BUNDLE ADJUSTMENTS
AND TRI-DIMENSIONAL
COORDINATE DETERMINATION *

J. M. GAUNT

Stanford Linear Accelerator Center,

Survey and Alignment Group,

Stanford University, Stanford, CA 94309

Abstract

Since its purchase in 1986, the KERN ECDS-PC system has become an essential tool in the work of SLAC's survey group. The principal element of its computational software is the adjustment system (in fact analogous to a photogrammetical bundle adjustment) which, unfortunately, has proved inflexible to the varied demands of precision accelerator alignment.

With the refusal of KERN to part with the source code, it was decided to produce an "in-house" adjustment system that could cater for the survey specialities of SLAC. This report documents the mathematical and theoretical aspects of the resulting programs; "3DCD" (3-dimensional coordinate determination) and "INTER2" (an interim program used to convert the raw data logged by the ECDS-PC system to a formatted input file for 3DCD).

In addition, "SETOUT" provides an option of setout angle calculation, using output from the adjustment program.

(Submitted for Publications)

Contents

List of Figures

1. INTRODUCTION

In the field of particle accelerator engineering, probably more so than other industrial applications, the precision and accuracy required for the construction of machine elements must be exceedingly high. Thus continuous and stringent quality control is of high priority and it is in this application that the ECDS-PC system has proved to be indispensable, either replacing or complementing existing industrial optical tooling techniques. Its greatest assets are the ease with which a 3-dimensional local reference system can be established, the automatic data collection to an on-site computer, and the accuracy and operational simplicity of the electronic theodolites.

The basic configuration of the KERN ECDS-PC system consists of two KERN E2 theodolites (up to eight can be used concurrently), a portable IBM PC with hard disk, and several interface boxes and cables allowing communications between the instruments and the computer. At SLAC the PC and other accessories are mounted on an electric cart.

It is not the intention of this document to prevaricate on the operations of the ECDS system, yet prior to a discussion concerning the adjustment software it is necessary to illuminate the two types of reference system permissible. The "object" system is used if three (or more) points on the object surface (i.e., the element to be surveyed) have known coordinates. Theodolite positions and orientations are obtained by resecting from these "control points," while, further "object points" can be coordinated by theodolite intersection.

The "local" system is established more arbitrarily: the simplest (and most common) method is to define coordinate axis by setting the coordinates and orientation parameters of one theodolite station to zero, and obtain a scale from observations to a subtense bar. Thus the seven parameters necessary to form a 3-dimensional datum are obtained with respect to which the network is adjusted.

In fact, in terms of adjustment, both cases can be united in an analogy to the method of photogrammetical bundle adjustments (1) where colinearity equations,

4

survey ground control, and tie points determine the relative orientation and position of individual plates in a photographic block. Cameras represent theodolite stations, ground control symbolizes known object points, and tie points correspond to intersected object points. The ECDS adjustment program does in fact adopt the bundle approach, incorporating an ingenious conversion of measured horizontal and vertical angles to imaginary photo coordinates. On the other hand, 3DCD avoids this route, hence no further references to photogrammetry are necessary. However, methods of structuring the large normal matrices involved in bundle adjustments have been adopted, and form the basis of the structure in 3DCD.

The principal arguments for an alternative to the ECDS adjustment program centered around the desire for self-determination rather than continued dependency upon external and inflexible powers. In fact few complaints can be leveled at the present ECDS system except the unavailability of the source code renders adaptations and expansions of the system impossible, unless, of course, one crosses KERN's palm with a considerable amount of silver. In the long run the cheapest and most flexible option was to generate our own software. To a certain extent the present capabilities of 3DCD are similar to those of the ECDS-PC system, although the former permits "theodolite-to-theodolite" (parallel collimation) observations, statistical analysis of the final coordinates, and output of angular residuals (rather than the latter's meaningless values). But, hopefully, the structured design of the program (despite being written in FORTRAN) will facilitate further adaptations as the need arises.

It is envisaged that data collection software will be written at SLAC and combined with 3DCD to form a complete coordinate determination system, totally independent from KERN's. Until then program INTER2 forms an interface between ECDS raw data and SLAC's adjustment program.
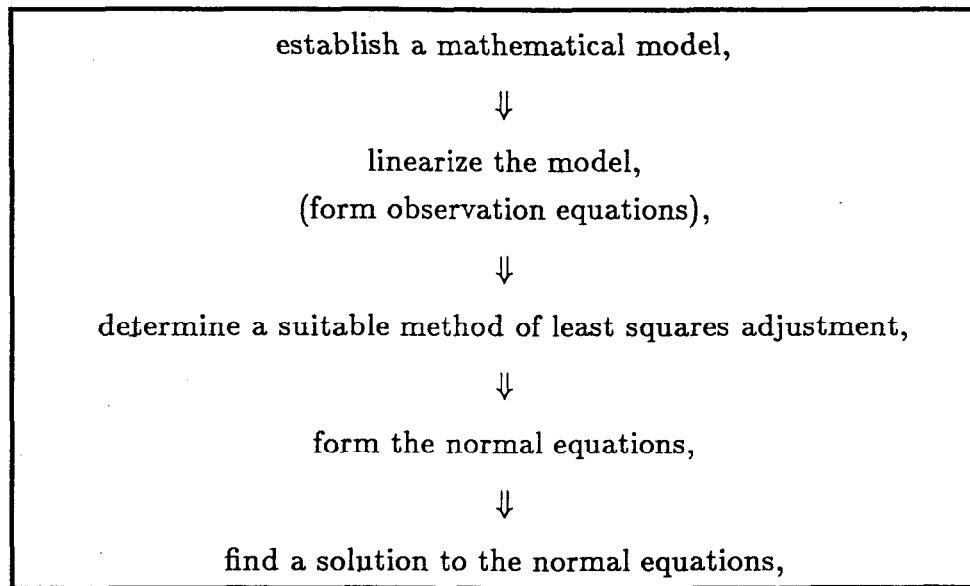
Similarly, other peripheral programs (e.g., circle fits, coordinate transformations, etc.) may be added to the overall package. At present only one such

program exists; SETOUT. This permits the calculation of setout angles to ideal coordinates in an "object" system.

2. 3DCD — A THREE–DIMENSIONAL ADJUSTMENT

2.1 Introduction

It appears that the solution of most least squares problems follows the pre-described flowchart:

establish a mathematical model,

$\Downarrow$

linearize the model,
(form observation equations),

$\Downarrow$

determine a suitable method of least squares adjustment,

$\Downarrow$

form the normal equations,

$\Downarrow$

find a solution to the normal equations,

and, indeed, this pattern is adhered to here. All the mathematical equations associated with these five sections will be derived, but it is assumed that the reader is already familiar with the underlying principles of least squares analysis and survey network adjustments.

2.2 The Mathematical Model

Before deriving any equations it is necessary to introduce and define coordinate systems, rotations, observable quantities, and parameters upon which the model can be based.

Each theodolite station is defined within the model by its position coordinates with respect to the network reference system, and three relative orientation parameters, i.e., the theodolites are assumed non-leveled. (Conventionally the vertical Z axis is defined by the direction of the gravity vector, assumed constant over the network and parallel to the Z axis of the reference system. Hence the orientation of the perpendicular XY plane about the Z-axis is the only relative orientation parameter.) On the other hand, object points involve just the three coordinate parameters. Hence the number of unknowns in the network can, at present, be expressed as

$$u = 6m + 3n \qquad (2.21)$$

where m is the number of theodolite stations, n the object points.

Coordinate reference systems are all right-handed and the rotations about their axes are positive if clockwise when viewed from the origin along the positive axis (Fig. 1). The three relative theodolite orientations parameters are rotations about the theodolite axes $(x,\ y,\ z)$ that transform the theodolite system so as to render it parallel to the network reference system $(X,\ Y,\ Z)$. They have the following notations:
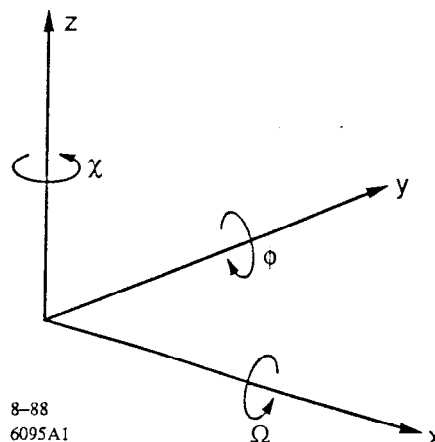

Fig. 1

8

$$R_x = \Omega = \text{rotation about the } x\text{-axis} = \text{pitch}$$

$$R_y = \phi = \text{rotation about the } y\text{-axis} = \text{roll}$$

$$R_z = \chi = \text{rotation about the } z\text{-axis} = \text{yaw} \quad .$$

They are applied in the order χ, then ϕ, then Ω.

The observations involved in the model mainly consist of horizontal and vertical angles, although, in the case of local reference systems, a small number of distances will be included. Horizontal angles, μ are measured clockwise in the xy plane with 0 position along the y-axis, while vertical angles, λ are measured in the perpendicular plane. Thus μ lies between 0–400 gons, λ between $\pm$ 100 gons. (Angles defined under a different system can easily be transformed to a compatible form.) The number of independent observations, n must be greater than or equal to the sum of unknown parameters u. In the former case least squares analysis is required to yield a unique solution. The redundancy, r, of the model is:

$$r = n - u \quad . \tag{2.22}$$

At this stage we can establish equations that combine parameters and observables to form our mathematical model. Consider a pointing from a theodolite station with coordinates (X_T, Y_T, Z_T) to an object point (X_o, Y_o, Z_o). The cosine direction vector with respect to the network reference system and as a function of the parameters is:

$$\underline{V} = \frac{1}{D} \begin{pmatrix} X_o - X_T \\ Y_o - Y_T \\ Z_o - Z_T \end{pmatrix} ,$$

where D is the distance between the two points. An equivalence of this vector can also be calculated from the observed angles (fig. 2). With respect to the

theodolite system its components are:

$$x = \sin \mu \cos \lambda$$

$$y = \cos \mu \cos \lambda \quad .$$

$$z = \sin \lambda$$
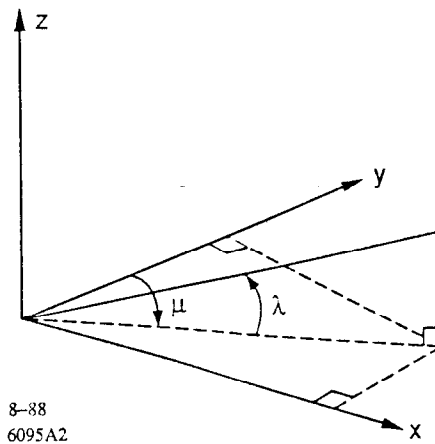

Fig. 2

Multiplying by the rotation matrix $R = R_x o R_y o R_z$, this vector is transformed to the network reference system, whence the equation:

$$R \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{D} \begin{pmatrix} X_o - X_T \\ Y_o - Y_T \\ Z_o - Z_T \end{pmatrix} \tag{2.23}$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega & -\sin\Omega \\ 0 & \sin\Omega & \cos\Omega \end{pmatrix} \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\chi & -\cos\phi\sin\chi & \sin\phi \\ \cos\Omega\sin\chi + \sin\Omega\sin\phi\cos\chi & \cos\Omega\cos\chi - \sin\Omega\sin\phi\sin\chi & -\sin\Omega\cos\phi \\ \sin\Omega\sin\chi - \cos\Omega\sin\phi\cos\chi & \sin\Omega\cos\chi + \cos\Omega\sin\phi\sin\chi & \cos\Omega\cos\phi \end{pmatrix}$$

or

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

Since R is orthogonal, its inverse is equivalent to its transpose, and equation 2.23 can be rewritten as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{D} \begin{pmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{pmatrix} \begin{pmatrix} X_o - X_T \\ Y_o - Y_T \\ Z_o - Z_T \end{pmatrix} \tag{2.24}$$

with

$$D^2 = (X_o - X_T)^2 + (Y_o - Z_T)^2 + (Z_o - Z_T)^2 \quad . \tag{2.25}$$

2.3 Linearized Observation Equations

To produce equations from which observation equations can easily be derived, expand (2.24) and (2.25) to obtain the following:

$$A = r_{11}(X_o - X_T) + r_{21}(Y_o - Y_T) + r_{31}(Z_o - Z_T)$$

$$B = r_{12}(X_o - X_T) + r_{22}(Y_o - T_T) + r_{32}(Z_o - Z_T) \qquad (2.31)$$

$$C = r_{13}(X_o - X_T) + r_{23}(Y_o - Y_T) + r_{33}(Z_o - Z_T) \quad .$$

Let:

$$E = A^2 + B^2$$

$$F = \sqrt{E}$$

$$G = C/D^2 \quad .$$

Then noting that:

$$\sin \lambda = z$$

$$\tan \mu = x/y$$

we can express the three types of observations as individual functions of the parameters as follows:

$$D^2 = (X_o - X_T)^2 + (Y_o - Y_T)^2 + (Z_o - Z_T)^2 \qquad (2.32a)$$

$$\mu = \tan^{-1}(A/B) \qquad (2.32b)$$

$$\lambda = \sin^{-1}(C/D) \quad . \qquad (2.32c)$$

Clearly all three are non-linear. Since the generalized methods of least squares adjustment apply to expressions concerning linearized parameters, equations (2.32)

must themselves be linearized. This is achieved by truncating their Taylor series expansions to first order terms only (2). For a function of n variables, $y = f(x_1, x_2, ..., x_n)$, the linear Taylor series approximation in the neighborhood of $x^o = (x_1^2, x_2^o, ..., x_n^o)$ is given by

$$f(x^o) + \frac{\delta y}{\delta x_1}|_{x^o} \cdot dx_1 + \quad ... \quad + \frac{\delta y}{\delta x_n}|_{x^o} \cdot dx_n$$

where $dx = (dx_1, dx_2, ..., dx_n)$ the vector of increments applied to x^o.

In our case $(x_1, x_2, ..., x_n)$ is the vector of parameters, $(X_T, Y_T, Z_T, \Omega, \phi, \chi, X_o, Y_o, Z_o)$, and x_o is the vector of their initial approximate values. (The method of obtaining approximates is discussed in Chapter 3.) dx is the vector of unknown increments that, when calculated from the least squares solution, will be added to the approximate parameters. It must be stressed that the linear Taylor expansion is only an approximation to the value of a function within the vicinity of x^o. Therefore an iterative approach is necessary: dx is determined, added to x^o, to produce $x^o = x^o + dx$, which is then fed back into the linear observation equations, and so on, until the size of dx becomes sufficiently small as to be negligible. The number of iterations required is dependant upon how close the initial approximate values are to their true values $\hat{x}$. [Figure 3 helps to convey this principle: as dx decreases, the closer the linear approximation of the function at x^o is to the actual function $y^o = f(x^o)$].
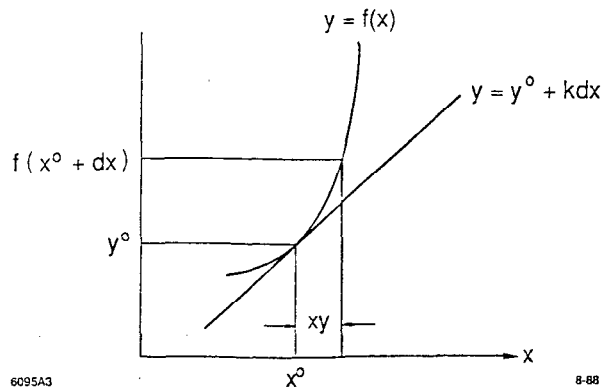

Fig. 3

13

As an example, the term referring to the X_T parameter in the vertical angle observation equation will be derived. Rearrange (2.32c) to give:

$$D \sin \lambda = C = r_{13}(X_o - X_T) + r_{23}(Y_o - Y_T) + r_{23}(Z_o - X_T) \quad .$$

Taking the partial derivatives of both sides:

$$\sin \lambda dD + D \cos \lambda d\lambda = r_{31}(dX_o - dX_T) + r_{23}(dY_o - Y_T)$$
$$+ r_{33}(dZ_o - dZ_T) + (X_o - X_T)\left(\frac{\delta r_{31}}{\delta \phi} + \frac{\delta r_{31}}{\delta \phi} + \frac{\delta r_{31}}{\delta \omega}\right) + \text{etc.}$$

Noting that $D \cos \lambda = \sqrt{(A^2 + B^2)} = F$; and $\sin \lambda = C/D$ then:

$$\frac{C}{D}dD + Fd\lambda = r_{31}(dX_o - dX_T) + \ldots \quad .$$

Isolating the dX_T term, and noting its coefficient (h_1) in the distance observation equation is $-(X_o - X_T)/D$, then

$$d\lambda = \frac{1}{F}(-r_{31} + (X_o - X_T)\frac{c}{D^2})dX_T + \ldots = \frac{1}{F}(G(X_o - X_T) - r_{31})dX_T + \ldots \quad .$$

The complete linearized observation equations are of the form (3):

$$d\lambda = g_1 dX_T + g_2 dY_T + g_3 dZ_T + g_4 d\Omega + g_s d\phi + g_6 d\chi + g_7 dX_o + g_8 dY_o + g_9 dZ_o$$

$$d\mu = h_1 dX_T + h_2 dY_T + h_3 dZ_T + h_4 d\Omega + h_5 d\phi + h_6 d\chi + h_7 dX_o + h_8 dY_o + h_9 dZ_o$$

$$dD = f_1 dX_T + f_2 dY_T + f_3 dZ_T + f_4 dX_o + f_s dY_o + f_6 dZ_o$$

where

$$g_1 = (1/F)(G(X_o - X_T) - r_{13}) = -g_7$$

$$g_2 = (1/F)(G(Y_o - Y_T) - r_{23}) = -g_8$$

$$g_3 = (1/F)(G(Z_o - Z_T) - r_{33}) = -g_9$$

$$g_4 = (1/F)(-r_{33}(Y_o - Y_T) + r_{23}(Z_o - Z_T))$$

$$g_5 = (1/F)((X_o - X_T)(-r_{23}\sin\Omega + r_{33}\cos\Omega) + (Y_o - Y_T)(r_{13}\sin\Omega) - (Z_o - Z_T)r_{13}\cos\Omega)$$

$$g_6 = 0$$

$$h_1 = (1/E)(r_{12}A - r_{11}B) = -h_7$$

$$h_2 = (1/E)(r_{22}A - r_{21}B) = -h_8$$

$$h_3 = (1/E)(r_{32}A - r_{31}B) = -h_9$$

$$h_4 = h_3(Y_o - Y_T) - h_2(Z_o - Z_T)$$

$$h_5 = (X_o - X_T)(-h_3\cos\Omega + h_2\sin\Omega) - (Y_o - Y_T)h_1\sin\Omega + (Z_o - Z_T)h_1\cos\Omega$$

$$h_6 = 1$$

$$f_1 = -(X_o - X_T)/D = -f_4$$

$$f_2 = -(Y_o - Y_T)/D = -f_5$$

$$f_3 = -(Z_o - Z_T)/D = -f_6 \quad .$$

A general, and abbreviated form of these equations is

$$dl = \dot{B}\dot{\delta} + \bar{B}\bar{\delta} \qquad (2.33)$$

where

$\dot{\delta}$ = vector of unknown increments to the theodolite station parameters

$\bar{\delta}$ = vector of unknown increments to the object point parameters

$\dot{B}, \bar{B}$ = the coefficients of the observation equations.

Also let:

$\hat{l} =$ the observation consistent with the final parameter values, $\hat{x}$

$v =$ the observation's residual

$l_c =$ value calculated as a function of the parameters

$l_o =$ the observed value .

Then

$$\hat{l} = l_c + dl$$
$$v = \hat{l} - l_o \qquad . \qquad (2.34)$$
$$dl = (l_o - l_c) + v$$

Hence (2.34) can be written as:

$$B_1 \dot{\delta} + B_2 \bar{\delta} = (l_o - l_c) + v \qquad (2.35)$$

or

$$B\delta = f + v \quad . \qquad (2.36)$$

If all observation equations are united under the format of 2.36, then B is a matrix of order $n \times u$, where n is the total number of observations and u the parameters, δ is a vector of dimension u, and f and v are vectors of dimension n.

2.4 The Unified Least Squares Approach

The observation equation (2.36) is in the form that requires probably the most simplified application of least squares: only condition equations are involved and

each condition contains only one observation (2). Minimization of the function

$$\phi = v^t W v$$

with respect to the parameters, x, produces normal equations of the form

$$N'\hat{x} = t' \qquad (2.41)$$

where

$$N' = B^t W B \qquad (2.42)$$

$$t' = B^t W f$$

and $\hat{x} = N'^{-1}t'$ is the vector of parameter unknowns.

At this stage it is necessary to give a brief explanation concerning the weight matrix, W. Each observation is assigned an *a priori* standard error, σ_{l_i}, its value based upon the capabilities of the instrument used and a degree of operational experience. For the observed angles the default standard error is 0.0005 gons. All observations are assumed independent of one another, i.e.:

$$Cov(l_i, l_j) = 0 \qquad \forall l_i \neq l_j \quad .$$

Therefore the *a priori* variance-covariance matrix, Q_{ll}, is an $n \times n$ diagonal matrix, the *ith* diagonal term being $\sigma_{l_i}^2$ where l_i is the *ith* observation. The weight matrix is its inverse:

$$W = Q_{ll}^{-1} \quad .$$

Finally there remains the task of establishing the reference datum, i.e. fixing the seven parameters that will position, orientate, and scale the network. As mentioned in the introduction (section 1), depending on whether a local or

object system is used, this can be done by fixing coordinates, theodolite orientation parameters, or distances. In the case of distance, this is simply achieved by giving the observation a very large weight (small standard error) thus restricting any variation. For parameters, two approaches are open. First, one can eliminate the fixed parameter from the normal equations. Or, second, the parameters can also be treated as observed quantities: applying a large weight to an "observed" parameter will hold it invariant in the adjustment. The latter, the unified approach, has been chosen because it has the advantage of generality: if each parameter approximate value is accompanied by an a-priori standard error it can be constrained to any degree by the user, independent of input and computational routines.

Mikhail (2) gives two derivations of normal equations that correspond to the unified approach. Here let us consider the observation equation of the *ith* parameter in the format of (2.34)

$$dx_i = (x_o - x_c)_i + v_i \qquad (2.43)$$

and its contribution to the normal equations, where x_o is the initial approximation, x_c the initial approximation updated after successive iterations, and dx the unknown parameter incrementation of the current iteration. Grouping all the parameter observations, assumed independent, together the equation:

$$Idx = f_x + v_x$$

is obtained, I being the u $\times$ u identity matrix and the other elements $u \times 1$ vectors. Its contribution to the normal matrix is:

$$I^t W_{xx} I \equiv W_{xx}$$

and to the right-hand-side vector

$$I^t W_{xx} f \equiv W_{xx} fx$$

where W_{xx} is the weight matrix of the observed parameter. Hence the normal

18

equation becomes:

$$(N' + W_{xx})\hat{x} = (t' + W_{xx}f_x)$$

or

$$N\hat{x} = t \quad .\qquad\qquad (2.44)$$

The components of equations (2.44) and (2.42) have the same dimensions, hence the introduction of the unified approach has no effect upon the format of the normal equations. N is a $u \times u$ matrix, u being the number of parameters, whilst $\hat{x}$ and t are vectors of dimension u.

2.5 Structure of the Normal Equations

The format of the normal equations is based upon that used in bundle adjustments: the vector of parameter unknowns, x, is arranged in such a way that the non-zero coefficients of the corresponding normal matrix lie within a structured pattern, resulting in reduced storage requirements and greater efficiency of solution algorithms. Consider any observation of type "theodolite-object" (the only type that occurs in the analogous bundle adjustment) of the form derived in 2.35, i.e.,

$$B_1\dot{\delta} + B_2\bar{\delta} = f + v$$

or

$$(B_1 : B_2) \begin{pmatrix} \dot{\delta} \\ \bar{\delta} \end{pmatrix} = f + v \qquad\qquad (2.51)$$

where B_1 (order 6×1) contains the coefficients of the theodolite parameters, and B_2 (order 3×1) those of the object point. Also assume that the vector of parameters Δ is ordered so that all those associated with theodolites are at its head, followed by those of the object points. Finally, partition the normal

equation into:

$$N = \begin{pmatrix} \dot{N} & \overline{N} \\ \overline{N^t} & \ddot{N} \end{pmatrix} \qquad t = \begin{pmatrix} \dot{T} \\ \ddot{T} \end{pmatrix}$$

where N is square of order $m \times 6$ (m = number of theodolites) and N is square of order $n \times 3$ (n = number of object points). Both $\dot{N}$ and $\ddot{N}$ are symmetric since N is symmetric. Now the contribution of 2.51 to the normal equation is given by:

normal matrix:

$$B^T W B = \begin{pmatrix} B_1^t \\ B_2^t \end{pmatrix} W (B_1 : B_2) = \begin{pmatrix} B_1^t W B_1 & B_1^t W B_2 \\ (B_1^t W B_2)^t & B_2^t W B_2 \end{pmatrix}$$

right-hand side:

$$B^T W f = \begin{pmatrix} B_1^t \\ B_2^t \end{pmatrix} W f = \begin{pmatrix} B_1^t W f \\ B_2^t W f \end{pmatrix} \quad .$$

The $B_1^t W B_1$ term will be assigned to matrix $\dot{N}$, as only theodolite coefficients are concerned. It is in fact a 6×6 matrix whose diagonal lies on the main diagonal of $\dot{N}$. The cross product term $B_1^t W B_2$ is located in $\overline{N}$ and is a 6×3 matrix. $B_2^t W B_2$, referring only to object point parameters, is a 3×3 matrix lying on the diagonal of $\ddot{N}$. Since (at present) theodolites cannot observe theodolites, object points likewise, we are at the bundle adjustment structure of N, where both $\dot{N}$ and $\ddot{N}$ have strictly diagonal forms, see fig. 4. In past work at SLAC it has often been desired to incorporate collimation type observations between theodolites: an observation of type "theodolite-theodolite." Clearly the diagonal structure of $\dot{N}$ can no longer be maintained as the cross term $B_1^t W B_2$ will also lie within it. This is easily resolved by storing the entire upper half of $\dot{N}$ (symmetric) as a vector (see Appendix 1). At first this measure appears rather wasteful, but in fact a matrix of this size has to be stored and inverted during the solution

procedure anyway. Therefore the addition of "theodolite-theodolite" capabilities has no detrimental effect upon storage requirements and execution speed.

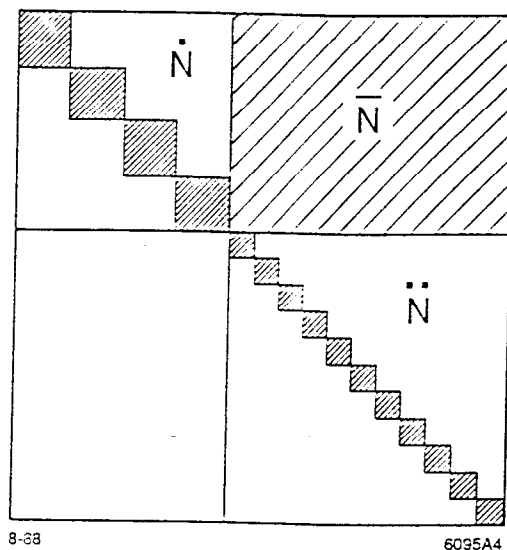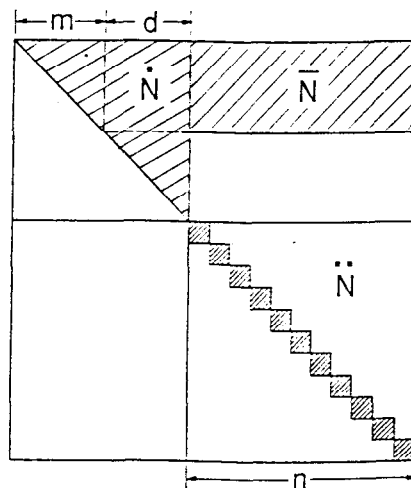8-88   6095A4

Fig. 4

The one remaining problem is the inclusion of distances. These observations could possibly be of the types "theodolite-theodolite" and "theodolite-object" (both already catered for) or, as is usually the case, "object-object." Such an observation type would destroy the diagonality of $\bar{N}$, usually the largest partition of N. One obvious solution is to group together object points (in the parameter vector) that are involved in distance observations. The resulting format would henceforth require 3×3 partitioning of the normal matrix — a most unpleasant thought. A more attractive alternative is to ensure all distances are of the type "theodolite-theodolite" by transferring object points included in distances to the vector of theodolite parameters. Although inelegant, this proposal alleviates the necessity to sort observations into three different groups and allows retention of the less cumbersome 2×2 partitioning. In fact, the principal normal forming and solving algorithms remain virtually unchanged by structural

21

alterations caused by the addition of distances observations. One disadvantage does occur, however, with the introduction of dummy orientation parameters assigned to the transferred object points. They are assigned zero values and do not affect the adjustment, although the size of $\dot{N}$ is unnecessarily increased. This could be avoided, but only to the detriment of simplistic algorithms. Alternately these additional parameters can be eliminated by deleting the necessary rows and columns from $\dot{N}$. In 3DCD neither of these cures are presently used because the number of distances in a network is relatively small and, hence, the reduction of efficiency is minimal. The final structure of the normal equations is shown in fig. 5. Note that $\overline{N}$ is divided into $m \times n$ submatrices (of order 6×3 each), where m is the number of theodolite stations (not including distance stations), and n the number of object points. These submatrices are referred to as $\overline{N}_{ij}$: if one considers the pointing from theodolite station i to the jth object point, then the $B_1^t \, W B_2$ term is added to $\overline{N}_{ij}$. Similarly the non-zero diagonal of $\ddot{N}$ is divided into n 3×3 submatrices. In fact each submatrix is symmetric hence only the upper triangular section is stored in vectors of length 6. Each submatrix is referred to as $\ddot{N}j$.

m = Number of theodolite station parameters
d = Number of distance point parameters
n = Number of object point parameters

Fig. 5

2.6 The Solution Procedure

First, let us clarify that:

m = number of true theodolite stations

d = number of distance related points transferred to the list of theodolites

n = remaining number of object points

$c = m + d$.

From 2.44, the solution of the normal equations is given by

$$\triangle = N^{-1}t \quad . \tag{2.61}$$

Let M be the inverse of the normal matrix, such that:

$$\begin{pmatrix} \dot{N} & \overline{N} \\ \overline{N}^t & \tilde{N} \end{pmatrix} \begin{pmatrix} \dot{M} & \overline{M} \\ \overline{M}^t & \tilde{M} \end{pmatrix} = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} \tag{2.62}$$

where I_1, I_2 are identity matrices of dimension $m + d$ and n, respectively. Expanding 2.62 gives the following equations,

$$\dot{N}\dot{M} + \overline{N}\,\overline{M}^t = I_1 \qquad a)$$
$$\dot{N}\overline{M} + \overline{N}\tilde{M} = 0 \qquad b)$$
$$\overline{N}^t\dot{M} + \tilde{N}\overline{M}^t = 0 \qquad c)$$
$$\overline{N}^t\overline{M} + \tilde{N}\tilde{M} = I \qquad d)$$

and

$$c) \qquad \Rightarrow \qquad \tilde{N}\overline{M}^t = -\overline{N}^t\dot{M}$$
$$\Rightarrow \qquad \overline{M}^t = -\tilde{N}^{-1}\overline{N}^t\dot{M}$$
$$a) \qquad \Rightarrow \dot{N}\dot{M} - \overline{N}\tilde{N}^{-1}\overline{N}^t\dot{M} = I$$
$$\Rightarrow \dot{M} = (\dot{N} - \overline{N}\tilde{N}^{-1}\overline{N}^t)^{-1}$$
$$d) \qquad \Rightarrow \tilde{N}\tilde{M} = I - \overline{N}^t\overline{M}$$
$$\Rightarrow \tilde{M} = \tilde{N}^{-1} - \tilde{N}\overline{N}^t\overline{M}$$

Hence, to summarize, the inverse normal matrix consists of the partitions

$$\dot{M} = (\dot{N} - \overline{N}\tilde{N}^{-1}\overline{N}^t)^{-1}$$
$$\overline{M} = -\dot{M}\overline{N}\tilde{N}^{-1}$$
$$\tilde{M} = \tilde{N}^{-1} - \tilde{N}^{-1}\overline{N}^t\overline{M}$$

(2.63)

Equation 2.61 is now written in the form

$$\begin{pmatrix} \dot{\delta} \\ \tilde{\delta} \end{pmatrix} = \begin{pmatrix} \dot{M} & \overline{M} \\ \overline{M}^t & \tilde{M} \end{pmatrix} \begin{pmatrix} \dot{T} \\ \tilde{T} \end{pmatrix}$$

which, upon expansion, yields:

$$\dot{\delta} = \dot{M}\dot{T} + \overline{M}\tilde{T}$$
$$= \dot{M}(\dot{T} - \overline{N}\tilde{N}^{-1}\tilde{T})$$

(2.64)

and

$$\tilde{\delta} = \overline{M}^t\dot{T} + \tilde{M}\tilde{T}$$
$$= -(\dot{M}\overline{N}\tilde{N}^{-1})^t\dot{T} + (\tilde{N}^{-1} - \tilde{N}^{-1}\overline{N}^t\overline{M})\tilde{T}$$
$$= \tilde{N}^{-1}(\tilde{T} - \overline{N}^t\overline{M})\tilde{T} - \tilde{N}^{-1}\overline{N}^t\dot{M}\dot{T}$$
$$= \tilde{N}^{-1}(\tilde{T} - \overline{N}^t\dot{\delta})$$

(2.65)

Equations 2.64 and 2.65 express the values of the solution vector. Described below are the necessary computational steps required to arrive at numerical values.

1. Inversion of $\tilde{N}^{-1}$: $\tilde{N}$ is divided into symmetric submatrices of order 3×3, stored as 6×1 vectors. Each submatrix is individually inverted (see 3 below), i.e.,

$$\begin{pmatrix} \tilde{N}_1 & & 0 \\ & \ddots & \\ & & \ddots \\ 0 & & \\ & & \tilde{N}_n \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{N}_1^{-1} & & 0 \\ & \ddots & \\ & & \ddots \\ 0 & & \\ & & \tilde{N}_n^{-1} \end{pmatrix}$$

2. Forming $R = \dot{N} - \overline{N}\ddot{N}^{-1}\overline{N}^t$:

$$\overline{N}\ddot{N}^{-1} = \begin{pmatrix} \overline{N}_{11} & \cdots & \cdots & \overline{N}_{1n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \overline{N}_{m1} & \cdots & \cdots & \overline{N}_{mn} \end{pmatrix} \begin{pmatrix} \ddot{N}_1^{-1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddot{N}_n^{-1} \end{pmatrix}$$

$$\underset{6m,\,3n}{} \qquad \underset{3n,\,3n}{}$$

$$= \begin{pmatrix} \overline{N}_{11}\ddot{N}_1^{-1} & \cdots & \cdots & \overline{N}_{1n}\ddot{N}_n^{-1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \overline{N}_{m1}\ddot{N}_1^{-1} & \cdots & \cdots & \overline{N}_{mn}\ddot{N}_n^{-1} \end{pmatrix}$$

$$\underset{6m,\,3n}{}$$

$$(\overline{N}\ddot{N}^{-1})\overline{N}^t = \overline{N}\ddot{N}^{-1} \begin{pmatrix} \overline{N}_{11}^t & \cdots & \cdots & \overline{N}_{m1}^t \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \overline{N}_{1n}^t & \cdots & \cdots & \overline{N}_{mn}^t \end{pmatrix} = \begin{pmatrix} r_{11} & \cdots & \cdots & r_{1m} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ r_{m1} & \cdots & \cdots & r_{mm} \end{pmatrix}$$

where

$$r_{ij} = \sum_{k=1}^n \overline{N}_{ik}\ddot{N}_k^{-1}\overline{N}_{jk}$$

R is symmetric, therefore only the upper half need by calculated. To avoid storing $\overline{N}\ddot{N}^{-1}\overline{N}^t$ each submatrix, r_{ij}, is evaluated, placed in a 6×6 temporary matrix and then subtracted from the relevant section of $\dot{N}$ (stored in vector form).

3. Inversion of R: R is a symmetric matrix of order $(n+d) \times (n+d)$ stored in vector format (see Appendix 1). The method of inversion chosen uses the modified Cholesky $U^t D U$ decomposition, where D is a diagonal matrix and U is an upper triangular matrix with all elements on the leading diagonal

equal to 1. The inverse is given by:

$$\dot{N}^{-1} = U^{-1}D^{-1}(U^{-1})^t$$

Let:

$$U = \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1c} \\ & 1 & u_{23} & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad D = \begin{pmatrix} d_{11} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & d_{cc} \end{pmatrix} \quad .$$

Then the algorithms for obtaining U and D from the elements (n_{ij}) of $\dot{N}$ are as follows:

$$d_{ii} = n_{ii} - \sum_{k=1}^{i-1} d_{kk} u_{ki}^2 \qquad i: 1 \to c$$

$$u_{ij} = \left(n_{ij} - \sum_{k=1}^{i-1} d_{kk} u_{ki} u_{kj}\right)/d_{ii} \qquad \begin{array}{l} i: 1 \to c \\ j: i \to c \end{array} \quad .$$

The inverse of U, denoted as V, is given by:

$$v_{ij} = u_{ij} - \sum_{k=i+1}^{j-i} u_{ik} v_{kj} \qquad \begin{array}{l} i: c \to 1 \\ j: c \to i+1 \end{array}$$

where V is of the form,

$$\begin{pmatrix} 1 & v_{12} & v_{13} & \cdots & v_{1c} \\ & 1 & v_{23} & & \\ & & 1 & & \\ & & & \ddots & 1 \end{pmatrix} \quad .$$

4. Calculation of $Q = \dot{T} - \overline{N}\ddot{N}^{-1}\ddot{T}$: The evaluation of $\overline{N}\dot{N}^{-1}$ has already been

obtained in 2). Therefore:

$$
Q = \begin{pmatrix} Q_1 \\ \vdots \\ Q_c \end{pmatrix} = \begin{pmatrix} \dot{T}_1 \\ \vdots \\ \dot{T}_c \end{pmatrix} - \begin{pmatrix} \overline{N}_{11}\check{N}_1^{-1} & \cdots & \overline{N}_{1n}\check{N}_n^{-1} \\ \vdots & & \vdots \\ \overline{N}_{m1}\check{N}_1^{-1} & & \overline{N}_{mn}\check{N}_n^{-1} \end{pmatrix} \begin{pmatrix} \check{T}_1 \\ \vdots \\ \check{T}_n \end{pmatrix}
$$
$$
\quad 6c,1 \qquad\quad 6c,1 \qquad\qquad\qquad 6m,3n \qquad\qquad\qquad 3n,1
$$

and so

$$
Q_i = \dot{T}_i - \sum_{k=1}^{n} \overline{N}_{ik}\check{N}_k^{-1}\check{T}_k \qquad i : l \to m
$$
$$
\quad = \dot{T}_i \qquad\qquad\qquad\quad i : m+1 \to c
$$

Each submatrix Q_i is a 6×1 vector and overwrites the vector $\dot{T}_i$ in terms of storage. Note that the dimension of $\overline{N}\check{N}^{-1}\check{T}$ is smaller than that of $\dot{T}$, hence some elements of $\dot{T}$ remain unchanged.

5. Produce $\dot{M}(\dot{T} - \overline{N}\check{N}^{-1}\check{T}) = \dot{M}Q = \dot{\delta}$: This is a straightforward matrix multiplication adapted to the structures of $\dot{M}$ and Q. The vector $\dot{\delta}$ is divided into c individual 6×1 vectors such that $\dot{\delta}_i$ contains the increments to the ith theodolite station parameters.

6. Calculate $S = \check{T} - \overline{N}^t \dot{\delta}$:

$$
\begin{pmatrix} S_1 \\ \vdots \\ \vdots \\ S_n \end{pmatrix} = \begin{pmatrix} \check{T}_1 \\ \vdots \\ \vdots \\ \check{T}_n \end{pmatrix} - \begin{pmatrix} \overline{N}^{t_{11}} & \cdots & \cdots & \overline{N}^{t_{m1}} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \overline{N}^t_{1m} & \cdots & \cdots & \overline{N}^{t_{mn}} \end{pmatrix} \begin{pmatrix} \dot{\delta}_1 \\ \vdots \\ \vdots \\ \dot{\delta}_m \end{pmatrix}
$$
$$
\quad 3n,1 \qquad\qquad\qquad 3n,6m \qquad\qquad\qquad 6m,1
$$

and so

$$
S_i = \sum_{k=1}^{m} \overline{N}^t_{ki}\dot{\delta}_k \quad .
$$

7. Produce $\tilde{N}^{-1}(\tilde{T} - \overline{N}^t\dot{\delta}) = \tilde{N}^{-1}S = \tilde{\delta}$:

$$
\begin{pmatrix} \tilde{\delta}_1 \\ \vdots \\ \vdots \\ \tilde{\delta}_n \end{pmatrix} = \begin{pmatrix} \tilde{N}_1^{-1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \tilde{N}_n^{-1} \end{pmatrix} \begin{pmatrix} S_1 \\ \vdots \\ \vdots \\ S_n \end{pmatrix} = \begin{pmatrix} \tilde{N}_1^{-1}S_1 \\ \vdots \\ \vdots \\ \tilde{N}_n^{-1}S_n \end{pmatrix}
$$

$$3n, 1 \qquad\qquad 3n, 3n \qquad\qquad 3n, 1 \qquad\qquad 3n, 1$$

$\tilde{\delta}$ is segmented into 3×1 vectors. Each ith subvector is simply evaluated by forming $\tilde{N}_i^{-1}S_i$ and refers to the parameters of the ith object point.

8. $\dot{\delta}$ and $\tilde{\delta}$ are now used to update the parameter values and the next iteration is initiated by returning to the formation of the observation equations. This procedure is repeated until convergence to a specific preset tolerance occurs. A solution is deemed convergent if the average absolute value of all coordinate parameters is less than 0.01 mm, i.e., if $t = m + d + n$ (the total number of points in the adjustment) then:

$$\sum_{i=1}^{t} |X_i| < 0.01 \times t \quad .$$

Similarly for Y and Z coordinate parameters.

2.7 Inversion of the Normal Matrix and Production of Statistics

Once the least squares solution has been obtained it is necessary to produce various indications of how "good" the fit is. The first source of information is the calculation of the observational residuals produced by comparing observed values with those calculated from the final parameter values. Examining the residuals allows one to detect inaccurate pointings. Also from the residuals one can produce a statistic called the reference standard error given by:

$$\sigma_o = \sqrt{\frac{v^t W v}{r}}$$

where r is the degrees of freedom in the adjustment. In the case of a unified least

squares approach r is equivalent to the number of "real" observations (angle and distances). This statistic clearly expresses the average of the ratios of *a priori* standard errors of observations with the actual residual obtained. Therefore the expected value of σ_o is unity if all observations are consistent with the adjusted parameters.

Finally some idea of the precision of the adjusted parameters is required. The covariance-variance matrix of the parameters, as all least squares texts will prove [see (2)], is given by

$$Q_{\triangle\triangle} = \sigma_o^2 N^{-1} \quad .$$

Therefore the standard error of each parameter is obtained from the diagonal of the inverse normal matrix, already derived in equations 2.63. Recall the partitioned rotation of N^{-1}, i.e.:

$$\begin{pmatrix} \dot{M} & \overline{M} \\ \overline{M} & \ddot{M} \end{pmatrix}$$

and that $\dot{M}$ has already been produced during the solution procedure. Hence there remains the formation of

$$\overline{M} = -\dot{M}\overline{N}\ddot{N}^{-1}$$

and

$$\ddot{M} = \ddot{N}^{-1} - (\overline{N}\ddot{N}^{-1})^t \overline{M} \quad .$$

The procedure for their calculation takes the following steps:

1. Formulation of $\overline{N}\ddot{N}^{-1}$:

$$= \begin{pmatrix} \overline{N}_{11} & \cdots & \cdots & \overline{N}_{1n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \overline{N}_{n1} & \cdots & \cdots & \overline{N}_{mn} \end{pmatrix} \begin{pmatrix} \ddot{N}_1^{-1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & N_n^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{N}_{11}\ddot{N}_1^{-1} & \cdots & \overline{N}_{1n}\ddot{N}_n^{-1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \overline{N}_{m1}\ddot{N}_1^{-1} & \cdots & \overline{N}_{mn}\ddot{N}_n^{-1} \end{pmatrix}$$
$$6m, 3n$$

The submatrices $\ddot{N}_i^{-1}$ were evaluated during the solution procedure. Each 6×3 submatrix $\overline{N}_{ij}\ddot{N}_j^{-1}$ is formed in turn and overwrites $\overline{N}_{ij}$ in the program.

2. Produce $-\dot{M}\overline{N}\ddot{N}^{-1} = \overline{M}$: Recall that $\dot{M}$ is a $6c \times 6c$ square symmetric matrix and $(\overline{N}\ddot{N}^{-1})$ is of order $6m \times 3n$. Obviously the dimensions are not compatible for multiplication due to the distance points that have no contribution in $\overline{N}$. Therefore partition $\dot{M}$ and $\overline{N}\ddot{N}^{-1}$ as follows

$$\dot{M} = \begin{pmatrix} \dot{m} & \overline{m} \\ \overline{m}^t & \ddot{m} \end{pmatrix} \quad \overline{N}\ddot{N}^{-1} = \begin{pmatrix} n_1 \\ 0 \end{pmatrix}$$

Then

$$\dot{M}\overline{N}\ddot{N}^{-1} = \begin{pmatrix} \dot{m}n_1 \\ \overline{m}n_1 \end{pmatrix}$$

We will see below that only the matrix product $\dot{m}n_1$ of order $6m \times 3n$ is necessary and is calculated by a multiplication algorithm adapted to the respective structures. The produce is stored in a system of submatrices similar to that initially used for $\overline{N}$ (see section 2.5).

3. Produce $(\overline{N}\breve{N}^{-1})^t\overline{M}$: Reverting to the partitioned notation introduced above, this produce is expressed as

$$(n_1^t \; 0) \begin{pmatrix} \dot{m}n_1 \\ \overline{m}^t n_1 \end{pmatrix} = n_1^t \dot{m}n_1$$

thus indicating that the prior formation of $\overline{m}^t n_1$ was unnecessary. In terms of submatrix notation, the product is:

$$\begin{pmatrix} \overline{N}_{11}^t & \cdots & \overline{N}_{m1}^t \\ \vdots & & \vdots \\ \overline{N}_{1n}^t & \cdots & \overline{N}_{mn}^t \end{pmatrix} \begin{pmatrix} \overline{M}_{11} & \cdots & \overline{M}_{1n} \\ \vdots & & \vdots \\ \overline{M}_{n1} & \cdots & \overline{M}_{mn} \end{pmatrix} = P$$

$$3n, 6m \qquad\qquad 6m, 3n$$

of which the ijth term is

$$P_{ij} = \sum_{k=1}^{m} \overline{N}_{ki}^t \overline{M}_{kj} \quad . \tag{2.71}$$

Note that the 3×6 submatrices $\overline{N}_{ij}^t$ actually contain $(\overline{N}_{ij}N_j^{-1})$ formed in step 1). Only the diagonal terms (P_{ii}) need to be calculated since we are concerned with standard deviations of final parameters and not their covariances.

Now that $\dot{M}$ and $\breve{M}$ are available the standard errors are obtained by locating the relevant diagonal term, taking its square root and multiplying by σ_o.

2.8 Relative Errors

Program 3DCD also permits the calculation of transversal relative errors between selected pairs of object points. (Note: distance points, i.e., scale bar targets, and theodolite stations have not been included in relative error determinations.)

Suppose the ith and jth object points have been selected. The first step is to produce the necessary 6×6 submatrix of the lower-right partition of the inverse normal matrix $(\breve{M})$ i.e.,

$$Q_{ij} = \begin{pmatrix} P_{ii} & P_{ij} \\ P_{ij}^t & P_{jj} \end{pmatrix} \quad .$$

The expression for the P submatrices have already been derived in eq. 2.71. Since P_{ii} and P_{jj} are calculated during the production of the absolute standard errors, it remains to obtain P_{ij}.

Let the upper-half of the symmetric P_{ii} matrix be represented as:

$$P_{ii} = \begin{pmatrix} q_{x_i x_i} & q_{y_i x_i} & q_{x_i z_i} \\ & q_{y_i y_i} & q_{y_i z_i} \\ & & q_{z_i z_i} \end{pmatrix}$$

Similarly for P_{jj}. Also:

$$P_{ij} = \begin{pmatrix} q_{x_i x_j} & q_{x_i y_j} & q_{x_i z_j} \\ q_{x_j y_i} & q_{y_i y_j} & q_{y_i z_j} \\ q_{x_j z_i} & q_{y_j z_i} & q_{z_i z_j} \end{pmatrix}$$

From these matrices two error values are obtained as follows. A relative error in the Z direction is obtained directly from the Q_{ij} matrix.

$$\sigma_{Z_{ij}}^2 = \sigma_o^2 (q_{z_i z_i} + q_{z_j z_j} - 2 q_{z_i z_j})$$

where σ_o is the reference standard error (see section 2.7). The second value is the transversal error of the relative error ellipse in the XY plane i.e., the distance from the center of the ellipse to its perimeter, perpendicular to the line connecting

points i and j (see fig. 6). Let:

$$Q_{xx} = q_{x_i x_i} + q_{x_j x_j} - 2q_{x_i x_j}$$

$$Q_{yy} = q_{y_i y_i} + q_{y_j y_j} - 2q_{y_i y_j}$$

$$Q_{xy} = q_{y_i x_i} + q_{y_j x_j} - q_{x_i y_j} - q_{x_j y_i}$$

From these three values the relative error ellipse parameters are calculated:
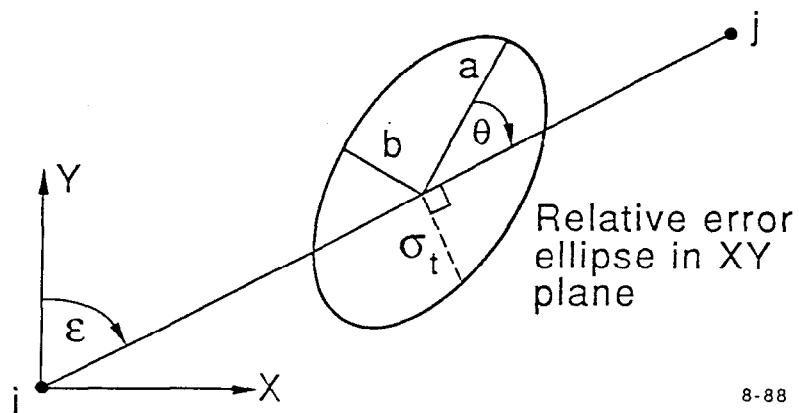
$$L = \sqrt{\left(4Q_{xy}^2 + (Q_{xx} + Q_{yy})^2\right)}$$

$$a = \sigma_o (Q_{xx} + Q_{yy} + L)^{1/2}/\sqrt{2}$$

$$b = \sigma_o (Q_{xx} + Q_{yy} - L)^{1/2}/\sqrt{2}$$

$$2\theta = tan^{-1}\left(\frac{2Q_{xy}}{Q_{xx} - Q_{yy}}\right)$$

where a and b are the semi-major and semi-minor axis respectively and θ is the angle between the semi-major axis and the x axis (see fig. 6).

Fig. 6

If ϵ is the grid bearing of j from i, then the angle, E, between the semi-major axis and the line connecting i and j is given by

$$E = \frac{\pi}{2} + \epsilon - \theta$$

and the transversal error is obtained from:

$$\sigma_t^2 = a^2 \cos^2 E + b^2 \sin^2 E \quad .$$

3. INTER2 — A DATA FORMATTING PROGRAM

3.1 Program Description

3DCD was written with the intention of combining it with a SLAC produced data collection package thus forming a system similar, but completely independent, to the ECDS-PC system. Until this is achieved, program INTER2 acts as an interface between the raw data collected by the ECDS system and the 3DCD adjustment program. One of the main functions of INTER2 involves the calculation of approximate parameters for theodolite stations and object points.

It must be mentioned that INTER2 will only accommodate a limited range of coordinate reference system constraints. In the case of an "object system" (see section 1), there are no problems, but for "local" systems the program expects the datum to be defined by fixing the six parameters of a single theodolite station (as is most often the case). INTER2 is capable of converting any ECDS raw data file into a formatted input file for 3DCD, even including "theodolite-theodolite" observations, as long as right-handed coordinate systems are maintained and units are always gons and millimeters.

3.2 Theodolite Approximate Parameters

During the observation of a network using the ECDS system each theodolite must undergo an initial orientation procedure from which approximate parameters for that theodolite can be calculated. This procedure differs slightly for the two different reference system types:

Object System — From the coordinates of the object control points, the direction of the X-axis is estimated, the theodolite turned along that direction, and its horizontal circle set to zero. This defines the yaw (rotation about Z-axis) as 0. The zenital angle is also recorded. The theodolite is then turned through approximately 300 gons (defining the Y-axis) and the zenital and horizontal angles noted once again. Finally an object control point is observed and its distance from the theodolite guestimated.

Local System — As above except that the sights along the X and Y axis are automatically assigned horizontal directions of 0 and 300 gons respectively and zenital angles of 100 gons. The reference theodolite is then observed and its distance guestimated.

In the object system the axis orientation observations in the vertical circle are used to calculate estimates of the roll (rotation about Y-axis) and pitch (rotation about X-axis). Clearly the telescope should be set to a horizontal position during the above procedures to ensure reasonable approximations of roll and pitch. The rotations are simply obtained by converting the X-axis and Y-axis vertical angles into radians. It is assumed that these angles are relatively small, hence the effect of applying roll first to the theodolite axis before calculating pitch is deemed unneccesary.

In order to approximate the coordinates of the theodolite stations consider the cosine direction vector, v, obtained from the zenital and horizontal directions, μ and λ, to a reference point with known coordinates (X, Y, Z).

$$\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\mu \sin\lambda \\ \sin\mu \sin\lambda \\ \cos\lambda \end{pmatrix} \tag{3.21}$$

Using the rotation matrix $R = R_x \ o \ R_y \ o \ R_z$ (where R_z is an identity matrix) described in section 2.2, the vector $\underline{v}$ is transformed to the network reference system, i.e., making it compatible with X, Y, Z. Multiplying by the estimated

36

distance, d, we obtain

$$\begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix} = d \cdot R\underline{v}$$

and subtracting the result from the values X, Y, Z gives the approximate coordinates of the theodolite station:

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} - \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}$$
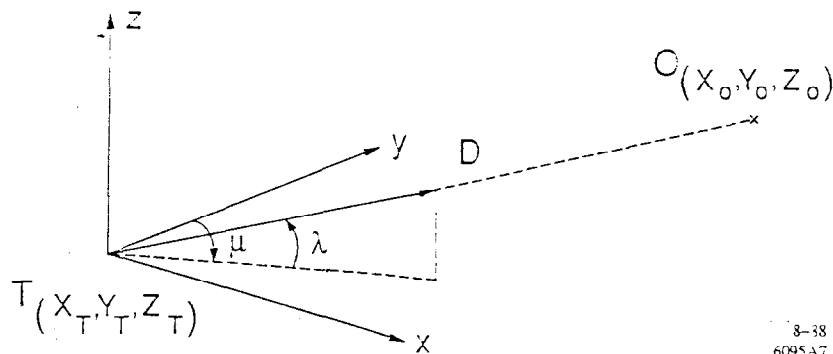
(see fig. 7).

Fig. 7

3.3 Object Point Approximate Coordinates

Once the theodolite station approximate paramters have been obtained, INTER2 reads the pairs of horizontal and zenital angles from which it determines a list of object points.

— The approximate coordinates of each object point are determined by individual least squares adjustments that assume the theodolite stations fixed, and that minimize the sum of the perpendicular distances, D_i, from the cosine direction
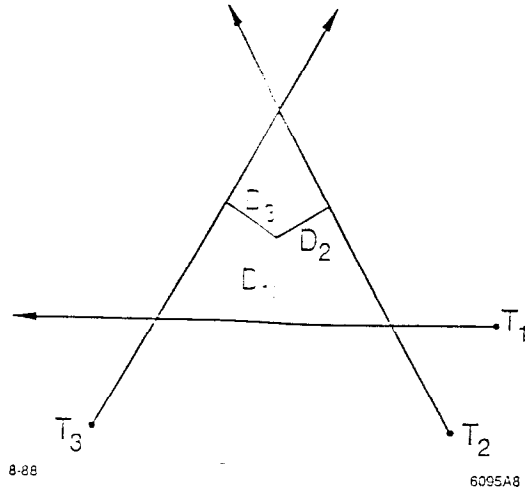
37

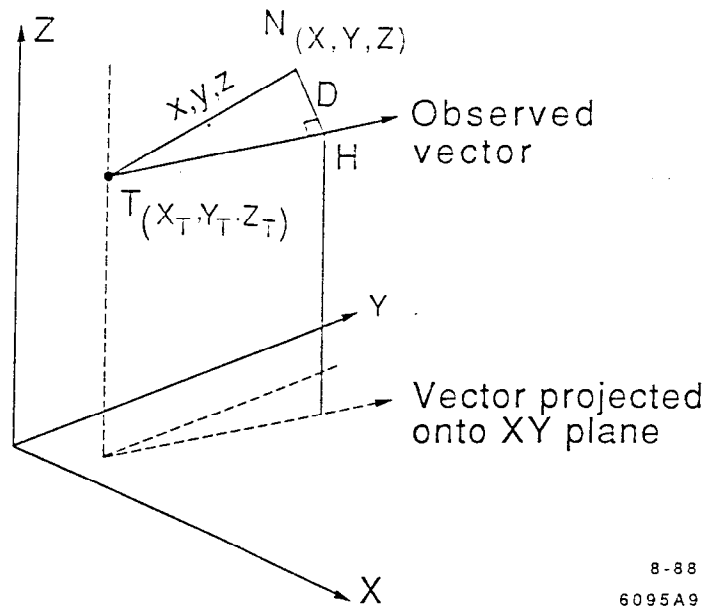vector of each angle observation pair (see fig. 8).

Fig. 8

Fig. 9

First, for each observation pair to object point N, calculate the cosine direction vector as in 3.21, and transform it from the theodolite system to the network reference system by applying the rotation R described in section 3.1. The result is the vector x_N, y_N, z_N. Now, referring to fig. 9, T is the theodolite station and TH the vector x_N, y_N, z_N. Let TN have cosine directions x, y, z. In the right-angled triangle THN:

$$TN^2 = (X - X_T)^2 + (Y - Y_T)^2 + (Z - Z_T)^2$$

where X, Y, Z are the unknown object point coordinates and X_T, Y_T, Z_T the theodolite station coordinates. The angle between the sides TN and TH is defined by

$$\cos \theta = \frac{TH}{TN}$$

This is also the angle between the vectors TN and TH, given by:

$$\cos \theta = x_N \cdot x + y_N \cdot y + z_N \cdot z$$

and

$$x = \frac{X - X_T}{TN} \qquad y = \frac{Y - Y_T}{TN} \qquad z = \frac{Z - Z_T}{TN}$$

Now,

$$TH = TN \cos \theta = TN \left(x_N \cdot x + y_N \cdot y + z_N \cdot z \right)$$

$$= TN \left(x_N \frac{(X - X_T)}{TN} + y_N \frac{(Y - Y_T)}{TN} + z_N \frac{(Z - Z_T)}{TN} \right)$$

$$= x_N(x - x_T) + y_N(Y - Y_T) + z_N(Z - Z_T)$$

Writing

$$h_N = x_N X_T + y_N Y_T + z_N Z_T \qquad \text{(all known quantities)}$$

then,

$$TN = x_N X + y_N Y + z_N Z - h_N \quad .$$

So:

$$D^2 = TN^2 - TH^2$$
$$= (X - X_T)^2 + (Y - Y_T)^2 + (Z - Z_T)^2 - (x_N X + y_N Y + z_N Z - h_N)^2$$

D will vary as a function of coordinates X, Y, Z. This variation may be described by the partial derivatives of D^2 with respect to X, Y, Z.

$$-\frac{1}{2}\delta D^2 / \delta X = (X - X_T) - (x_N X + y_N Y + z_N Z - h_N) \cdot x_N$$
$$= (x_{N^2} - 1)X + x_N y_N Y + x_N z_N Z - (x_N h_N - X_T)$$

Similarly for δY and δZ. The least squares method will minimize $\sum D_i^2$ if the sum of the partial derivative equations above for each observation is equal to 0, i.e.,

$$\begin{pmatrix} \sum(x_N^2 - 1) & \sum x_N y_N & \sum x_N z_N \\ \sum x_N y_N & \sum(y_N^2 - 1) & \sum y_N z_N \\ \sum x_N z_N & \sum y_N z_N & \sum(z_N^2 - 1) \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \sum(x_N h_N - X_T) \\ \sum(y_N h_N - Y_T) \\ \sum(z_N h_N - Z_T) \end{pmatrix}$$

or $Nx = t$. Note that N is symmetric, hence is stored as a vector of dimension 6 (see Appendix 1). Its inverse is calculated (as for the inverse of $\dot{M}$ in section 2.6) and hence the values of X, Y, Z.

4. SETOUT — CALCULATION OF SETOUT ANGLES

In certain industrial survey situations misaligned elements need to be shifted to their ideal positions (with respect to an "object" reference system). One method of achieving this is to calculate setout angles for theodolites of known position, intersect the ideal position by turning the theodolites through these angles, and adjust the element position so that it coincides (as close as observational accuracy permits) with that of the intersected point.

If the "object" system has been measured using the ECDS system and adjusted via 3DCD, the theodolite coordinates and orientation parameters are predetermined and it is a simple task to obtain the setout angles.

Recalling eq. (2.24) and letting X_o, Y_o, Z_o now be the ideal coordinates, then the cosine direction vector of the setout pointing is obtained in the local theodolite system, i.e.:

$$\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin\mu\cos\lambda \\ \cos\mu\cos\lambda \\ \sin\lambda \end{pmatrix}$$

where μ is the horizontal setout angle, λ the vertical. Therefore:

$$\lambda = \sin^{-1} z$$

$$\mu = \tan^{-1}(x/y)$$

Note that μ is the horizontal angle with respect to the Y-axis, and is simply converted to its equivalent with respect to the X axis by subtracting 100 gons. Similarly λ is converted to a zenital angle by subtracting it from 100 gons.

Bibliography

1. Burnside, C. D. (1985), *Mapping from Aerial Photographs*, John Wiley & sons, New York.

2. Mikhail, E. M. (1976), *Observations and Least Squares*, Harper and Row, New York.

Further program documentation

Programmers charged with updating the software will find extensive in-line documentation contained within the source code listings, as well as variable lists for all three programs in files 3DCD.DOC, INTER2.DOC, and SETOUT.DOC.

Appendix 1. Vector Storage of Symmetric Matrices

Since its contributions are of the form $B^t W B$, the normal matrix of a least squares adjustment is symmetric i.e., $(n_{ij}) = (n_{ji})$. In this case it is only necessary to store the upper-triangular part of the matrix and the most space efficient storage is that of a vector representation.

Let the upper-triangle of the matrix N be:

$$\begin{pmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ & n_{22} & n_{23} & n_{24} \\ & & n_{33} & n_{34} \\ & & & n_{44} \end{pmatrix}$$

Then its column-wise vector equivalent is:

$$\begin{pmatrix} n_{11} & n_{12} & n_{22} & n_{13} & n_{23} & n_{33} & n_{14} & n_{24} & n_{34} & n_{44} \end{pmatrix}$$

The ijth term of N can be located in the vector via the expression:

$$\text{Vector index} = \frac{I \cdot (I - 1)}{2} + J$$

where $I = max(i, j)$, $J = min(i, j)$. Similarly, if the dimension of N is $n \times n$, then the length of the vector equivalent is:

$$l = \frac{n \cdot (n + 1)}{2}$$

Appendix 2. User's Guide

INTER2:

Two input files, both created by the ECDS system, are required for INTER2, one containing the raw observation data, the other the network reference control data, i.e., fixed parameters and scale-bar data. To invoke the program, simply type INTER2, and a prompt panel appears on the screen requesting the name of the observation data file (input file). There are few limitations on the naming of this file, although it must not have a file extension (see MS-DOS reference manual) .INP. The suggested convention is:

fileid.RAW

where fileid indicates the location of the survey network. The control file is then expected to have the same filename as the input file with extension.DAT, hence under the suggested convention this would be:

fileid.DAT

Finally the output is written to:

fileid.INP

therefore the reason for excluding .INP as a file extension for the raw datafile.

Notes:

1. The character length of point names in INTER2 is limited to eight, whilst those of ECDS can be up to sixteen. Renaming of points, therefore, may well be necessary.

2. Theodolite stations are still identified by integer indices. Due to the possibility of "theodolite-theodolite" observations, object points can no longer use the same 0-9 identifiers.

3. Ten theodolites stations are permitted in INTER2, in comparison to eight in the ECDS. If the maximum number is necessary then the tenth-theodolite should be indexed as 0 (zero).

44

INTER2 produces a input file for 3DCD, including the calculation of approximate parameters for both the object points and theodolite stations. Therefore one can now run:

3DCD:

Typing 3DCD invokes the adjustment program, producing a prompt panel requesting the name of the input file, i.e.,

fileid.INP

Three output files can be produced:

fileid.OUT

fileid.DBG

fileid.STN

the first containing the overall adjustment results, the second certain debugging information (see below), and the third station parameters for use in SETOUT.

During the program execution, the user is informed of the progress of computations via output to the screen panel. The number of the current iteration is shown, along with the sum of the absolute X, Y, and Z coordinate increments from the previous iteration. The user will also be told whether the solution has converged or diverged, although this will of course be obvious from the aforementioned X, Y, Z incrementations.

There are three output options available when executing 3DCD: this option is entered on the sixth line of the input file after the specifier "LISTING =". They are:

LONG — default option: observation residuals and final parameter values are written to fileid.OUT.

DEBUG — as well as the output written under option LONG, a second output file — fileid.DBG is created containing the observation residuals

45

and parameter incrementations after each iteration. This information is of assistance in detecting erroneous observations and blunders.

STATS — this option adds a list of the standard errors of the final X, Y, Z coordinates of both theodolite stations and object points.

Notes:

1. Each approximate parameter in the input file has an assigned *a priori* standard error with which we can vary the degree of constraint on points.

2. Theodolite station names can be up to eight characters in length.

3. All distance units are in millimeters, angles are given in gons.

4. When using option STATS one can also obtain relative error parameters between selected pairs of points. At the bottom of the input file created by INTER2 is a heading: "Relative errors". The user ccan add a list of pairs of points using the format

$$A8, \; 2X, \; A8$$

e.g.,

POINT1 ⊓⊓⊓⊓ POINT2

Note that all pairs must consist of object points only: theodolite stations and distance points are not accomodated and will be ignored if entered by accident.

5. A batch program is available that executes both INTER2 and 3DCD, permitting editing of files between programs. Type RUN3DCD and follow the prompts.

SETOUT:

Two input files are necessary, the first containing the theodolite station parameters from the 3DCD adjustment and named:

fileid.STN

The second must be created by the user and contains the list of setout points and their ideal coordinates. A suggested name is:

fileid.IDL

Data should be entered into this file using the format:

$$A8, \ 2X, \ 3F15.5$$

facilitated by noting that the decimal points of the X, Y, Z coordinate s should fall on the 20th, 35th and 50th columns respectively. The output file is automatically assigned the name:

fileid.SET

Notes:

1. The setout coordinates obviously must be in the same system as the theodolite stations. Therefore it is unlikely that SETOUT should ever be used in conjuction with a network measured in a "local" system.