x,1)|4\rangle \end{aligned} \tag{2.19}$$

and analogously for the "anti-block" $\bar{f}$. In principle, $f_{12,34}(p|x)$ is completely specified given $c, \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_p$; In practice, for generic values of the conformal dimensions, we only know the first few terms in the power series which, plugging

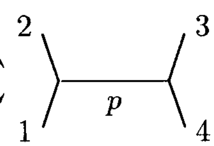
(2.9) into (2.19), begins

$$f_{12,34}(p|x) = x^{\Delta_p - \gamma/3}[1 + \mathcal{O}(x)] \quad . \tag{2.20}$$

The power series expansion will have a finite radius of convergence if the operator product expansions of the theory do; the convergence of the OPE's has been argued to follow from conformal invariance, and is assumed in the bootstrap approach. Note also that the conformal blocks are analytic functions of x except at the possible branch points $x = 0, 1, \infty$ and on the branch cuts which join them; more precisely, each block is a single branch of some analytic function. $x = 0, 1, \infty$ correspond to those points where two or more of the operators defining our amplitude fuse together.

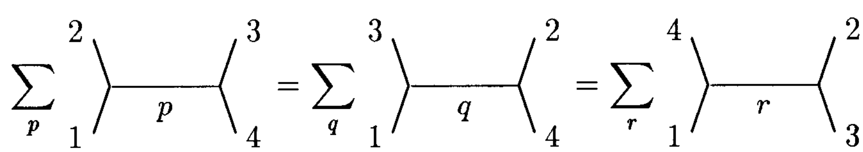
In calculating the general four-point amplitude we have chosen a particular decomposition, (2.14), which can be represented diagrammatically as

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \sum_p$$



(2.21)

There are, however, two equally good alternative decompositions we could use which, if our theory is consistent, had better give the same results. This demand is the crossing symmetry constraint of BPZ:



(2.22)

or, explicitly in terms of conformal blocks,

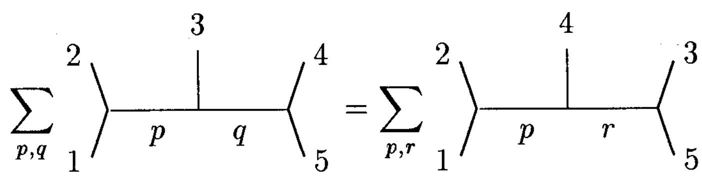
$$\sum_p C_{12p}\, C_{34p}\, f_{12,34}(p|x)\bar{f}_{12,34}(p|\bar{x}) = \sum_q C_{13q}\, C_{24q}\, f_{13,24}\left(q|\frac{1}{x}\right)\bar{f}_{13,24}\left(q|\frac{1}{\bar{x}}\right)$$
$$= \sum_r C_{14r}\, C_{23r}\, f_{14,23}(r|1-x)\bar{f}_{14,23}(r|1-\bar{x}) \quad , \tag{2.23}$$

where we have used (2.12) to re-express,

$$\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)} = \frac{1}{x} \quad , \frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_3)(z_2-z_4)} = 1-x \quad . \tag{2.24}$$

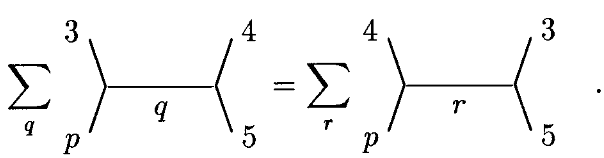
There are no additional factors in (2.23) because of the symmetric choice of prefactors in (2.11); this will prove somewhat more convenient than the asymmetric convention for the conformal blocks defined by BPZ.

It is not difficult to see that if all of the four-point amplitudes in a theory are crossing symmetric, then in fact there is a unique result for every N-point amplitude regardless of how we choose to decompose it. For example,



(2.25a)

follows immediately from the four-point result



(2.25b)

Thus the problem of constructing and classifying all consistent conformal field theories on the plane boils down to calculating the conformal blocks for arbitrary conformal dimensions and central charge and determining the possible complete sets of these which satisfy (2.23) for some values of the structure constants, C_{ijk}. Unfortunately, the general conformal blocks are not known and the power series expansions of them are of little direct use for imposing crossing symmetry; the expansion for $f_{12,34}(x)$ does not in general converge for x near 1 or ∞ about which the series for $f_{14,23}(1-x)$ and $f_{13,24}(1/x)$ converge, respectively. We will address, and is some special cases be able to sidestep, this difficulty in section 5.

A brief aside on the subject of correlation functions on higher genus surfaces is in order. The work of Cardy, in particular, has demonstrated that physically meaningful constraints on the operator content of a CFT arise from the demand that it can be consistently formulated on a torus [12]. One might worry that the conformal bootstrap is incomplete, ignoring as it does everything but the correlation functions on the plane. This is correct, but in practice the genus one restrictions are of only secondary importance and can be satisfied *comparatively* easily given a *complete* solution for a CFT on the plane. Without entering into details, the necessary new ingredient on the torus is the invariance of correlation functions under the modular group generated by $\tau \to \tau+1$ and $\tau \to -1/\tau$, where τ is the modular parameter of the torus. In practice, invariance under $\tau \to \tau+1$ is insured if all of the operators in the CFT have integer spin ($\Delta - \bar{\Delta} \in Z$). Invariance under $\tau \to -1/\tau$ serves as a completeness condition, necessitating that any operator which is mutually local with respect to all of the operators in the theory (that is no branch cuts appear in any OPE's) is also included. We know of no rigorous proof of this last statement, but it has so far proved to be the case that any CFT completely consistent on the plane has been formulated consistently on the torus, perhaps after some suitable truncation or filling out of the operator spectrum. There is no denying the efficacy of examining modular invariance on the torus for isolating a complete, consistent set of operators in a given CFT; we only wish to make the point that consistency on the plane — guaranteed by the conformal bootstrap — is the primary consideration in classifying all CFTs. Finally let us note that demanding a consistent formulation of a CFT on a Riemann surface of genus greater than one gives no new conditions after we have guaranteed consistency on the plane and torus. This has been argued for some time in the context of string theories (see e.g. [13] and references therein) and has recently been shown more rigorously by Moore and Sieberg for RCFTs [5] and, independently, by Sonoda [14] for CFTs.

3. CONSTRAINTS ON RATIONAL CONFORMAL FIELD THEORIES

The most thoroughly studied class of solutions to the conformal bootstrap equations (2.23), and indeed the only one studied in BPZ, is the set of discrete minimal models. These include all of the unitary theories with $c < 1$ [15] as well as a countably infinite set of nonunitary models which nonetheless possess well-behaved crossing symmetric amplitudes (and are of physical relevance in two-dimensional critical phenomena). The possible modular invariant combinations of minimal model characters have been found and fall into an ADE classification: two infinite series and three exceptional invariants [16]. Fateev and Dotsenko obtained the structure constants for the diagonal (A series) modular invariants by explicitly solving the crossing symmetry constraints [17]. Each of these diagonal minimal theories contains only a finite number of primary fields, each of which is uniquely specified by its conformal dimension. As a consequence, all of the necessarily finite number of conformal blocks (or antiblocks) appearing in a particular factorization of any four-point amplitude are linearly independent functions of x (or $\bar{x}$). BPZ found these to be a basis of solutions of certain differential equations in x (or $\bar{x}$) satisfied by the correlation functions as a consequence of the existence of null states in the Virasoro algebra for the minimal models. One should not be surprised then to find that the sets of conformal blocks appearing in different factorizations of a given four-point amplitude are alternative bases for the same space of functions and hence linearly related to each other. In fact, this property follows from the analytic structure of the crossing symmetry constraint, (2.23), alone. That is, any linearly independent sets of holomorphic and anti-holomorphic functions satisfying

$$\sum_{i=1}^{N_f} f^i(x)\bar{f}^i(\bar{x}) = \sum_{j=1}^{N_g} g^j(x)\bar{g}^j(\bar{x}) \quad , \tag{3.1}$$

necessarily satisfy

$$f^i(x) = \sum_J M^{ij} g^j(x) \tag{3.2a}$$

$$\bar{f}^i(\bar{x}) = \sum_j \bar{M}^{ij} \bar{g}^j(\bar{x}) \quad , \tag{3.2b}$$

where M and $\bar{M}$ are square ($N_f = N_g$) invertible matrices with constant entries. To see this, consider the N_f equations obtained by applying $\frac{\partial^n}{\partial \bar{x}^n}$ to (3.1) for $n = 0, \cdots, N_f - 1$, as a simultaneous system of linear equations for the f^i. The determinant of the matrix of coefficients is just the Wronskian for the set of functions $\bar{f}^i$,

$$\bar{W}(\bar{x}) \equiv \begin{vmatrix} \bar{f}^1 & \dots & \bar{f}^{N_f} \\ \partial \bar{f}^1 & \dots & \partial \bar{f}^{N_f} \\ \vdots & & \vdots \\ \partial^{N_f - 1} \bar{f}^1 & \dots & \partial^{N_f - 1} \bar{f}^{N_f} \end{vmatrix} \quad . \tag{3.3}$$

The rows of (3.3), or for that matter the columns, are linearly independent because the N_f functions are, and hence $\bar{W}(\bar{x})$ is not identically zero. Thus we may invert our system of equations to obtain a solution of the form (3.2a). For f (which depends only on x) to be a nonzero solution, M^{ij} must be independent of $\bar{x}$, [which is self consistently guaranteed by (3.2b)]. The analogous manipulations may be used to solve for $\bar{f}^i, g^i$, and $\bar{g}^i$ which establishes (3.2b) as well as the fact that M and $\bar{M}$ are square and invertible.

The chief utility of the relations (3.2) is, as we will see, that they allow us to say a great deal about the purely holomorphic half of a conformal field theory without any specific knowledge of the anti-holomorphic half. Moreover, most of these results follow solely from the analytic structure of the correlation functions — not from the detailed definition of the conformal blocks given in (2.19). It is largely this fact, and a desire to exploit our extensive knowledge of analytic functions on Riemann surfaces, that motivates the study of so called Rational Conformal Field Theories (RCFT) [2-10].

In direct analogy with the minimal models of BPZ, we define RCFT's to be any theory in which any amplitude is decomposable into a *finite* sum of products of linearly independent holomorphic and anti-holomorphic "chiral blocks." These chiral blocks consist of the contributions to the amplitude of a single "primary" field and all of its "descendents" where these terms are now defined relative to some extended algebra of holomorphic fields which contains the Virasoro algebra and whose generators are restricted to have integer conformal dimensions. For example, if the Virasoro algebra is extended by a Kac-Moody algebra the chiral blocks are "current" blocks and the RCFT is one of the Wess-Zumino-Witten models [18]. The reader should be warned that the precise definition of RCFT seems to vary from author to author depending on their needs at hand. The definition of RCFT's we have given, while somewhat lax, is sufficient for our immediate purpose which is to rederive Vafa's constraints on the dimensions of primary fields as a function of their fusion rule algebra [4]. These will follow solely from the analytic structure of the four-point crossing symmetry constraints (2.23) when only a finite number of chiral blocks appear, each with singular expansions of the form (2.20). Thus we don't need to require that the total number of primary fields appearing in our theory is finite, which is typically included in a definition of RCFT's, (that is we include what have been dubbed quasi-rational CFTs [9]) but we need the restriction that generators of the extended chiral algebra have only integer dimensions, which is relaxed in some definitions. In section 4 we will strengthen Vafa's relations, turning them into inequalities which again will be valid for any RCFT as we have defined them.

Now, if we consider any four-point amplitude $\langle\phi_1\phi_2\phi_3\phi_4\rangle$ of primary fields in a RCFT the chiral blocks appearing in different factorizations, (i.e., in the s, t and u channels) are linearly related,

$$f^i_{12,34}(x) = \sum_{j=1}^{N}(C^{13,24}_{12,34})_{ij} f^j_{13,24}(1/x) \qquad (3.4a)$$

$$= \sum_{j=1}^{N}(C^{14,23}_{12,34})_{ij} f^j_{14,23}(1-x) \quad , \qquad (3.4b)$$

where $i, j = 1, 2 \dots N$ label the N independent chiral blocks and the C's are constant matrices. Collecting the f^i's into N-dimensional vectors, F, we can write (3.4) in shorthand notation as

$$F_0 = C_0^\infty F_\infty = C_0^1 F_1 \quad . \qquad (3.5)$$

The index $0, 1, \infty$ indicates, that F_0, for example, has as components the blocks $f^i_{12,34}(x)$ each of which has an expansion of the form (2.20) about $x = 0$,

$$F^i_0(x) \equiv f^i_{12,34}(x) \approx x^{d^i_0 - \gamma/3}(a_0 + a_1 x + a_2 x^2 + \ldots) \quad , \tag{3.6a}$$

where $\gamma = \sum \Delta_i$ [the sum of the dimensions of the external states as in (2.11)] and we have written d^i_0 for the conformal dimension of the primary field appearing as the intermediate state in the i^{th} chiral block in the "0" channel. Similarly, we have

$$F^i_\infty(x) \equiv f^i_{13,24}(1/x) \approx (1/x)^{d^i_\infty - \gamma/3}(b_0 + b_1(1/x) + \cdots) \text{ for } x \to \infty \tag{3.6b}$$

$$F^i_1(x) \equiv f^i_{14,23}(1-x) \approx (1-x)^{d^i_1 - \gamma/3}(c_0 + c_1(1-x) + \cdots) \text{ for } x \sim 1 \quad . \tag{3.6c}$$

Note that the exponent in (3.6) is precisely the same as in the conformal block case, (2.20) because, by assumption, the generators of the chiral algebra in a RCFT have integer dimensions and so all of the descendent fields contributing to the i^{th} chiral block have dimensions d^i + integer. Unlike the conformal block case, however, it is possible in a RCFT that in some amplitudes the primary intermediate state does not appear, in which case the leading singularity, d^i+ integer $- \gamma/3$, is contributed by a descendent field (i.e., the a, b, c coefficients in (3.6) need not all be non-zero). Also, we cannot in general define all of the primary fields to be self-conjugate and at the same time maintain their status as primary fields under the extended algebra of the RCFT.

In large part, we know the analytic structure of the chiral blocks. Each block is a branch of an analytic function of x with the possible branch points at $0, 1, \infty$.

Furthermore, we know the singularity structure at each of these points. If we analytically continue the functions $F_c^j(x)$ about a closed path encircling the branch point $x = c$ but no others, it is clear from (3.6) that we will pick up only a phase: $F_c^j(x) \to e^{2\pi i(d_c^j-\gamma/3)}F_c^j(x)$. This operation of analytically continuing about a closed path is known in the theory of differential equations as a monodromy transformation. Defining M_c to be the monodromy operator about $x = c$, the result above translates into the statement that M_c, acting on F_c, can be represented by a diagonal matrix, $\mathcal{M}_c$,

$$M_c(F_c) = \mathcal{M}_c F_c \equiv \begin{pmatrix} e^{2\pi i(d_c^1-\gamma/3)} & & 0 \\ \vdots & & \vdots \\ \vdots & \ddots & \vdots \\ 0 & & e^{2\pi i(d_c^N-\gamma/3)} \end{pmatrix} \begin{pmatrix} F_c^1 \\ \vdots \\ F_c^N \end{pmatrix} \quad . \tag{3.7}$$

We can also determine how the functions F_c^i transform under M_d where d is another of the points $0, 1, \infty$. To do this we use the matrix C_c^d to express F_c^i in the F_d basis in which M_d is diagonal; perform the monodromy transformation, and then transform back to the F_c basis, i.e.,

$$M_d(F_c) = M_d\left(C_c^d F_d\right) = C_c^d \mathcal{M}_d F_d = C_c^d \mathcal{M}_d (C_c^d)^{-1} F_c \quad . \tag{3.8}$$

If we analytically continue F_0, say, about a closed path not containing $x = 0, 1$, or ∞ such as C_1 in figure 1 we will, of course, leave F_0 unchanged. This result is unaltered if we deform the contour on the Riemann sphere in a region where F_0 is analytic. In particular we can deform C_1 in figure 1 to C_2 which clearly represents a successive operation of M_0, M_1 and M_∞. Thus, we have the identity

$$M_\infty(M_1(M_0(F_0))) = F_0 \tag{3.9}$$

which, using (3.8), becomes

$$C_1^\infty \mathcal{M}_\infty (C_1^\infty)^{-1} C_0^1 \mathcal{M}_1 (C_0^1)^{-1} \mathcal{M}_0 = I \quad , \tag{3.10}$$

with I the $N \times N$ identity matrix.

As it stands, (3.10) is of little practical use because knowing the C matrices is virtually tantamount to having a complete solution to the theory, requiring as it does detailed knowledge of the chiral blocks and structure constants. We can, however, eliminate the C dependence from (3.10) by taking the determinant (following Vafa [4]); This leaves det $(M_\infty M_1 M_0) = 1$ or, using the explicit expressions (3.7),

$$\sum_{i=1}^{N} (d_0^i + d_1^i + d_\infty^i) = N\gamma \qquad (\text{mod } 1) \quad . \tag{3.11}$$

For a given four-point function in a RCFT, (3.11) equates the dimensions of the primary fields appearing as intermediate states summed over the N independent chiral blocks and three independent channels with N times the sum of the dimensions of the four external primary fields, up to some integer. The detailed knowledge of the structure constants, C_{ijk}, dropped out of (3.11) with the departure of the C matrices, but a remnant remains in the knowledge of *which* chiral blocks appear for a given correlation function. For this reason, we introduce the fusion rule coefficients, N_{ijk}, which have been exploited to great success by Verlinde [3]. N_{ijk} is a non-negative integer denoting the number of independent ways the two primary fields ϕ_i and ϕ_j can fuse in their operator product expansion to give ϕ_k or one of its descendents ($\phi_i\phi_j \sim \sum_k N_{ijk}\phi_k$). For the diagonal minimal models the N_{ijk} are all either zero or one depending on whether C_{ijk} vanishes or not, because apart from the overall factor C_{ijk}, the OPE of two primary fields (of the Virasoro algebra) fusing into a third is completely fixed by the conformal symmetry. This is why, in particular, there is a unique conformal block for each intermediate state primary field (once the factorization and external fields are specified). The N_{ijk} are also either zero or one for the non-diagonal minimal models; We must be careful in this case, however, to include all of the purely holomorphic fields of the model in the extended algebra used to define which fields are primary, in order to maintain the one to one correspondence between primary fields and chiral blocks [8,9]. For a general RCFT this one to one correspondence need not hold; there can be multiple independent parameters specifying the coupling between a given three primary fields and hence more than one independent chiral block labeled by a single intermediate state primary field (again for fixed external states). N_{ijk} is just the multiplicity of independent three-point couplings for ϕ_i ϕ_j and ϕ_k, and $N_{ijk}N_{\ell mk}$ is the corresponding number of *independent* chiral blocks $f^n_{ij,\ell m}(x)$ with the intermediate state primary field ϕ_k.

We can now rewrite the sum over independent chiral blocks in (3.11) as a sum over different intermediate state primary fields multiplied by the appropriate multiplicities. For our generic four-point amplitude, $\langle\phi_1\phi_2\phi_3\phi_4\rangle$, we then have

$$\begin{aligned}&\sum_{\text{primaries}} (N_{12i}N_{34i} + N_{14i}N_{23i} + N_{13i}N_{24i})d^i = \\ &\sum_i N_{12i}N_{34i}(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) \pmod 1 \quad ,\end{aligned} \tag{3.12}$$

which is precisely Vafa's condition [4].

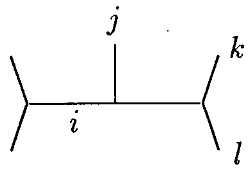
The derivation of (3.12) we have given above is essentially directly equivalent to that given by Vafa, although we have avoided any sophisticated mathematics, as advertised. The modular transformations on the punctured sphere (i.e., Dehn twists) of ref. [4] are just the monodromy transformations considered here. We have gained two minor benefits in the present derivation, however. First, we have seen quite explicitly how Vafa's conditions are included in the constraints of crossing symmetry of four-point amplitudes given by BPZ (as we knew that they must be); In particular we see what information was lost in taking the determinant of the original matrix equation to reach (3.12). Second, we have traded Vafa's somewhat mysterious identity between certain sequences of Dehn twists for two familiar results: that projective symmetry reduces the four-point functions in a conformal field theory to functions of a single anharmonic ratio [x in (2.12)], and that we can deform the contour C_1 to C_2 in figure 1 without changing the result of analytically continuing around it.

An obvious question to ask is whether we can obtain any new relations by similar examinations of higher point functions on the sphere. We argued in section 2 that crossing symmetry of four-point amplitudes guarantees the same for higher point functions. We have seen above, however, that (3.12) represents just a small part of the crossing symmetry constraints; It is not clear *a priori* whether the higher point analogues of (3.12) might not give some complimentary information. The analysis given above is easily extended to M-point amplitudes. These are functions of $M-3$ independent anharmonic ratios whose singularity structure is once again known from the appropriate OPE's. We will not present the computation here, but the result,

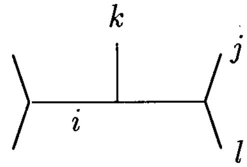
$$\sum_{\substack{\text{blocks}\\ i}} \left(d^i_{12} + d^i_{13} \cdots d^i_{1M} - (M-4)\Delta_1 - \sum_{j=1}^{M} \Delta_j \right) = 0 \qquad (\text{mod } 1), \qquad (3.13)$$

in fact gives no new constraints on the conformal dimensions not already contained in (3.11). The only complication which arises in demonstrating this is determining which particular combinations of constraints of the form (3.11) reproduces (3.13) (we remind the reader that there is an *a priori* different condition of the form (3.11) for *each* nonvanishing four-point amplitude).

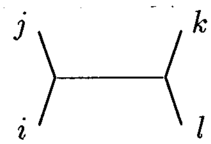
Finally, let us note in passing that the duality matrices, C^d_c, which we have eliminated from (3.10) to obtain Vafa's constraints, are treated as the principle variables of interest (in place of the chiral blocks) in the recent systematic formulation of "modular geometry" advanced by Moore and Seiberg [5]. They examine the consistency conditions these matrices must satisfy if all amplitudes in a RCFT are crossing symmetric and factorize properly. For example, it is clear from the analysis we have given of the four-point functions that for consistency we must have, e.g., $C^1_0 C^\infty_1 = C^\infty_0$ [c.f., (3.5)]. Moore and Seiberg have found a complete set of such consistency conditions; unlike the full crossing symmetry constraints of BPZ, new conditions on the duality matrices arise from examining five-point functions. Much of the considerable effort (and accompanying mathematics) involved in their treatment enters in defining and manipulating chiral vertex operators. These allow one to relate the duality matrices involved in different amplitudes and thus define precisely the notion that the matrices relating



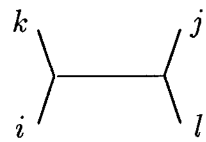
and



for example, are the same as those relating,



and



.

The same framework applies for arbitrary genus g amplitudes as well, the chiral blocks in this case are also functions of $3g$-3 complex moduli. In appropriate corners of moduli space these reduce to chiral blocks on the plane. For example, the chiral block with intermediate state ϕ_j for the two-point amplitude $\langle\phi_i(z)\phi_i(w)\rangle$ on the torus becomes a four-point chiral block on the plane for $\langle\phi_i\phi_i\phi_j\phi_j\rangle$ in the limit Im $(\tau) \to \infty$. Alternatively, in the limit that $z \to w$, the residue of the most singular piece of the chiral block is the character χ_j which is a block for the one-loop vacuum amplitude. It is these correspondences which underlie Verlinde's results relating the behavior of the characters in a RCFT under modular transformations to the fusion rule algebra and duality matrices on the plane [3]; this has been shown clearly and quite explicitly in [7].

4. INEQUALITY CONSTRAINTS ON CONFORMAL DIMENSIONS OF PRIMARY FIELDS

Consider the constraints (3.11) or (3.12) for the special case of a four-point amplitude in a RCFT with only a single chiral block contributing,

$$d_0 + d_1 + d_\infty + M = \gamma \qquad M \in Z \quad . \tag{4.1}$$

In this case we must have $F_0(x) = F_1(x) = F_\infty(x)$ (or atleast equality up to a constant factor). We know the behavior of this function [call it $F(x)$] at each of the points, $x = 0, 1, \infty$, at which it is not analytic [see (3.6)], and this fixes its form to be

$$\begin{aligned} F(x) &= x^{d_0-\gamma/3}(1-x)^{d_1-\gamma/3}\left(a_0 + a_1 x + \ldots a_M x^M\right) \\ M &= \gamma - d_0 - d_1 - d_\infty \quad . \end{aligned} \tag{4.2}$$

Here the a_i are coefficients undetermined by the known analytic structure and M is fixed by the condition that $F(x) \sim (1/x)^{d_\infty - \gamma/3}$ as $x \to \infty$. We now see the significance of the integer M appearing in the condition (4.1) obtained from crossing symmetry: it is the order of the finite polynomial in the expression (4.2) for the chiral block. Furthermore, M cannot be arbitrary. If $F(x)$ is to have the correct analytic structure about $x = 0$, M must be non-negative. Thus, (4.1) in fact becomes an inequality which limits how big the intermediate state conformal dimensions can be for a given sum, γ, of the external state conformal dimensions.

Can we turn the general result, (3.11), for an arbitrary (but finite) number, N, of chiral blocks contributing to the four-point amplitude into an inequality? To extend the argument given above to this case we need some combination of all the chiral blocks which has simple analytic structure which we can identify. We have already introduced such an object in section 3, the Wronskian of the chiral blocks ($c = 0, 1, \infty$),

$$W_c(x) \equiv \begin{vmatrix} F_c^1(x) & F_c^2(x) \cdots & F_c^N(x) \\ \partial F_c^1 & \partial F_c^2 & \vdots \\ \vdots & \ddots & \vdots \\ \partial^{N-1} F_c^1 & \cdots & \partial^{N-1} F_c^N \end{vmatrix} . \tag{4.3}$$

The point of using a determinant is that it changes by at most some constant factor when we perform any linear algebraic operations on its rows or columns. In particular, given that F_0, F_1, and F_∞ are linearly related, we know that $W_0(x) \propto W_1(x) \propto W_\infty(x)$. We also know the behavior of the most singular part of $W(x)$ as $x \to 0, 1$ or ∞ (it, of course, is regular elsewhere on the Riemann surface of the blocks). For example,

$$W(x) \approx x^{\sum_i (d_0^i) - N/3\gamma - N(N-1)/2} (a_0 + a_1 x + a_2 x^2 + \ldots) \tag{4.4}$$

as $x \to 0$. The third factor in the exponent is just the number of derivatives appearing in a given term of the determinant (4.3). The expression for $W(x)$ near $x = 1$ is analogous to (4.4), while for large x we have

$$W(x) \approx \left(\frac{1}{x}\right)^{\sum_i d_\infty^i - N/3\gamma + N(N-1)/2} \left(b_0 + b_1 \left(\frac{1}{x}\right) + b_2 \left(\frac{1}{x}\right)^2 + \cdots\right) . \tag{4.5}$$

The derivatives with respect to x make $W(x)$ *less* singular as $x \to \infty$ by powers of $(1/x)$ but more singular as $x \to 0$ or $x \to 1$. The same logic which led to the single block result (4.2) now gives

$$W(x) = x^{\sum_i d_0^i - N/3\gamma - N(N-1)/2}(1-x)^{\sum_i d_1^i - N/3\gamma - N(N-1)/2}(a_0 + a_1 x + \ldots a_M x^M) \tag{4.6}$$

$$M = N\gamma + N(N-1)/2 - \sum_{i=1}^{N}(d_0^i + d_1^i + d_\infty^i) \quad M \in Z \geq 0 \quad . \tag{4.7}$$

Once again M must be a non-negative integer for $W(x)$ to exhibit the correct behavior about $x = 0$. Thus we have not only reproduced the result (3.11) but turned it into an inequality. In terms of the fusion rule coefficients, the sum of intermediate state dimensions over the N-independent chiral blocks is as before in (3.12).

In any of the diagonal minimal models, the most singular contributions to the conformal blocks in a given channel are made by Virasoro primary fields, each with a distinct conformal dimension. As a consequence, $W(x)$ is of precisely the form given in (4.6) with a_0, a_m and $\sum_0^M a_i$ all non-zero. In a rational conformal field theory,this is not necessarily the case; $W(x)$ may not be as singular as implied in (4.6) for arbitrary a_i and so the relevant inequality may be *stronger*, i.e., $M \geq$ some positive integer. For example, in a RCFT it is possible, as we have indicated before, to have multiple independent chiral blocks contributing to a particular four-point amplitude which all involve the same intermediate primary state, or possibly different primary states all with the same conformal dimensions, and hence share the same leading singularity structure. In this case, the most singular contributions to the chiral block cancel in the determinant defining W. In our generic four-point amplitude, $\langle\phi_1\phi_2\phi_3\phi_4\rangle$, the number of independent chiral blocks in a given channel with a particular intermediate primary of dimension d^i was given in (3.12). Let us use a shorthand notation for the sum of these multiplicities over different primary states ϕ_i with the same dimension Δ_ℓ, suppressing the dependence on the external states,

$$N_0^\ell \equiv \sum_{\Delta_i=\Delta_\ell} N_{12i}\, N_{34i}; \quad N_1^\ell \equiv \sum_{\Delta_i=\Delta_\ell} N_{14i}\, N_{23i}; \quad N_\infty^\ell \equiv \sum_{\Delta_i=\Delta_\ell} N_{13i}\, N_{24i} \quad . \tag{4.8}$$

Consider, now the expression for $W(x)$ defined in a given channel, $W_c(x)$, [i.e., (4.3)]. If N_c^ℓ is bigger than one, then N_c^ℓ of the column vectors in the definition of W_c have the same leading singularity structure. Without changing W_c (aside from a constant) we can take linear combinations of the N_c^ℓ columns such that the most singular term in the expansion for the F_c's cancels in $N_c^\ell - 1$ columns, the next most singular is absent from $N_c^\ell - 2$ columns, and so on down to the last column in which we have canceled off the $N_c^\ell - 1$ most singular terms. We can perform the same procedure on other sets of columns corresponding to other $N_c^j > 1$. We then find that $W(x)$ about $x = 0$, for example, is not so singular as implied in (4.4), but at most given by

$$W(x) \approx x^{\sum_i N_0^i d^i - N/3\gamma - N(N-1)/2 + \sum_i N_0^i(N_0^i - 1)/2}(a_0 + a_1 x + \ldots) \quad , \tag{4.9}$$

as $x \to 0$. Thus considering the behavior in all three channels we find a stronger inequality than (4.7). In terms of the fusion rules,

$$\begin{aligned} N\gamma + N(N-1)/2 &= M + \sum_{\ell,c} N_c^\ell \Delta_\ell \\ M \geq K &\geq \sum_{c=0,1,\infty} \sum_i \frac{1}{2}[N_c^i(N_c^i - 1)]; \quad M, K \in Z \geq 0 \quad . \end{aligned} \tag{4.10}$$

The lower bound for K may be greater than given in (4.10) if the leading contributor to a chiral block is a descendent rather than a primary field, (and hence the singularity in that channel is softer than naively expected) or if $N_c^\ell > 1$ and there are other blocks contributing with dimensions $\Delta = \Delta_\ell + n, n \in Z \leq N_c^\ell + 1$. The argument leading to (4.10) is easily extended in these cases.

The inequalities we have derived are quite simple to impose in any RCFT for which we know the fusion rules and conformal dimensions. Let us consider some simple examples to see, in particular, how stringent the inequalities can be and qualitatively what information they carry. Consider first a single free boson. The generic four-point amplitude of primary fields is

$$A = \left\langle e^{ik_1\phi(z_1)} e^{ik_2\phi(z_2)} e^{ik_3\phi(z_3)} e^{ik_4\phi(z_4)} \right\rangle \quad , \sum_{i=1}^{4} k_i = 0 \quad . \tag{4.11}$$

The conformal dimension of $e^{ik_i\phi}$ is $k_i^2/2$, and the structure constants just enforce momentum conservation in three-point functions. In each channel only a single conformal block appears. The primary fields which are the intermediate states in these blocks are

$$\begin{aligned} \text{“0”channel} &: e^{i(k_1+k_2)\cdot\phi} \\ \text{“1”channel} &: e^{i(k_1+k_4)\cdot\phi} \\ \text{“}\infty\text{”channel} &: e^{i(k_1+k_3)\cdot\phi} \quad . \end{aligned} \tag{4.12}$$

Thus (4.7) reduces to,

$$M = \frac{1}{2}\sum k_i^2 - \frac{1}{2}((k_1+k_2)^2 + (k_1+k_4)^2 + (k_1+k_3)^2) = -k_1 \sum k_i = 0 \quad . \tag{4.13}$$

The inequality given in (4.7) is saturated for this case. This fact implies, according to (4.2), that the conformal block is given by,

$$
\begin{aligned}
F(x) &= x^{\frac{1}{2}(k_1+k_2)^2-\gamma/3}(1-x)^{1/2(k_1+k_4)^2-\gamma/3} \\
\gamma &= \frac{1}{2}\sum k_i^2 \quad .
\end{aligned}
\tag{4.14}
$$

To put this in a more familiar form we can use (2.11), (2.12) and (2.18) to obtain the holomorphic half of the full four-point amplitude in (4.11),

$$
A = \prod_{i<j}(z_i - z_j)^{k_i \cdot k_j} \quad . \tag{4.15}
$$

Notice that we have obtained the four-point amplitude purely from the knowledge of the fusion rule algebra and the conformal dimensions of the states involved (including intermediate ones). In general, any four-point amplitude depending on a single chiral block can be determined in the same fashion up to a finite number of parameters (the coefficients a_i in 4.2). We will further exploit this fact in section 5. Finally, note that if we had chosen self-conjugate primary fields (sines and cosines in place of simple exponentials) it would have proved inconvenient; the amplitudes would be more complicated and two conformal blocks would contribute to each.

Consider next the first of the unitary minimal models, the Ising model. There are three primary fields of the Virasoro algebra I, σ, ψ with dimensions 0, 1/16, 1/2. The fusion rules are $\psi\psi \sim I; \sigma\sigma \sim I + \psi; \psi\sigma \sim \sigma$ and the obvious ones involving the identity, I. The nontrivial four-point functions are $\langle\psi\psi\psi\psi\rangle$, $\langle\psi\psi\sigma\sigma\rangle$, $\langle\sigma\sigma\sigma\sigma\rangle$. Only a single conformal block contributes to each of the first two. In these cases (4.7) gives

$$
\begin{aligned}
\langle\psi\psi\psi\psi\rangle &: M = 4\Delta_\psi = 2 \\
\langle\psi\psi\sigma\sigma\rangle &: M = 2\Delta_\psi = 1 \quad .
\end{aligned}
\tag{4.16}
$$

Two conformal blocks contribute to $\langle\sigma\sigma\sigma\sigma\rangle$, giving

$$\langle\sigma\sigma\sigma\sigma\rangle : M = 8\Delta_\sigma + 1 - 3\Delta_\psi = 0 \quad . \tag{4.17}$$

Note that the Ising fusion rules and Vafa's constraint ($M \in Z$, unrestricted) alone, constrain Δ_ψ and Δ_σ to satisfy

$$2\Delta_\psi \in Z \quad , \Delta_\psi = 8\Delta_\sigma \quad (\text{mod } 1) \quad . \tag{4.18}$$

The fact that $M \geq 0$ tells us considerably more, in particular if $\Delta_\sigma = 1/16, 3/16$ or $5/16$, then Δ_ψ can only be equal to $1/2$.

Finally, a more general but equally simple example. Consider an RCFT which contains some self-conjugate primary fields ϕ_1 and ϕ_2 which fuse to only a single primary field, e.g., $\phi_1\phi_2 \sim \phi_3$. The amplitude $\langle\phi_1\phi_1\phi_2\phi_2\rangle$ will be built from a single chiral block. In two channels the intermediate primary state is ϕ_3 and in the third channel it can only be the identity, I. This follows because $\phi_1\phi_1 \sim I + \cdots$, $\phi_2\phi_2 \sim I + \cdots$, and by crossing symmetry the same number of blocks contribute to each channel (in this case one). Equation (4.10) gives

$$0 \leq M = 2(\Delta_1 + \Delta_2 - \Delta_3) \quad , \tag{4.19}$$

so, in particular, $\Delta_3 \leq \Delta_1 + \Delta_2$. This simple case illustrates the basic content of the inequalities we have derived: in any amplitude the average dimension of the intermediate state primary fields appearing in different channels must not be too large compared to the dimensions of the external states. If the external state dimensions are small, then some intermediate state dimensions can be relatively large only if others are small or there are many intermediate primaries appearing.

Finally, we should mention the work of Mathur, Mukhi and Sen which inspired our use of the Wronskian in the derivation of the inequalities given above. These authors examine the Wronskian determinant formed from the genus one characters of a RCFT (derivatives taken with respect to the modular parameter τ) and use its known modular properties to find an inequality quite complementary to the one we have presented here,

$$\begin{gathered}\frac{n(n-1)}{2} + \frac{nc}{4} - 6\sum_{i=1}^{n} h_i = \ell \\ \ell \in Z = 0 \text{ or } \geq 2 \quad .\end{gathered} \tag{4.20}$$

Here n is the total number of primary fields in the RCFT, h_i their conformal dimensions (assumed distinct) and c is the central charge. The less restrictive version of (4.20) with ℓ an arbitrary integer was also given in Vafa's paper. Unlike the inequalities we've derived from amplitudes on the plane, (4.20) is valid only for a RCFT with a finite number of primary fields and gives only a single condition for any given RCFT. It does, however, constrain c which (4.7) does not.

5. REVIVING THE CONFORMAL BOOTSTRAP

The constraints of Vafa, their generalizations given in the previous section, and the genus one constraints of Mathur, Mukhi and Sen provide convenient handles for directly studying the objects of most immediate interest in rational conformal field theories, the central charge, conformal dimensions and fusion rules of primary fields. They give us only partial information, however, and if our goal is the classification of all conformal field theories then we shall have to do better. The most popular line of investigation currently seems to revolve around the twin hopes that: (1) RCFT's can be completely classified using our detailed knowledge of the analytic structure of amplitudes defined on arbitrary genus Riemann surfaces and (2) that RCFT's form an in some way natural, dense subset of all conformal field theories. The work of Moore and Seiberg mentioned briefly in section 3 represents the most explicit and systematic formulation along these lines to date. From a practical point of view, their results are much more complicated and involve much less convenient variables than the examples listed above, but one must expect this if we are trying to completely classify all RCFT's. Unfortunately, however, their consistency conditions on the duality matrices are not sufficient in themselves to do this job. At best, they constrain the values of conformal dimensions only modulo one and of c modulo eight, and thus do not include the inequalities of ref. [6] or those given here. This should not be too surprising in that the constraints of ref. [5] follow purely from consistency conditions on fitting chiral blocks together to form amplitudes. As BPZ showed, however, we have much more detailed knowledge at our disposal. All chiral blocks are built from conformal blocks which are completely fixed (in principle) by the conformal symmetry.

As an alternative approach to the problem of classifying conformal field theories we would like to suggest that it is time to reexamine the conformal bootstrap of BPZ. The chief stumbling block involved is our lack of explicit expressions for the general conformal blocks. We will not remedy this situation here (unfortunately). Instead we confine ourselves to examining four-point amplitudes which are almost completely fixed by their analytic structure. In short, we will test the feasibility of employing methods such as those discussed in sections 3 and 4 to attack the conformal bootstrap perturbatively. The basic trick will be to look at four-point amplitudes with only a single conformal block contributing. As discussed in the previous section these amplitudes are fixed up to a finite set of parameters [c.f. (4.2)], which can be further reduced if the amplitude has additional symmetry. We will then employ the explicit expansions of the conformal blocks discussed in section 2 to some finite order to completely fix these unknown parameters and thereby find constraints on the allowed dimensions and central charges. We note in passing that many of the computations and manipulations used below have appeared, working towards a different end, in [19].

To begin, we restrict our attention to conformal field theories which include at least one primary field, ψ, such that the OPE of ψ with ψ contains only a single Virasoro primary (which must necessarily be the identity operator, I) and its descendents. As in the previous two sections we will focus only on the holomorphic degrees of freedom, imagining for example that we are considering a theory completely symmetric in its holomorphic and anti-holomorphic parts. By assumption, only a single conformal block, $F(x)$, contributes to the amplitude $\langle\psi\psi\psi\psi\rangle$. In each "channel" the intermediate state appearing is the identity, $d_0 = d_1 = d_\infty = 0$. We know, then, the form of $F(x)$ completely from its analytic structure [cf., (4.2)],

$$F(x) = x^{-4/3\delta}(1-x)^{-4/3\delta}(1 + a_1 x + \ldots a_M x^M); \quad M = 4\delta \quad , \tag{5.1}$$

where δ is the conformal dimension of ψ which we see must necessarily satisfy $4\delta \in Z$. We can restrict F further by using the fact that the amplitude $\langle\psi\psi\psi\psi\rangle$ is completely symmetric in its external legs and hence must be invariant under permutations of z_1, z_2, z_3 and z_4. Given our symmetric definition of the conformal blocks this translates into the condition,

$$F(x) = \alpha_1 F(1-x) = \alpha_\infty F(1/x) \quad . \tag{5.2}$$

We have included real constants, α_1 and α_∞, because the anti-holomorphic block $\bar{F}(\bar{x})$could contribute a compensating factor under the given transformation leaving the total amplitude invariant as required. If we call the undetermined M^{th} order polynomial appearing in (5.1), $P_M(x)$, then (5.2) gives,

$$P_M(x) = \alpha P_M(1-x) = \beta P_M(1/x) x^M \quad . \tag{5.3a}$$

$$P_M(0) = 1 \quad . \tag{5.3b}$$

α and β are constants (trivially related to α_1 and α_∞) and (5.3b) serves to normalize P and guarantee that P includes no overall factors of x or $1-x$. In general, there are solutions only if $\alpha = (-1)^M$ *and* $\beta = 1$; there is no solution for P_1. It is not difficult to prove that

$$P_2(x) = 1 - x + x^2 \tag{5.4}$$

$$P_3(x) = 1 - \frac{3}{2}x - \frac{3}{2}x^2 + x^3 \tag{5.5}$$

and all higher order polynomials are sums of products of these two of the appropriate order,

$$P_M = \sum_{\substack{I,J \\ 2I+3J=M}} a_{IJ} P_2^I P_3^J \quad ; \quad \sum a_{IJ} = 1 \quad . \tag{5.6}$$

Thus, for example, $P_4 = P_2^2$, $P_5 = P_2 P_3$ and there is a one parameter family of solutions for $P_6, aP_2^3 + (1-a)P_3^2$.

Our strategy is now to compare the almost fully-determined expression for the conformal block [(5.1) and (5.6)] with however many terms in the explicit expansion for the conformal block (2.19) we have stomach to calculate. This computation simplifies for the case at hand ($\Delta_i = \delta, i = 1,2,3,4; \Delta_p = 0$) and we give the result to fifth order here,

$$\begin{aligned} F(x) = f_{\delta\delta,\delta\delta}(I|x) = x^{-4/3\delta}(1-x)^{-4/3\delta}[1-x]^{2\delta}\Bigg[1 + \frac{2\delta^2}{c}x^2 \\ + \frac{2\delta^2}{c}x^3 + \frac{\delta^2}{(\frac{5}{2}c^2+11c)}(5\delta^2+2\delta+\frac{9}{2}c+20)x^4 \\ + \frac{2\delta^2}{(\frac{5}{2}c^2+11c)}(9+2c+2\delta+5\delta^2)x^5 + \mathcal{O}(x^6)\Bigg] \quad . \end{aligned} \tag{5.7}$$

Here c is the Virasoro central charge. BPZ derive this expansion only up to order x^2 but for arbitrary intermediate state [1]. The product of the two square brackets must equal $P_{4\delta}$ in our case, so we Taylor expand the first bracket and compare the product with $P_{4\delta}$ order by order. The first few orders may be used to fully determine $P_{4\delta}$ [i.e., the a_{IJ} in (5.6)] for small enough δ, and subsequent orders serve to give c and rule out inconsistent solutions.

P_1 does not exist so there are no consistent solutions with $\delta = 1/4$. Comparing (5.7) taking $\delta = 1/2$ with P_2 gives the order x^2 constraint $1/2c = 1$ or $c = 1/2$. The coefficients of the higher powers of x must vanish (to agree with P_2), and this is indeed found to be the case for $c = \delta = 1/2$ to the order we have given in (5.7). The Ising model has $c = 1/2$ and a spin 1/2 field which when fused with itself gives just the conformal family of the identity so we are certainly not surprised to have found this solution. What is perhaps surprising, however, is that $c = 1/2$ is the only consistent solution. That is if a conformal field theory includes a dimension 1/2 primary field which when fused with itself gives only the conformal family of the identity, then the central charge must be 1/2 regardless of whatever the rest of the conformal field theory might be. This illustrates one of the strengths of the conformal bootstrap: we can learn a good deal from a single four-point amplitude without making any assumptions about the remainder of the theory. Note that a theory built by tensoring the Ising model with some other CFT is not a counterexample to the above result as might seem to be the case. This operation indeed leaves the Ising correlators unchanged and increases c, but the conformal symmetry is generated in the tensor product case by the sum of the individual stress energy tensors. There are then an infinite number of fields in the tensor product theory appearing in the $\psi\psi$ OPE which are primary with respect to this total stress energy tensor.

We proceed by substituting $\delta = 3/4$ into the product of the square brackets in (5.7) and equating this, order by order in x, with P_3. In analogous fashion to the case above we find a single consistent solution, $c = -3/5$, which agrees with one of the nonunitary minimal models of BPZ. The minimal models are each characterized by a pair of positive co-prime integers p and q, with central charges and allowed conformal dimensions of primary fields given by

$$c = 1 - \frac{6(p-q)^2}{pq} \tag{5.8}$$

$$\Delta_{rs} = \Delta_{q-r,p-s} = \frac{(rp-sq)^2 - (p-q)^2}{4pq} \quad ; \quad \begin{matrix} 1 \le r \le q-1 \\ 1 \le s \le p-1 \end{matrix} \quad . \tag{5.9}$$

An algorithm for finding the fusion rules for these models is given in BPZ. One finds, in particular, that the primary fields with dimensions

$$\Delta_{1,p-1} = \Delta_{q-1,1} = \frac{1}{4}(p-2)(q-2) \quad , \tag{5.10}$$

fused with themselves give only the identity operator. Thus each of the minimal models must eventually appear in our current exercise.

Equating $P_4 = P_2^2$ with the expansion from (5.7) with $\delta = 1$ we find a new development. Matching the x^2 coefficients requires $c = 1$, which also guarantees that the x^3 coefficients coincide, but for these values the coefficients of x^4 and higher powers do not agree; Accordingly, there is no consistent solution for $\delta = 1$. This conflicts with the commonly held belief that the only Virasoro primary fields in the conformal field theory of a single free boson (which has $c = 1$) are the identity, the exponentials $:e^{ik\phi}:$, and $\partial\phi$. If this were indeed the case then $\partial\phi$ would be a dimension one primary field with $[\partial\phi] \times [\partial\phi] \sim [I]$. In fact, however, these are not the only primary fields. An infinite set of certain linear combinations of products of ϕ derivatives are also primary under the Virasoro algebra and appear in the $\partial\phi \times \partial\phi$ OPE. The one of lowest dimension is $: \partial^3\phi\partial\phi + (\partial\phi)^4 - 3/2\partial^2\phi\partial^2\phi :$, of dimension four, accounting for why our assumption that $[\partial\phi] \times [\partial\phi]$ fused only to $[I]$ broke down first for the x^4 term in the conformal block expansion. There are not enough Virasoro descendents of I and $\partial\phi$ to account for all of the independent combinations of ϕ derivatives because not all of them are independent. That is there are Virasoro null states for $c = 1$ and $\Delta = \frac{1}{4}N^2, N \in Z$ as can be seen from the Kac determinant formula. On the other hand, if we extend the Virasoro algebra to include the generator $\partial\phi$, then the only primary fields under this extended algebra are I and the exponentials $e^{ik\phi}$.

We can continue the present exercise for ever larger δ; the results for the lowest few values which give consistent solutions to the perturbative bootstrap to order x^5 are summarized in Table 1. The minimal models are labeled by their (p,q) values (c.f. (5.8)). Beginning with $\delta = \frac{3}{2}$ the possibility for multiple solutions arises. Recall that P_6 is determined by (5.6) up to a single free parameter. Comparing with (5.7) for $\delta = 3/2$ we can use the x^2 coefficient to express this parameter in terms of c. Equating x^3 coefficients turns out to give a redundant constraint but equating x^4 terms gives a quadratic equation for c with the two solutions given in Table 1. There are no consistent solutions for $\delta = \frac{7}{4}$; as for $\delta = 1$, the solution valid to order x^3 (in this case it is $c = 49$) breaks down at order x^4. Table 1 stops at $\delta = 11/4$ because P_{12} has two undetermined parameters and the expansion in (5.7) up through x^5 is insufficient to fix both these parameters and constrain c.

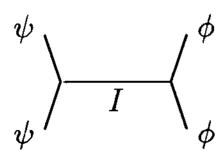
All of the examples of known consistent models appearing in Table 1 are members of the minimal series. The minimal models may, in fact, be the only solutions to the perturbative bootstrap in the class considered above which are consistent to all orders in the perturbative bootstrap, or there may be additional solutions for larger values of δ where the multiplicities of possible solutions are much greater. As yet we have not managed to settle this question; more detailed knowledge of the conformal blocks seems to be required. Unfortunately, a solution consistent to some finite order in the x expansion (such as the unknown $c = -17$ and $c = 127$ entries in Table 1) might easily break down at some still higher order (as the $c = 1$ and $c = 49$ cases did). These failed models may be healthy CFTs with additional primary fields appearing (as was the case for the free boson) or they might be unsalvageable.

So far we have only considered four-point amplitudes $\langle\psi\psi\psi\psi\rangle$ with $\psi\psi \sim I$. The methods described, however, have a much larger domain of applicability. Other amplitudes involving only a single conformal block may be examined in entirely analogous fashion. Again the amplitude is fixed up to a finite number of parameters by the known analytic structure. These parameters are fixed by matching with the expansion of the conformal block in one channel to some finite order; matching higher order terms in the expansion or matching with the expansions of the conformal blocks for the other two channels then gives constraints on the central charges and dimensions which may appear in a consistent CFT.

To treat amplitudes involving not one but a finite number of conformal blocks we can use the same methods, focusing not on the amplitude itself but on the Wronskian determinant made from the conformal blocks appearing in the amplitude. The Wronskian (as discussed in section 4) is again fixed up to a finite number of terms by its known analytic structure. For example, assuming Ising model conformal dimensions and fusion rules the Wronskian for $\langle\sigma\sigma\sigma\sigma\rangle$ is completely fixed [c.f. (4.6),(4.17)]; Agreement with the conformal block expansions requires $c = 1/2$. In general, this analysis requires the expansions (to some order) for all of the conformal blocks appearing in the amplitude. Alternatively, we could use the method of ref. [6] to find (again up to a finite number of parameters) the differential equation satisfied by the conformal blocks, solve this in a power series for a given block and compare this with the expansion found from (2.19) for that block; this approach requires the expansion of only a single conformal block but in general it must be known to a higher order in x because the differential equation involves more undetermined parameters than the corresponding Wronskian.

To some extent the conformal bootstrap may be treated perturbatively in RCFTs as well. If the extended chiral algebra is explicitly known, then the chiral blocks may be calculated order by order and the bootstrap handled just as above with conformal blocks replaced by chiral blocks. Alternatively, the reverse procedure might prove feasible. That is, starting with some chiral block (or set of chiral blocks) we might, by examining conformal block expansions order by order, deduce which Virasoro primaries contribute and hence learn about the extended chiral algebra.

Finally, we are not limited to examining a single amplitude and constraining the central charges consistent with it; we can deduce a great deal about the rest of the operator content as well. For example, let us return to the amplitude $\langle\psi\psi\psi\psi\rangle$ with $\psi\psi \sim I$. For any primary field ϕ in the theory, the amplitude $\langle\psi(z_1)\psi(z_2)\phi(z_3)\phi(z_4)\rangle$ will be nonvanishing; even if the $\psi\phi$ OPE is entirely non-singular we still have $\langle\psi\psi\phi\phi\rangle = \langle\psi\psi\rangle\langle\phi\phi\rangle \neq 0$. In addition, in the "0" channel only a single conformal block can contribute represented by the diagram,



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and so the amplitude completely factorizes in its holomorphic and anti-holomorphic parts. Accordingly, only a single chiral block appears in the cross channel ("1" or "∞") but this might include any number of intermediate state Virasoro primary fields with conformal dimensions differing from each other by integers. Let us group all of the candidate primary fields in the theory into families, $\{\phi_i\}$, such that each member of the i^{th} family has dimension Δ_i modulo 1, and $\Delta_i \neq \Delta_j$ modulo 1 for any two different families. Further, label the primary field in the i^{th} family with the least dimension (which we take to be Δ_i) ϕ_i°. In a sensible theory each Δ_i is finite so a ϕ_i° exists for each family; if several fields have dimension Δ_i we take any one of them to be ϕ_i°. Now consider $\langle\psi\psi\phi_i^\circ\phi_i^\circ\rangle$. As noted above, in the 0 channel only the conformal block of the identity contributes; in the other two channels only primaries within a single family, say $\{\phi_j\}$, can contribute. The leading singularity of the amplitude as $x \to 1$ or ∞ is contributed by the intermediate state (or states) with least dimension which we will denote ϕ_j^k with dimension $\Delta_j + k, k \in Z \geq 0$. The singularity structure fixes the single block to be of the form

$$F(\phi_i^\circ; x) = x^{-\gamma/3}(1-x)^{d-\gamma/3}(1 + a_1 x + \cdots a_M x^M) \quad , \tag{5.11}$$

$$\begin{aligned} \gamma = 2(\Delta_i + \delta) \quad &; \quad d = d_1 = d_\infty = \Delta_j + k \\ M = M_i^\circ &\equiv 2(\Delta_i + \delta - \Delta_j - k) \quad . \end{aligned} \tag{5.12}$$

At this point it seems that M_i° could be any positive integer, allowing an infinite set of values for Δ_i, but this is not the case. To see this we need to consider $\langle\psi\psi\phi_j^k\phi_j^k\rangle$. The intermediate primary state in the 0 channel is again I; in the cross channel only one family of primary fields may contribute and this must be $\{\phi_i\}$. Furthermore, the member of $\{\phi_i\}$ which contributes to the amplitude with the least conformal dimension is clearly ϕ_i° so we have $F(\phi_j^k; x)$ of the same form as (5.11) with $\gamma = 2(\Delta_j + k + \delta)$; $\quad d = \Delta_i$ and

$$M = M_j^k \equiv 2(\Delta_j + k + \delta - \Delta_i) \quad . \tag{5.13}$$

The sum of (5.12) and (5.13) gives us the necessary restriction

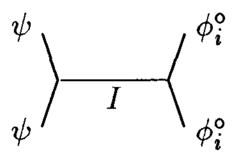
$$M_i^\circ + M_j^k = 4\delta \quad . \tag{5.14}$$

Since the M's must be positive integers there are only 4δ possible choices for M_i°. We cannot similarly restrict the value of a generic M_ℓ^m because ϕ_ℓ^m need not appear as the leading intermediate state primary field in any amplitude of the theory.

For each value of M_i° we can compare (5.11) with the appropriate conformal block expansion to fix the a_i coefficients and the allowed values of Δ_i. The amplitude is symmetric under the interchanges $z_1 \leftrightarrow z_2$ or $z_3 \leftrightarrow z_4$ which translates into

$$F(\phi_i^\circ; x) \propto F\left(\phi_i^\circ; \frac{x}{x-1}\right) \quad . \tag{5.15}$$

This cuts the number of independent coefficients in (5.11) in half. To proceed we need the conformal block expansion for



.

Using the definition (2.19), this is found to quadratic order to be

$$\begin{aligned} f_{\psi\psi,\phi\phi}(x) = x^{-\gamma/3}(1-x)^{d-\gamma/3}\Big[& 1 + (d-\delta-\Delta)x + \\ & + \left\{\frac{1}{2}(\delta+\Delta-d)(\delta+\Delta-d-1) + \frac{2}{c}\delta\Delta\right\} x^2 + \mathcal{O}(x^3)\Big] \quad , \end{aligned} \tag{5.16}$$

where we have pulled out the same factors as in (5.11) in order to facilitate a direct comparison. To illustrate the method, consider the first case in Table 1, $\delta = c = 1/2$. From (5.14) the possible values for M_i° are 0,1,2. $M_i^\circ = 0$ requires that all of the coefficients in (5.16) vanish, which to second order gives $\Delta = 0$ and $d = \delta$. This is just the case $\phi = I$, which of course tells us nothing new. $M_i^\circ = 2$

implies [c.f., (5.14)] that the intermediate state in the 1 and ∞ channels is the identity which in turn requires $\phi = \psi$. The remaining possibility is $M_i^\circ = 1$. For this case the polynominal in (5.11) is [making use of (5.15)] $1 - \frac{1}{2}x$. Comparison with the x coefficient in (5.16) gives $d = \Delta$; setting the coefficient of x^2 to vanish then fixes $\Delta = 1/16$. This we recognize as the spin field, σ, in the Ising model. The possibility remains of additional fields, ϕ^k with dimensions $1/16 + k, k \in Z > O$. But we know from the Ising model solution that the conformal block $f_{\psi\psi,\sigma\sigma}$ is equal to the conformal block in the cross channel $f_{\psi\sigma,\psi\sigma}$ with σ as the sole intermediate primary field. These functions depend only on c, Δ and δ and so we have learned that no primary fields of dimension $1/16 + k$ appear in our amplitude, hence they do not appear in the $\psi\phi$ OPE. But then at least one of these operators must appear as leading intermediate primary in $\langle\psi\psi\phi^k\phi^k\rangle$ and hence have $M^k = 0, 1, 2$. This contradicts $\Delta^k = 1/16 + k$, however, and so none of these fields can be present. I, ψ and σ are the only primary fields in the theory and, as we have seen, they satisfy the Ising model fusion rules. That we may regenerate a complete conformal field theory starting from a small piece is a deep and amusing feature of the conformal bootstrap.

The perturbative approach to the conformal bootstrap certainly requires more effort to apply then do the simple constraints described in sections 3 and 4. In return for this we receive much more detailed and explicit information. For example, all of the information in the Kac determinant formula, which determines the spectrum of dimensions in minimal CFTs, is generated order by order in the bootstrap exercises considered above. At the same time we have explicitly determined some of the four-point functions of the theory. The manipulations we have employed, in fact, provide the simplest means for calculating some four-point functions (and do not require knowledge of an operator representation, null states, etc.). As a final example of this, consider the correlation function of energy momentum tensors $\langle TTTT \rangle$ for *arbitrary* central charge. It is easy to show that the functional form in (5.1) is still valid for correlators of quasi-primary fields such as $T(z)$ (that is, fields primary under projective transformations but not the full Virasoro algebra). For this case $\delta = 2$ and the unknown polynominal, by symmetry, must be $P_8(x) = aP_2^3 + (1-a)P_3^2 P_2$, [using (5.3)–(5.6)]. The parameter, a, may be fixed in terms of the central charge c by comparing the first few singular terms in (5.11) with the result obtained using only the singular pieces in the TT OPE,

$$T(z)T(w) \approx \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w) \quad . \tag{5.17}$$

The final result for the holomorphic amplitude (with the factors in (2.11) restored) is

$$\begin{gathered}\langle T(z_1)T(z_2)T(z_3)T(z_4)\rangle = \left(\prod_{i<j}(z_i - z_j)^{-4/3}\right)\frac{c^2}{4}x^{-8/3}(1-x)^{-8/3} \\ \left[1 + x^8 + \left(\frac{8}{c}+6\right)(x^2+x^6) - \left(\frac{24}{c}+4\right)(x^3+x^5) + \left(\frac{32}{c}+3\right)x^4\right] \quad .\end{gathered} \tag{5.18}$$

Of course we could have obtained this result directly from the mode expansion for T and the Virasoro commutation relations but this would require considerably more effort and involve a correspondingly higher probability of error.

6. DISCUSSION

In sections 3, 4 and 5 we have obtained a number of results from reexamining conformal field theories on the plane. These, and the methods involved, should further our understanding of known CFTs, prove useful in calculating some amplitudes and hopefully shed light on the classification of all CFTs. The inequalities of section 4 provide stringent constraints on the conformal dimensions and fusion rules in any new candidate rational conformal field theories. The perturbative treatment of the conformal bootstrap, on the other hand, might lead to new non-rational CFTs — a class of theories which remains basically unexplored. To push this method further will require the conformal block expansions to much higher order than we have computed for the "feasibility study" of section 5, but there is no real obstacle to carrying this out (preferably by computer).

We must emphasize that the methods we have described are no substitute for knowing the conformal block functions. To quote BPZ, "...the computation of the conformal blocks...for general values of Δ_n's is the problem of principle importance for the conformal quantum field theory." [1] Knowledge of the conformal blocks would permit us to completely solve the exercises of section 5 which we have only nibbled at with the perturbative approach. Despite the lack of progress in this direction we do not feel that the determination of the conformal blocks is intractable, especially given all that we already know about them. The general conformal block is a function of c, five conformal dimensions and the anharmonic ratio x which we know how to calculate order by order in a singular expansion about $x = 0$. In addition, we know the exact function of x for some specific values of the dimensions and c. For example, for the $\langle\psi\psi\psi\psi\rangle$ block considered in section 5 we know $F(x)$ up to a finite number of parameters for an infinite set of values $\{c, \delta\}$, namely, those corresponding to the minimal models [c.f., (5.8), (5.10)]. It should be possible to find the function which interpolates between these points.

Even the restricted problem of classifying RCFTs probably requires some detailed information of the general conformal blocks. For example, given a solution to the reconstruction schemes of ref. [5] or ref. [6], (that is a well-behaved collection of crossing symmetric chiral blocks which may be extended to N point functions on arbitrary genus surfaces), the question remains whether the chiral blocks are in fact composed of crossing symmetric conformal blocks. One must explicitly find the extended chiral algebra and check that it indeed contains the Virasoro algebra and also that the factorization of amplitudes in terms of descendent fields behaves properly. It would be extremely interesting to learn what functions can be valid chiral blocks. Perhaps *any* function of x with branch points restricted to $x = 0, 1, \infty$ is a chiral block in some RCFT (or quasi-RCFT). Or, at the opposite extreme, perhaps all RCFTs can be represented as tensor products of minimal models (nonunitary ones included); the number of possible modular invariant combinations and symmetries which can be generated in this class of theories is at least very large, as has been demonstrated for tensor products of Ising models in the context of string theory [13].

We conclude with a comment on one of the principle motivations for studying the classification of CFTs, the hope that it will further our understanding of string theory. CFTs form the building blocks for the possible classical vacua of string theory, which are already known to constitute an enormous set. The most important problem in string theory is to understand the dynamics which may select between these possible vaccua. Definitive progress in this direction has been virtually nonexistent (despite much effort) and is likely to remain so as long as we lack the knowledge of the full space in which the dynamics operates and a convenient parameterization of it. This space of kinematics must at least include all of the unitary CFTs with integer c, but likely it will have to be expanded well beyond this class of solutions to provide a smoothly parameterized space on which the physically relevant dynamics might seem natural (and therefore be guessed). It may eventually prove prudent to relax some of the conditions on CFTs to obtain a more manageable space of string kinematics, in which case a true classification of CFTs will be unnecessary. For example, the general conformal blocks might themselves provide us with the necessary space; amplitudes in the full string theory may be expressible as weighted sums over all of the possible conformal blocks which might appear.

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Note added: After completion of this work we recieved ref. [20] which also derives the inequality constraints that we have given here in section 4. In addition this brought to our attention ref. [21] which includes a version of these constraints for the particular case of SU(2) WZW models.

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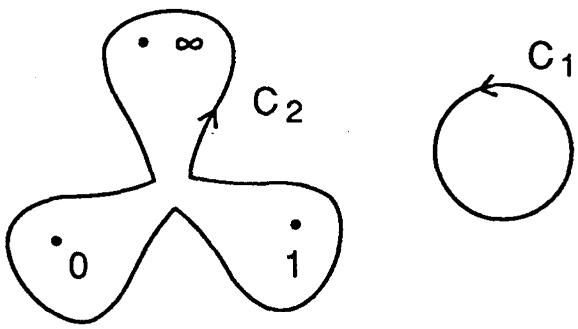

Fig.1. Analytically continuing F about C_1 or C_2 on the Riemann sphere leaves it unchanged.

δ	Possible c	Known Examples
$\frac{1}{2}$	$\frac{1}{2}$	(3,4) minimal (Ising)
$\frac{3}{4}$	$\frac{-3}{5}$	(3,5) minimal (nonunitary)
$\frac{5}{4}$	$\frac{-25}{7}$	(3,7) minimal (nonunitary)
$\frac{3}{2}$	$\frac{7}{10}$	(4,5) minimal (tricritical Ising)
$\frac{3}{2}$	$\frac{-21}{4}$	(3,8) minimal (nonunitary)
2	$\frac{-44}{5}$	(3,10) minimal (nonunitary)
$\frac{9}{4}$	$\frac{-351}{33}$	(3,11) minimal (nonunitary)
$\frac{9}{4}$	-17	?
$\frac{5}{2}$	$\frac{-13}{14}$	(4,7) minimal (nonunitary)
$\frac{11}{4}$	$\frac{-187}{13}$	(3,13) minimal (nonunitary)
$\frac{11}{4}$	127	?

Table 1: Some solutions to the perturbative bootstrap to order x^5.