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DISCRETE FIELD THEORIES AND SPATIAL PROPERTIES OF STRINGS*†

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ABSTRACT

We use the ground-state wave function in the light-cone gauge to study the spatial properties of fundamental strings. We find that, as the cut-off in the parameter space is removed, the strings are smooth and have a divergent size. Guided by these properties, we consider a large-N lattice gauge theory which has an unstable phase where the size of strings diverges. We show that this phase exactly describes free fundamental strings. The lattice spacing does not have to be taken to zero for this equivalence to hold. Thus, exact rotation and translation invariance is restored in a discrete space. This suggests that the number of fundamental short-distance degrees of freedom in string theory is much smaller than in a conventional field theory.

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1. SPATIAL PROPERTIES OF STRINGS

Many systems in statistical mechanics and field theory exhibit string-like excitations. Superconducting fluxoids, domain boundaries in 2+1 dimensional magnets, Nielsen-Olesen vortices and electric flux tubes in QCD are a few examples. In many respects they are similar to the idealized objects of string theory, which we will refer to as fundamental strings. There are, however, significant differences. In particular, these field theories do not have massless gauge bosons or gravitons. This raises the following interesting question: can fundamental string behavior occur in a field theoretic or statistical mechanical system? A related question is whether string theory can be simulated on a computer which stores information in terms of local degrees of freedom and evolves it according to near neighbor interactions on a discrete lattice, as in lattice gauge theory. The purpose of this work is to show that the answer is positive. Our construction of such a theory will be guided by a study of the spatial properties of fundamental strings. We will show that these properties are highly peculiar. In particular, both the average length and the average size of the region occupied by a fundamental string are divergent. Motivated by this, we will assume that the field theoretic system we are looking for must have an instability which creates the divergence in the size of strings. This instability would ordinarily exclude it as a sensible theory. Indeed, as in the conventional description of fundamental strings, almost everything one might think of calculating diverges, except for the spectrum and the S-matrix. All these features may seem surprising, but they are unavoidable in any theory of fundamental strings. In order to display them, we consider what the typical fundamental string looks like in space-time.

We will begin by considering the wave function of a string in the light-cone frame. The points of a closed string are parametrized by σ running from 0 to 1. The parameter is defined so that the total longitudinal momentum P^+ is uniformly distributed over σ. The light-cone hamiltonian for the transverse coordinates of the string is

$$H = \frac{1}{2}\int_0^1 d\sigma\Big(-\frac{\delta}{\delta X^i(\sigma)}\frac{\delta}{\delta X^i(\sigma)} + (\frac{\partial X^i}{\partial\sigma})^2\Big) \tag{1.1}$$

The $D-2$ transverse coordinates are free fields with mode expansions

$$X^i(\sigma) = X^i_{cm} + \sum_{n>0}(X^i_n\cos(2\pi n\sigma) + \bar{X}^i_n\sin(2\pi n\sigma)) \tag{1.2}$$

The wave function of the ground state of the string has the product form (dropping the superscript i)

$$\Psi(X(\sigma)) = \prod_n\Big((\frac{\omega_n}{2\pi})^{1/2}\exp(-\omega_n(X_n^2 + \bar{X}_n^2)/4)\Big) \tag{1.3}$$

with $\omega_n = 2\pi n$. Squaring this gives a probability distribution for the transverse position of the string. We will use this distribution to generate a statistical ensemble of strings and plot their transverse positions. To carry this out in practice, it is necessary to truncate the mode expansion at some maximum wave number N. This is one of the ways of introducing a cut-off in the parameter space of the string. Passage to the continuum limit is achieved as $N \to \infty$. This cut-off procedure can be thought of as introduction of time averaging into the measurement of transverse position of the string. Indeed, if the string is 'photographed' with light-cone time exposure τ, then all the modes with $\omega_n \gg 1/\tau$ average to zero and can be neglected. The cut-off is removed as $\tau \to 0$.

A string configuration is determined by a sequence of values of X^i_n and $\bar{X}^i_n$, with $n = 1, \ldots, N$ and $i = 1, \ldots, D-2$, sampled with probability

$$P(X^i_n) = (\frac{\omega_n}{2\pi})^{1/2}\exp(-\omega_n(X^i_n)^2/2) \tag{1.4}$$

and similarly for $\bar{X}^i_n$. Each such configuration defines a parametrized curve in $(D-2)$-dimensional space. By necessity we show projection of string onto 2 transverse dimensions. Each run consists of choosing $100 \times (D-2)$ random numbers from their respective probability distributions. For each run we compute curves with $N = 10$ to 50, with increments of 10. We adopt the following method: as we proceed, for example, from $N = 10$ to $N = 20$, the coefficients of the normal modes with the first 10 wave numbers are kept the same as for the $N = 10$ 'snapshot'. Therefore, for each run, increasing N corresponds to 'photographing' the same string with shorter exposure. The 'snapshots' generated by one such run are shown in figure 1. In fact, we find that different runs are overwhelmingly likely to produce 'snapshots' with the following features.

1) The size of the region in transverse space occupied by string grows slowly with N. A quantitative measure of this is r, the *rms* distance of a point on the string to its center of mass:

$$r^2 = \left\langle (\vec{X}(\sigma) - \vec{X}_{cm})^2 \right\rangle \tag{1.5}$$

Since there is no preferred point on the closed string, we can arbitrarily set $\sigma = 0$.

$$r^2 = (D-2) \left\langle (\sum_{n=1}^{N} X_n)^2 \right\rangle = (D-2) \sum_{n=1}^{N} \langle X_n^2 \rangle = \frac{D-2}{2\pi} \sum_{n=1}^{N} \frac{1}{n} \tag{1.6}$$

The *rms* radius of the string grows with the mode cut-off as $\sqrt{\log N}$.

2) The plots of total string length L *vs.* N appear to be linear. To show this analytically, we start with

$$\langle L \rangle = \int_0^1 \langle |\vec{v}| \rangle \, d\sigma, \tag{1.7}$$

where $\vec{v} = d\vec{X}/d\sigma$. By translation invariance in σ, $\langle L \rangle = \langle |\vec{v}|(\sigma = 0) \rangle$. Using

$$v^i(\sigma = 0) = \sum_{n=1}^{N} n \bar{X}_n^i \tag{1.8}$$

and the fact that each $\bar{X}_n^i$ is gaussian distributed, one can show that v^i is gaussian distributed with variance

$$\Sigma^2 = 2\pi \sum_{1}^{N} n = \pi N(N+1). \tag{1.9}$$

It is easy to show that this implies

$$\langle L \rangle = \langle |\vec{v}| \rangle \sim \Sigma \sim N \tag{1.10}$$

3) The string becomes space-filling in the limit $N \to \infty$. Since the *rms* radius grows only as $\sqrt{\log L}$,* the string tends to cover the same region of transverse space many times. As we remove the cut-off, the string passes arbitrarily close to any point in space.

4) The string is smooth. As the number of modes increases, there is no tendency to develop small-scale structure in space. Transverse line curvature is a quantitative measure of smoothness. We will now show that, in any number of transverse dimensions greater than two, the expectation value of curvature is completely cut-off independent.† The transverse extrinsic curvature is conveniently expressed as

$$\kappa = \frac{|\vec{a}_\perp|}{v^2} \tag{1.11}$$

where $\vec{a}_\perp$ is the component of $\vec{a} = d^2\vec{X}/d\sigma^2$ normal to $\vec{v}$. Since

$$\frac{d^2X^i}{d\sigma^2}(\sigma = 0) = -4\pi^2 \sum_{n=1}^{N} n^2 X_n^i, \tag{1.12}$$

eq. (1.8) implies that $\vec{a}$ and $\vec{v}$ are uncorrelated. Therefore,

$$\langle \kappa \rangle = \langle |\vec{a}_\perp| \rangle \left\langle \frac{1}{v^2} \right\rangle \tag{1.13}$$

★ In other words, the Hausdorff dimension of the fundamental string is infinite.

† In two transverse dimensions the expectation value of the extrinsic curvature diverges due to a relatively high likelihood of cusps.

We find that $\langle 1/v^2 \rangle \sim 1/\Sigma^2$ and $\langle |\vec{a}_\perp| \rangle \sim \widetilde{\Sigma}$, where

$$\widetilde{\Sigma}^2 = \sum_1^N n^3 = \left(\frac{1}{2}N(N+1)\right)^2 \sim \Sigma^4 \tag{1.14}$$

It follows that $\langle \kappa \rangle \sim \widetilde{\Sigma}/\Sigma^2$ is independent of the cut-off!

5) The entire string consists of $O(N)$ smooth 'loops'. The structure of a loop is roughly independent of N. Adding modes simply adds more loops. Furthermore, each loop occupies a fraction $\sim 1/N$ of the parameter space σ. Thus, in a fairly uniform manner the individual loops tend to carry a longitudinal momentum $\sim P^+/N$. To show that no 'accidents' occur as we proceed to high values of N, we have plotted in figure 2 the section of the string confined between $\sigma = 0$ and $2/N$ for $N = 20$ and $N = 500$. The remarkable similarity between the two can be qualitatively regarded as a statement of conformal invariance in our approach.

One may wonder if any of the effects discussed above are artifacts of the sharp mode cut-off. We have checked that this is not the case. Also, one can introduce a different regularization which is quite appealing on physical grounds. Imagine chopping the string into $2N+1$ segments, each one carrying longitudinal momentum $P^+/2N+1$. Each segment is replaced by an indivisible 'parton' m with transverse coordinates $X^i(m)$. A rough way to describe this is to say that the string cannot be subdivided into pieces with longitudinal momentum less than $P^+/2N+1$. The regulated string hamiltonian is

$$H = \frac{2N+1}{2} \sum_m \left(-\frac{\delta}{\delta X^i(m)} \frac{\delta}{\delta X^i(m)} + \left(X^i(m+1) - X^i(m)\right)^2 \right) \tag{1.15}$$

It is exactly diagonalized by Fourier modes (1.2) evaluated at discrete positions $\sigma(m) = m/2N+1$. The ground state wave function is given by (1.3) with frequencies $\omega_n = (4N+2)\sin(\pi n/2N+1)$. In figure 3 we show a typical picture of the regularized string ($N = 50$). Note that, as more and more discrete points crowd the σ-axis, the string never becomes continuous in space. We can show that, as

$N \to \infty$, the average distance in space between each pair of neighboring partons approaches a constant value a, which is of the order of the Planck length. This is responsible for the linear growth of the total length. It is also easy to show that the *rms* radius grows as $\sqrt{\log N}$. Thus, all the important information about the spatial properties of string can be obtained in the regularization where the string never becomes continuous in space. This suggests that, if the transverse space is replaced by a lattice with spacing of the order of a, the critical properties of strings will not be affected. In what follows we confirm this expectation by explicitly constructing a class of discrete field theories which give rise to the fundamental string behavior.

One may wonder whether any of the strange effects described above are observable. In particular, the infinite *rms* radius seems very unphysical. However, it does lead to an observable effect. Consider scattering of a high-energy string from a string at rest. The interaction is mediated by string exchange. In the light-cone frame of the fast string of energy E the lifetime of the interaction is of order $\tau = 1/E$. Oscillations with frequency $\gg 1/\tau$ average to zero. Thus, we retain a number of modes $\sim E$. This introduces a mode cut-off and gives an observable particle radius $\sim \sqrt{\log E}$. As the resolution is improved, the string 'expands'. This phenomenon leads to the well-known Regge behavior of scattering cross-sections satisfied by the dual amplitudes.[1)] Thus, the effect is indeed observable and presents no difficulty for scattering of strings by strings. On the other hand, if the string is scattered by a local external field, then the interaction is instantaneous. Therefore, the string must appear infinite. Thus, the fundamental strings cannot be consistently coupled to local external fields: they are either a theory of everything or of nothing.

The difference between the fundamental strings and the extended objects in the conventional field theory can be easily exhibited by studying the spatial distribution of the longitudinal momentum P^+. For a hadron, this could be measured by interaction with external gravitational field. The result is a non-singular form factor $F(q^2)$. For a fundamental string, an analogous form factor can be obtained by observing that the distribution of P^+ is measured by the vertex operator

$\partial_\alpha X^+ \partial^\alpha X^+ \exp(iq \cdot X)$, where α is the world sheet index. In the light-cone gauge $X^+ = \tau$ and the form factor reduces in $D = 26$ to

$$F(q^2) = \left\langle \int d\sigma \exp(iq \cdot X(\sigma)) \right\rangle \sim \exp(-\frac{6}{\pi} q^2 \log N) \tag{1.16}$$

In the limit $N \to \infty$ the form factor is non-vanishing only at $q = 0$. It follows that P^+ is smeared uniformly all over transverse space. This peculiar property applies not only to the ground state of the string but also to any finitely excited state. For any such state the change in the wave function relative to the ground state concerns only a finite number of normal modes and becomes negligible in the limit $N \to \infty$. The strange behavior of the gravitational form factors is possibly connected with the existence of the graviton: at least for the massless spin-2 state it could be foreseen on the basis of general principles. A theorem by Weinberg and Witten[2] states that, in a Lorentz invariant theory with a Lorentz invariant energy-momentum tensor, the gravitational form factor of a massless spin-2 particle must satisfy $F(q^2 \neq 0) = 0$. How do various theories get around this theorem? In Einstein gravity one cannot define a Lorentz invariant energy momentum tensor. On the other hand, in 26-dimensional string theory the form factors are Lorentz-invariant by construction, and the theorem must apply. It seems to require that, if the graviton is to exist in string theory, it must be smeared all over transverse space. This suggests that the fundamental string theory uses its infinite zero-point fluctuations to allow the existence of gravitons.

2. LATTICE THEORY ON THE LIGHT CONE

In this section we describe a light-cone lattice gauge theory due to Bardeen, Pearson and Rabinovici.[3] It has string-like excitations which become free in the large-N_c limit. This theory has a parameter, μ, which governs the average length of string in the ground state. For μ greater than the critical value μ_c the average length of string is finite in lattice units. For $\mu = \mu_c$ the average length of string diverges. We will be interested in the range $\mu < \mu_c$ where the theory is unstable with respect to infinite growth of strings. This instability is similar to the behavior of the fundamental strings discussed in the previous chapter. In fact, we will show that, for $\mu < \mu_c$, the strings in this light-cone lattice theory are identical to the fundamental strings. This exact equivalence does not require lattice spacing to be taken to zero.

We introduce the light-cone variables $x^+ = (x^0 + x^{D-1})/\sqrt{2}$, $x^- = (x^0 - x^{D-1})/\sqrt{2}$, and x^i, where i labels the $D-2$ transverse directions. Following Bardeen, Pearson and Rabinovici we replace the transverse space by a $(D-2)$-dimensional cubic lattice $\vec{n}$ with integer coordinates. On each directed link of the lattice L there is a unitary matrix-valued variable which satisfies $U_{ij}(L) = U^{\star}_{ji}(-L)$. In the light-cone quantization the variable x^+ is treated as time. After passing to the light-cone gauge $A_- = 0$ the gauge theory light-cone Lagrangian becomes

$$
\begin{aligned}
L =& \frac{1}{g^2}\int dx^- \{ \sum_{links} tr\big(\partial_\mu U(L)\partial^\mu U(-L)\big) + \sum_{plaq} tr\big(U(L_1)U(L_2)U(L_3)U(L_4)\big) + \\
& g^4 \int dx^{-\prime} \sum_{\vec{n}} |x^- - x^{-\prime}| J^m_-(\vec{n},x^-) J^m_-(\vec{n},x^{-\prime}) \}
\end{aligned}
\tag{2.1}
$$

where the index μ refers to the $+$ and $-$ directions and the lattice spacing has been set to 1 for convenience. $J^m_-(\vec{n})$ denotes the longitudinal momentum current at the site $\vec{n}$. To make the theory tractable, ref. 3 relaxes the condition of unitarity. Thus the matrices U are replaced by gM where $M(L)$ are general $N \times N$ complex matrices. In order to restore unitarity, ref. 3 introduces the effective potential for

M:

$$V(M) = \mu\ M_{ij}M^{\star}_{ij} + g^2\lambda\ M_{ij}M^{\star}_{ik}M_{lk}M^{\star}_{lj} \tag{2.2}$$

The speculation of ref. 3 is that the continuum limit of the $U(N_c)$ gauge theory can be obtained by tuning μ and λ along a renormalization group trajectory, as the lattice spacing is being taken to zero. We will not pursue this interesting idea here. Instead we will argue that, with no fine tuning, there exists a broad range of parameters where this theory exactly reproduces fundamental strings.

For our purposes it is necessary to add to the lagrangian other terms quartic in M which would be trivial if gM was unitary. These are the plaquette-like terms $tr(M^4)$ shown in figure 4. The trace is taken around all possible loops of length 4 and zero area. In fact the second term in (2.2) is of this type. All other terms of this type reside on pairs of links, L and K, beginning at the same vertex:

$$\delta L = -g^2\lambda' \int dx^- \sum_{L,K} tr\big(M(L)M(-L)M(K)M(-K)\big) \tag{2.3}$$

Using standard methods we derive the light-cone hamiltonian

$$\begin{aligned} P^- = g^2 \int dx^- \{ &\sum_{links} tr\big(\frac{\mu}{g^2}M(L)M(-L) + \lambda\ M(L)M(-L)M(L)M(-L)\big) + \\ \lambda' &\sum_{L,K} tr\big(M(L)M(-L)M(K)M(-K)\big) - \sum_{plaq} tr\big(M(L_1)M(L_2)M(L_3)M(L_4)\big) - \\ \int dx^{-\prime} &\sum_{\vec{n}} |x^- - x^{-\prime}| J^m_-(\vec{n}, x^-)J^m_-(\vec{n}, x^{-\prime}) \} \end{aligned} \tag{2.4}$$

The link fields may be decomposed into creation and annihilation operators:

$$M_{ij}(x^-) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2k}} \big(A_{ij}(k)\exp(-ikx^-) + B^{\dagger}_{ij}(k)\exp(ikx^-)\big) \tag{2.5}$$

which obey the commutation relations

$$[A_{ij}(k), A^{\dagger}_{kl}(q)] = [B_{ij}(k), B^{\dagger}_{kl}(q)] = \delta_{ik}\delta_{jl}\delta(k-q) \tag{2.6}$$

Each link is assigned a direction and $A^{\dagger}_{ij}(k)$ creates a string bit which carries longitudinal momentum k and points from index i to index j along the link's direction. Similarly, $B^{\dagger}_{ij}(k)$ creates a bit pointing from j to i and opposite to the assigned direction. An oriented string state is defined in the following way. Consider an oriented connected loop Γ consisting of links L_i, with i running from 1 to N. The Fock state associated with this loop is

$$tr\{O^{\dagger}(L_1, k_1)\ldots O^{\dagger}(L_N, k_N)\}|0> \tag{2.7}$$

where $O^{\dagger}_{ij} = N_c^{-1/2} A^{\dagger}_{ij}$ or $N_c^{-1/2} B^{\dagger}_{ji}$ depending on whether a given string bit points along or opposite the assigned direction.

An important feature of the $N_c \to \infty$ limit of the theory is that the light-cone hamiltonian (2.4) does not split or join strings. If we apply it to a one-string state of the form (2.7), we obtain another such state. Therefore, there exists a linear dynamical equation for one-string states, which is simply the light-cone Schroedinger equation. Such an equation should not be confused with the Migdal-Makeenko equation[4)] for the expectation value of the Wilson loop in a gauge theory. One conspicuous difference is that the Migdal-Makeenko equation is non-linear even in the large-N_c limit, where strings become free. We believe that all the useful information about the large-N_c limit of the theory in (2.4) is contained in the linear Schroedinger equation. On the first glance, solving this equation appears to be formidable. It is even difficult to write it down in an explicit form, since the action of various terms in the hamiltonian is quite complicated. Nevertheless, there is a phase of the theory, where the hamiltonian can be easily analyzed and shown equivalent to a free field hamiltonian. Fortunately, this is the fundamental string phase we are after. For now, let us compare various terms in (2.4) with (1.1), the hamiltonian of a free fundamental string. For example, the term

$$\mu \int dx^- \sum_L tr\big(M(L)M(-L)\big) \tag{2.8}$$

plays the role similar to the second term in (1.1). It accounts for the potential energy due to the string tension. The other terms in (2.4) can locally displace bits of string.* They are the analogues of the first term in (1.1); *i.e.*, they give rise to the string kinetic energy. Below we will argue that, in the phase of the lattice theory where strings are infinitely long, its light-cone hamiltonian (2.4) is, in fact, identical to (1.1). We will show this in detail for the lattice model in one transverse dimension. There, each lattice state can be mapped into a function $X(\sigma)$ used in the light-cone description of the fundamental strings. Let the string carry longitudinal momentum k_t. Starting with an arbitrary lattice site, we identify it with $X(\sigma = 0)$. The first link is represented by a segment between $\sigma = 0$ and $\sigma = k_1/k_t$. Thereafter, each link is represented by a segment with length in parameter space proportional to its longitudinal momentum. At the point σ_i the function $X(\sigma_i)$ is given by the lattice coordinates of the corresponding site. Between sites σ_i and σ_{i+1}, $X(\sigma)$ can be defined by a linear interpolation.

The phase structure of the model is governed by the string potential energy, which comes from the action of (2.8) and is $\sim \mu \sum_1^N k_i^{-1}$. It is clear that, for sufficiently large values of the string tension μ, the ground state will be dominated by states with small numbers of links, with each link carrying a significant fraction of the total longitudinal momentum. In this case we do not expect a behavior similar to the fundamental strings. As we decrease μ it becomes energetically favorable to have a larger number of links with smaller k. In fact, it is clear that, for a sufficiently negative μ, an instability develops which favors strings of infinitely many links each one carrying infinitesimal k. Evidently, in this case the σ-axis becomes densely populated suggesting the possibility of a continuum description in σ-space. Indeed, from here on 'continuum limit' will always refer to the continuum limit in parameter space, and not in real space.

In order to discuss this phase we need a regulator in the form of a minimum allowed longitudinal momentum. Following ref. 5 we will think of the matrices $M(x^-)$ as anti-periodic in the x^- direction. Then the k-space is discretized and

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⋆ The reader can check this by applying various terms in the hamiltonian (2.4) to simple string configurations.

the zero-momentum links are excluded so that each link carries an odd integer multiple of the minimum momentum k_m. The total length of string cannot exceed $k_t/k_m = N$ in lattice units. The limit $N \to \infty$ defines our lattice string theory.

3. EFFECTIVE HAMILTONIAN AND EQUIVALENCE TO A FREE FIELD

We will begin studying this model in the limit $\mu \to -\infty$. In this limit the ground state and low-lying excitations consist of strings of maximum possible length $k_t/k_m = N$. To solve for their wave functions, we use degenerate perturbation theory in the 'kinetic terms' of the hamiltonian. It turns out that the current-current interaction has vanishing matrix elements between the states of maximum allowed length. The only terms in the hamiltonian that act to move the string in the transverse dimension are the plaquette-like terms introduced above. Fortunately, the model based on these terms is soluble exactly. Below we describe the necessary construction in some detail.

If the maximum allowed length is N then the string configurations that need to be included are labeled by series of N pluses and minuses subject to the closed string constraint that their sum is zero. This requires N to be even. It is convenient to think of these configuration as series of Ising spins $\sigma_3(i)$. There are two types of terms that need to be taken into account. The first one,

$$\lambda' \sum_n \int dx^- M_{ij}(n) M_{jk}(n+1) M^\dagger_{lk}(n+1) M^\dagger_{il}(n), \tag{3.1}$$

is shown in figure 4b). The second one,

$$\lambda \sum_n \int dx^- M_{ij}(n) M^\dagger_{kj}(n) M_{kl}(n) M^\dagger_{il}(n), \tag{3.2}$$

is shown in figure 4c). It is not hard to show that, to leading order in N_c, the action of (3.1) on states in the spin representation is equivalent to

$$\lambda' N \sum_n \left(\frac{1+\sigma_3(n)\sigma_3(n+1)}{2} + \sigma_+(n)\sigma_-(n+1) + \sigma_-(n)\sigma_+(n+1)\right) \tag{3.3}$$

where in terms of the standard Pauli matrices

$$\sigma_+ = \frac{\sigma_1 + i\sigma_2}{2}, \qquad \sigma_- = \frac{\sigma_1 - i\sigma_2}{2} \tag{3.4}$$

and we have set $g^2 N_c = 1$. Similarly, the term (3.2) is represented by

$$\lambda N \sum_n (1 - \sigma_3(n)\sigma_3(n+1)) \tag{3.5}$$

Let us choose temporarily $2\lambda = \lambda' = 1$. Then, up to an additive constant, the hamiltonian is simply given by the quantum XY model:

$$H = N \sum_n (\sigma_+(n)\sigma_-(n+1) + \sigma_-(n)\sigma_+(n+1)) \tag{3.6}$$

Actually, since we have ignored the center of mass motion of strings, the above hamiltonian is only applicable to states that are translation invariant on the transverse lattice (have zero lattice momentum). A generalization of the effective hamiltonian to states of finite lattice momentum will be given below. But first let us show that the quantum XY model can be solved by introducing anti-commuting variables

$$\psi_+(n) = i^{-n} \prod_{m<n} \sigma_3(m)\sigma_+(n), \tag{3.7}$$

$$\psi_-(n) = i^{+n} \prod_{m<n} \sigma_3(m)\sigma_-(n). \tag{3.8}$$

These are the staggered fermions on the lattice in the parameter space of the string.[⋆] In terms of these variables the fermion number on a site $\frac{1}{2}\sigma_3(n)$ is given

⋆ These fermions have nothing to do with the world sheet fermions of superstrings.

by $\frac{1}{2}[\psi_+(n), \psi_-(n)]$. The transverse spatial separation between any two points on the string is then given by twice the fermion number contained between these two points. Since the string is closed, we must always work in the sector where the total fermion number is zero. We define a fermion doublet

$$\psi(n) = \sqrt{\frac{N}{2}} \begin{pmatrix} \psi_-(2n-1) \\ \psi_-(2n) \end{pmatrix} \tag{3.9}$$

Then in the continuum limit $N \to \infty$ the hamiltonian reduces to

$$H = i \int d\sigma \psi^\dagger \alpha \frac{\partial \psi}{\partial \sigma} \tag{3.10}$$

together with the boundary condition $\psi(\sigma = 1) = -\psi(\sigma = 0)$. α acts on Dirac indices as the Pauli matrix σ_1. It is well-known[6,7] that a Dirac fermion is equivalent to a periodic boson variable $\phi(\sigma)$ which is defined so that the fermion number density

$$\frac{1}{2}[\psi^\dagger, \psi](\sigma) = \frac{1}{\sqrt{\pi}} \frac{\partial \phi}{\partial \sigma} \tag{3.11}$$

From the previous discussion it follows that the separation between two points on the string is $\frac{2}{\sqrt{\pi}}(\phi(\sigma_1) - \phi(\sigma_2))$. Therefore the bosonized variable $\frac{2}{\sqrt{\pi}}\phi(\sigma)$ is a smeared version of the original lattice position $X(\sigma)$. The boson hamiltonian equivalent to (3.10) is

$$\frac{1}{2}\Pi_{cm}^2 + 2\pi \sum_m m(a_m^\dagger a_m + \tilde{a}_m^\dagger \tilde{a}_m) \tag{3.12}$$

Since the field ϕ is defined on a circle of radius $\frac{1}{2\sqrt{\pi}}$ the center of mass momentum is restricted to integer multiples of $2\sqrt{\pi}$. This is connected with the fact that we are studying the theory in the sector of zero lattice momentum: the states are invariant under discrete shifts of 2 lattice units. To introduce non-zero lattice momentum let us consider string states which pick up a phase $\exp(ip)$ when translated by one lattice unit. The effective hamiltonian for this system is

$$H = N \sum_n (\sigma_+(n)\sigma_-(n+1)\exp(2ip/N) + \sigma_-(n)\sigma_+(n+1)\exp(-2ip/N)) \quad (3.13)$$

To justify the appearance of phases in the above formula we note that whenever $\sigma_+(n)\sigma_-(n+1)$ acts on a string state, the center of mass moves $2/N$ lattice units to the right. Similarly, the conjugate term moves the center of mass to the left. In fact, the interaction (3.13) can be obtained from (3.6) by introducing a constant vector potential along the string. A gauge transformation

$$\sigma_-(n) \to \sigma_-(n)\exp(-2ipn/N), \qquad \sigma_+(n) \to \sigma_+(n)\exp(2ipn/N) \qquad (3.14)$$

reduces (3.13) to (3.6) at the expense of a non-trivial boundary condition on the continuum fermionic variables:

$$\psi(\sigma = 1) = -\psi(\sigma = 0)\exp(-2ip). \qquad (3.15)$$

Upon bosonization, this condition changes the constraint on the values of the center of mass momentum:

$$\Pi_{cm} = 2\sqrt{\pi}(n + \frac{p}{\pi}), \qquad (3.16)$$

where n is an integer. After inclusion of string states with non-zero lattice momentum, the zero mode in the boson hamiltonian has acquired continuous spectrum. As a result, (3.12) is the standard light-cone free string hamiltonian. The physical reason is the fact that, as $N \to \infty$, the possible positions of the string center of mass become continuous.

Now we would like to argue that relaxing the constraint $\lambda' = 2\lambda$ on the coefficients of the plaquette-like terms does not in general violate the free boson behavior demonstrated above. In fact, this adds a term

$$-cN \sum_n \sigma_3(n)\sigma_3(n+1) \tag{3.17}$$

to the effective hamiltonian (3.13). This term can be interpreted as introducing dependence on the extrinsic line curvature into the string hamiltonian.[8)] Depending on the sign of the coefficient c, it favors alignment or anti-alignment of the adjacent links. The continuum limit of this new hamiltonian is the Thirring model. For any $-1 \leq c < 1$, the Thirring model is equivalent to a free boson field. The only effect of changing c is rescaling of the string tension. Increasing c makes string wiggles look bigger on a fixed lattice scale. Therefore, pairs of adjacent links become more aligned. As $c \to 1$ the size of the wiggles diverges. For $c \geq 1$ alignment is favored too strongly and the relativistic free field behavior is no longer valid. On the other hand, for $c < -1$, anti-alignment is favored so much that the behavior of lattice strings is dominated by the lattice. Our discussion confirms the simple intuition that one should be allowed to take the spatial continuum limit in our lattice model without changing the equivalence with fundamental strings. However, if we increase the lattice spacing beyond the typical size of string wiggles in the 'fundamental' phase, a transition to the lattice-dominated phase takes place. Thus, for a broad range of parameters, the extrinsic curvature terms do not alter the critical behavior of strings.

In section 1 we demonstrated that the spatial behavior and energy levels of our lattice string theory are completely determined by the equivalence with a massless free field in σ-space. For example, this guarantees that the growth of the mean squared radius of string is logarithmic in the length of string. As in the usual string formalism, the ground state energy is quadratically divergent. In our system it has the negative sign. Conventionally, in the light-cone formalism this divergence is absorbed in renormalization of the speed of light.

Next consider the corrections to the limit $\mu \to -\infty$. Then the terms in the hamiltonian which act to decrease the total length of string become important.

For example, the plaquette-like terms can replace three links of longitudinal momentum k_m by one link of momentum $3k_m$. Such perturbations act locally on the string and in the continuum limit can be represented by a series of operators local in σ-space. The only operators that can affect the critical behavior in σ-space are renormalizable and super-renormalizable. It turns out that the only renormalizable operators that can be induced are just the two terms already present in the hamiltonian (1.1). Therefore, a range of μ must exist in which the effect of the corrections to the limit $\mu \to -\infty$ is absorbed in rescaling of the string tension. Since for large and positive μ there exists another phase of the theory where strings have finite length, there must be a critical value $\mu = \mu_c$ where a phase transition occurs.

A straightforward generalization of this lattice field theory can be given in any number of dimensions. If $D = 26$ then, in the phase where strings are infinite, all the finite energy spectrum is identical to the spectrum of the conventional bosonic string. The same equivalence applies to the expectation values of products of vertex operators.

4. CONCLUSIONS

Perhaps the most striking result of this work is that a discrete theory is completely equivalent to a free string in continuous space. No spatial continuum limit is required. This is possible because we are dealing with a theory of extended objects. Since their length diverges, the physics depends only on the critical properties of the effective 1+1-dimensional field theory induced on the string. In this way, theories with discrete and continuous values of transverse space coordinates end up in the same universality class. The physical reason is that the instability only allows string bits with vanishingly small longitudinal momenta. Typically, the light-cone time scale for motion of the i-th string bit is $\sim P^+(i)$, *i.e.*, the string bits move very rapidly and completely wash out any memory of the lattice. Evidently, this effect does not depend on the details of any particular lattice structure. We expect that there exists a large universality class of discrete theories which exhibit identical behavior for a broad range of parameters. They share one important ingredient – the instability.

Another interesting question is the connection between the theory discussed above and the $N_c = \infty$ QCD. The original work of ref. 3 suggested that QCD is a very special limit of this theory in which the spatial continuum limit is taken as the parameters are carefully tuned along some renormalization group trajectory. This should be contrasted with the fundamental string behavior which occurs for a broad range of parameters, without necessity of taking the lattice spacing to zero.

The construction of a discrete field theory which exactly reproduces bosonic strings makes it clear that string theory has vastly fewer short distance degrees of freedom in space than a conventional quantum field theory.⋆ This was already suggested by the smoothness of the string pictures of figure 1. The absence of the short-distance structure in strings is probably the reason why the fixed-angle scattering amplitudes fall off exponentially at high energy.[10)]

Our model required the number of colors N_c to be taken to ∞. It is interesting to study the new effects induced by a finite N_c. The string can then split and join as in the conventional picture of string interactions. The string coupling constant is proportional to $1/N_c$. It should be possible to determine whether the $1/N_c$ expansion reproduces the bosonic string amplitudes.

Acknowledgements: Section 1 of this talk is based on joint work with M. Karliner.[11)]

⋆ This was also suggested on very different grounds in the work of Atick and Witten (ref. 9).

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FIGURE CAPTIONS

1) Projections of string onto 2 transverse dimensions with mode cut-off varying from 10 to 50.

2) The section of string confined between $\sigma = 0$ and $2/N$ for $N = 20$ (solid line) and $N = 500$ (dashed line).

3) A typical configuration of the ground state of 101 partons connected by springs.

4) The plaquette-like terms which would be trivial if the link variables were unitary.

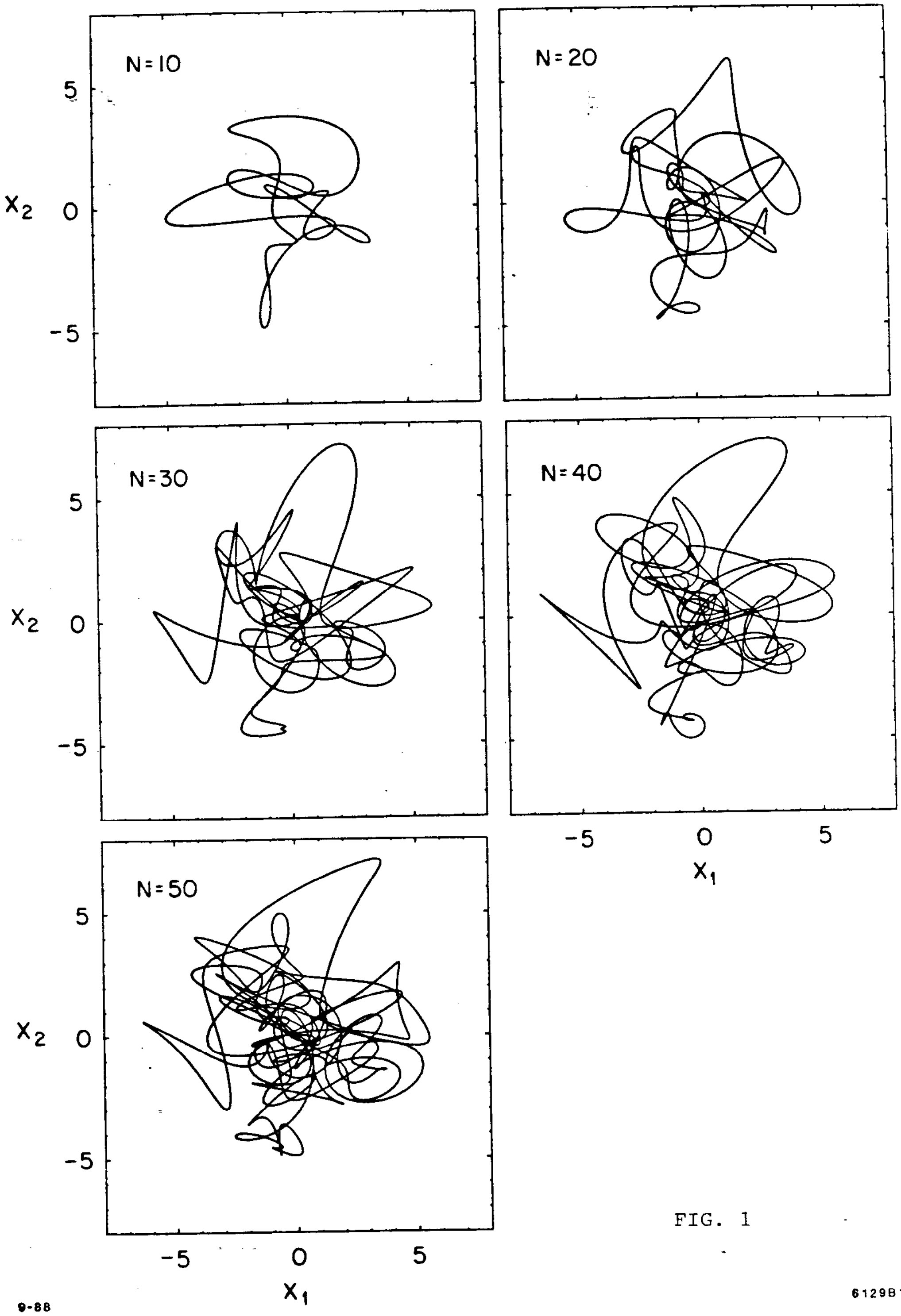

FIG. 1

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Fig 1

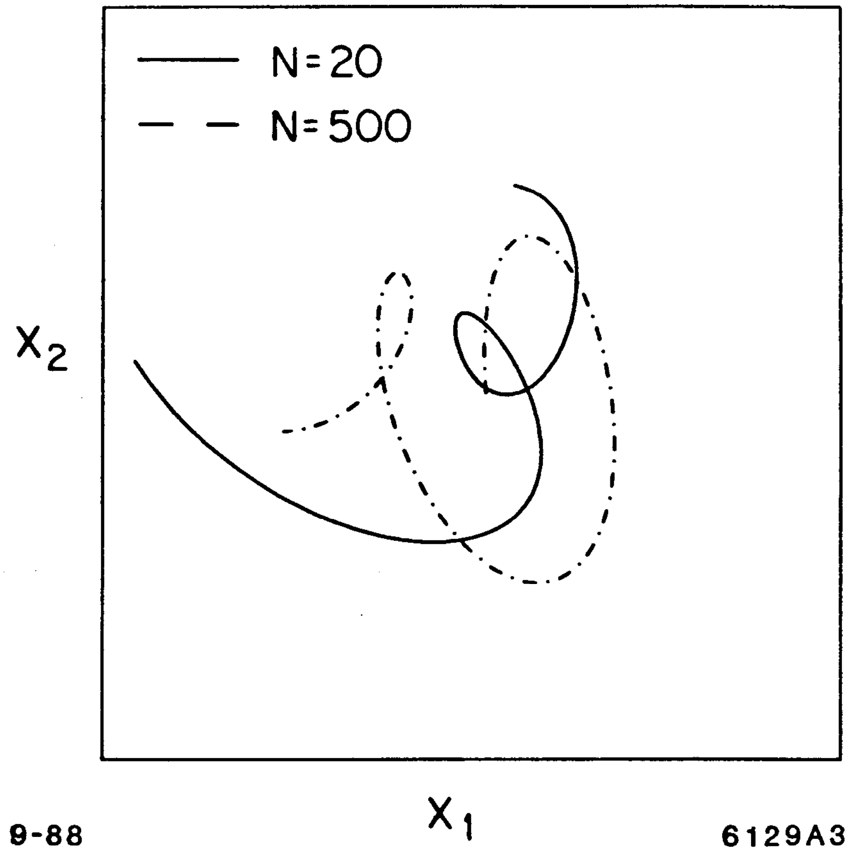

Fig. 2

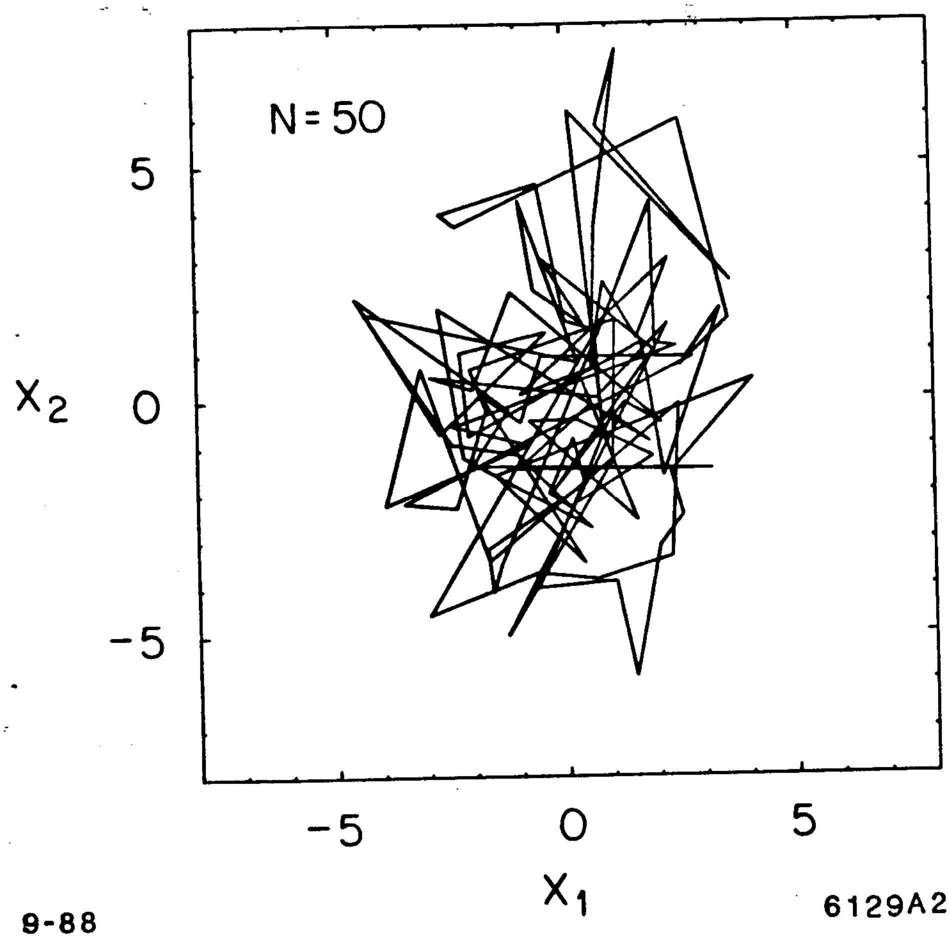

Fig. 3

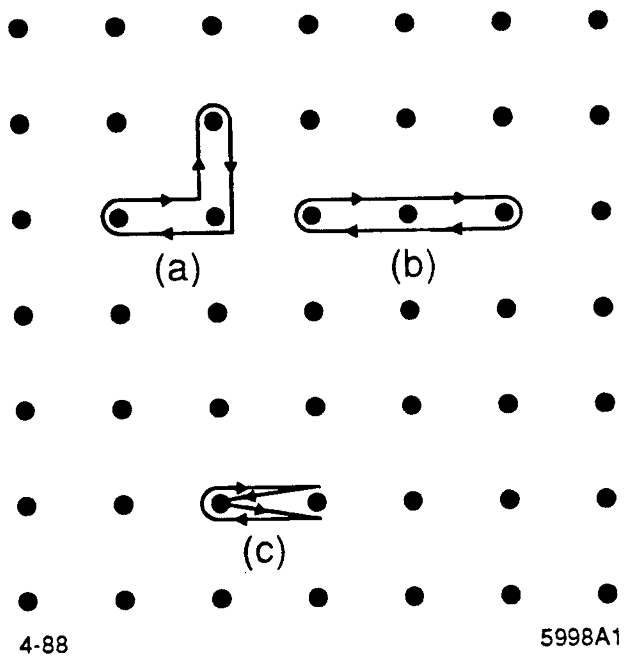

Fig. 4