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DEPOLARIZATION DUE TO BEAM-BEAM INTERACTION IN ELECTRON-POSITRON LINEAR COLLIDERS*

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ABSTRACT

We investigate two major mechanisms which induce depolarization of electron beams during beam-beam interaction in linear colliders. These are the classical spin precession under the collective field of the oncoming beam, and the spin-flip effect from beamstrahlung. Analytic formulas are derived for estimating these depolarization effects. As examples, we estimate the depolarization in the Stanford Linear Collider (SLC) and a possible future TeV linear collider (TLC). The effects are found to be negligibly small for SLC and not very large for TLC.

INTRODUCTION

Polarized beams in linear colliders could be an interesting option for high energy physics experiments. It seems to be easier to prepare longitudinally polarized electron beams in linear colliders than in storage rings. In a linear collider there is no need for the complicated spin rotator, which is necessary in a storage ring in order to orient the spins to their longitudinal directions and is a serious cause of depolarization. On the other hand, polarized positron beams may not be easy to achieve in linear colliders. Nevertheless, this is not essential for high energy experiments. The depolarization process can in principle occur in the damping ring, the linac, and the final focussing system in a linear collider. But these can be largely suppressed as long as the machine is carefully designed. In the present paper we discuss the depolarization due to beam-beam interaction, which is inherent for a linear collider and can not be alleviated.

There are two mechanisms of spin depolarization induced by the collective electromagnetic field of the oncoming beam, which is transverse to the beam trajectory. A longitudinally polarized spinor would precess classically under this

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field according to the BMT equation,[1] which can lead to a depolarization. It is well-known that the precession angle is γa times the deflection angle, where γ is the particle energy in units of the rest mass and $a = 0.0011596$ the coefficient of the anomalous magnetic moment of electron. One can therefore roughly estimate the importance of this effect with given beam parameters. Yet explicit formula is still needed for more quantitative evaluations.

The other mechanism of depolarization is the Sokolov-Ternov effect,[2] i.e., the spin-flip effect, during synchrotron radiation. This process tends to polarize spins along the direction of the field. Thus for storage rings the effect tends to polarize the beam in the transverse direction, and for linear colliders it tends to degrade the longitudinal polarization. In storage rings the polarization length, i.e. velocity of light times the polarization time, is of the scale of the solar system, and the effect is cumulated through long time of beam storage. On the contrary, the depolarization length in a linear collider due to the Sokolov-Ternov effect from beamstrahlung,[3] i.e., the radiation from beam-beam interaction, can be roughly estimated to be of the order milimeter or less, by applying the scaling law available in the classical limit. For quantitative estimations, however, it is necessary to apply the full quantum theory.

In this paper, depolarization by classical precession is discussed in the next section, and depolarization by spin-flip radiation in the following section. Depolarizations in the SLC and a TLC are estimated at the end of both sections.

PRECESSION IN THE BEAM-BEAM FIELD

Let us consider the collision of an electron and a positron bunch each consisting N particles with energy γmc^2. Define the coordinate system as follows: the electron (positron) comes to a collision along positive (negative) s-axis, whose origin is the collision point of the bunch centers. The $x-y$ plane is perpendicular to s-axis. Define $z_1(z_2)$ as the longitudinal coordinate in the electron (positron) bunch with the origin fixed at the center of each bunch and positive towards the bunch head. The time t is defined such that $t = 0$ at the instance when the two bunch centers collide. Obviously, $z_1 = s - t$ and $z_2 = -s - t$. The equation of motion of an electron with the initial condition (x, y, z_1) can be written as

$$\frac{d^2x}{dt^2} = -\frac{4r_eN}{\gamma}\, n_{L}(-2t - z_1)\, \frac{F_x(x,y)}{\sqrt{\sigma_x\sigma_y}} \quad , \tag{1}$$

and a similar equation for y. Here, $n_{L}(z)$ is the longitudinal distribution function normalized in such a way that $\int n_{L}(z)dz = 1$. σ_x and σ_y are the transverse *rms* beam size, r_e the classical electron radius and

$$F_x(x,y) = \sqrt{\sigma_x\sigma_y} \int \frac{x - X}{(x-X)^2 + (y-Y)^2} n_{T}(X,Y)\, dX dY \tag{2}$$

is the transverse force, where $n_T(x,y)$ is the transverse distribution function. For a round ($\sigma_x = \sigma_y = \sigma$) Gaussian beam, we have

$$F_x(x,y) = \frac{x}{r}\frac{1-\exp(-r^2/2\sigma^2)}{r/\sigma} \qquad (r^2 = x^2+y^2) \quad . \tag{3}$$

In the high energy region where $\gamma a \gg 1$, the equation of motion of the spin $\vec{s}$, defined in the rest frame, is

$$\frac{d\vec{s}}{dt} = \gamma a \left(\frac{d^2x}{dt^2}\vec{e}_y - \frac{d^2y}{dt^2}\vec{e}_x\right) \times \vec{s} \quad , \tag{4}$$

regardless of whether the field is electric or magnetic. Here, $\vec{e}_x$ and $\vec{e}_y$ are the unit vectors along x and y axes. Since the deflection angle is very small, the longitudinal spin component is nearly equal to $\vec{s}_z$. We assume that the initial polarization is longitudinal ($s_z = 1$) and the depolarization is not very large. Also, for the moment, we ignore the change of the field due to the pinch effect. Then Eq. (4) can be integrated as

$$s_z(t) = 1 - \frac{1}{2}(\gamma a)^2 \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] \quad , \tag{5}$$

and from Eq. (1) we have

$$\frac{dx}{dt} = -\frac{4r_eN}{\gamma}\frac{F_x(x,y)}{\sqrt{\sigma_x\sigma_y}}\int_{-\infty}^{t} n_L(-2t-z_1)dt \quad . \tag{6}$$

Therefore, the depolarization at time t is given by

$$\Delta P(t) = \frac{1}{2}\left[\gamma a \frac{4r_eN}{\gamma\sqrt{\sigma_x\sigma_y}}F(x,y)\int_{-\infty}^{t} n_L(-2t-z_1)dt\right]^2 \quad , \tag{7}$$

where

$$F^2(x,y) = F_x^2(x,y) + F_y^2(x,y) \quad . \tag{8}$$

Let us denote the average over the particle distribution by $\langle\ \rangle$. Then, the average final polarization is

$$\langle \Delta P \rangle = \langle \Delta P(t=\infty) \rangle = 2\frac{(ar_eN)^2}{\sigma_x\sigma_y}\langle F^2 \rangle \quad , \tag{9}$$

since $\int_{-\infty}^{\infty} n_L(-2t-z)dt = 1/2$. The average of F^2 has been derived before:[4]

$$\langle F^2 \rangle = \frac{1}{2}\log\frac{4}{3}\, f^2(R) \quad , \tag{10}$$

with

$$f(R) = \frac{2\sqrt{R}}{1+R}, \qquad R = \frac{\sigma_x}{\sigma_y} \quad . \tag{11}$$

Notice that the derivation was based on round Gaussian bunches while the form factor $f(R)$ for elliptic cylindrical distributions was multiplied *a posteri.* All the following formulas are derived in the same manner.

A more interesting quantity is the average longitudinal polarization during the collision, which is a luminosity weighted average. We denote this average by the square bracket [ ]. For any function of (x,y,s,t), we have

$$\begin{aligned}
[f] &= \frac{\int dxdydsdt\, n(x,y,s-t)n(x,y,-s-t)f(x,y,s,t)}{\int dxdydsdt\, n(x,y,s-t)n(x,y,-s-t)} \\
&= \frac{\int dxdydz_1dz_2\, n(x,y,z_1)n(x,y,z_2)f(x,y,\frac{z_1-z_2}{2},\frac{-z_1-z_2}{2})}{\int dxdydz_1dz_2\, n(x,y,z_1)n(x,y,z_2)} \quad ,
\end{aligned} \tag{12}$$

where

$$n(x,y,z) = n_T(x,y)n_L(z) \quad . \tag{13}$$

The average of ΔP is now

$$[\Delta P] = \frac{2}{3}\frac{(ar_eN)^2}{\sigma_x\sigma_y}[F^2] \quad , \tag{14}$$

where we have used the relation

$$\int dz_1dz_2\, n_L(z_1)n_L(z_2)\left[\int_{-\infty}^{-(z_1+z_2)/2} n_L(-2t-z_1)dt\right]^2 = \frac{1}{12}$$

for any longitudinal distribution. The luminosity-weighted average of F^2 is

$$[F^2] = \log\frac{9}{8}\, f^2(R) \quad . \tag{15}$$

Actually, the results do not depend on the longitudinal distribution function. The ratio of $\langle \Delta P \rangle$ and $[\Delta P]$ is

$$[\Delta P] = \frac{2}{3}\frac{\log(9/8)}{\log(4/3)}\langle \Delta P \rangle = 0.273 \langle \Delta P \rangle \quad . \tag{16}$$

Notice that the above results can also be expressed in more convenient forms. The luminosity, in the absence of disruption, is given by

$$L_o = \frac{f_{rep} N^2}{4\pi \sigma_x \sigma_y} \quad , \tag{17}$$

where f_{rep} is the repetition frequency. A comparison with Eq. (9) gives

$$\langle \Delta P \rangle = 0.386 \frac{L_o / f_{rep}}{10^{30} cm^{-2}} f^2(R) \quad . \tag{18}$$

Actually, a more physical scaling law is to relate $\langle \Delta P \rangle$ with the average number of radiated photons per electron, which can be given by

$$n_{cl} = \frac{5\sqrt{\pi}}{2\sqrt{3}}(\sqrt{2}-1)\frac{\alpha r_e N}{\sqrt{\sigma_x \sigma_y}} f(R) \quad , \tag{19}$$

where n_{cl} is the average number of photons calculated by the classical synchrotron radiation formula and α the fine structure constant. Then, we have

$$\langle \Delta P \rangle = \frac{12(3+2\sqrt{2})}{25\pi} \log\frac{4}{3} \left(\frac{a}{\alpha}\right)^2 n_{cl}^2 = 0.00647\, n_{cl}^2 \quad . \tag{20}$$

This simple relation suggests a more direct comparison of the depolarization and the number of photons. According to the classical radiation theory, the average number of photons per unit time is given by

$$\frac{dn_{cl}}{dt} = \frac{5}{2\sqrt{3}}\frac{\alpha \gamma}{\rho} \quad , \tag{21}$$

where ρ is the instantaneous radius of curvature of the orbit. On the other hand the precession angular velocity is

$$\frac{d\phi}{dt} = \frac{\gamma a}{\rho} \quad , \tag{22}$$

when the field is perpendicular to the spin. Therefore, we have

$$\phi(t) = \frac{2\sqrt{3}}{5}\frac{a}{\alpha} n_{cl}(t) \quad . \tag{23}$$

Thus, the final depolarization is

$$\langle \Delta P \rangle = \left\langle \frac{1}{2}\phi^2(t=\infty) \right\rangle = \frac{6}{25}\left(\frac{a}{\alpha}\right)^2 n_{cl}^2 \quad , \tag{24}$$

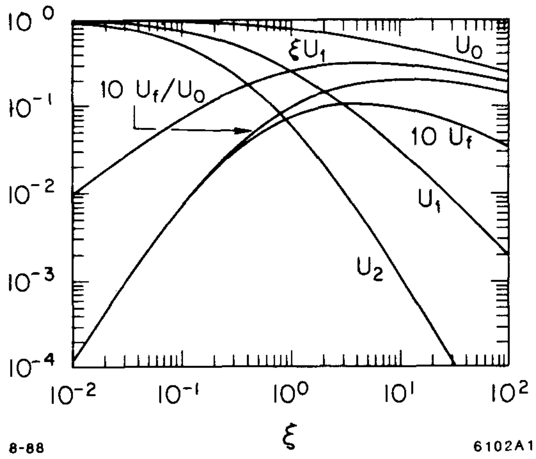

Fig. 1.

where we ignored the difference between $\langle n_{cl}^2 \rangle$ and $\langle n_{cl} \rangle^2$. By using the relation $a = \alpha/2\pi$ in quantum electrodynamics, we get

$$\langle \Delta P \rangle = \frac{3}{50\pi^2} n_{cl}^2 = 0.00608\, n_{cl}^2 \quad , \tag{25}$$

and

$$[\Delta P] = \frac{1}{600} n_{cl}^2 \quad . \tag{26}$$

Thus, in order to have negligible beam-beam depolarization, it is necessary that

$$n_{cl} \lesssim 4 \quad . \tag{27}$$

If the actual n_{cl} can be obtained by computer simulations or by other means, one can estimate the depolarization readily. The inequality in Eq. (27) is generally satisfied in several existing designs of linear colliders, although it is only marginal in some cases.

In the use of the above expressions, the following considerations should be taken:

First, when the ratio ξ of the critical energy of radiation to the initial beam energy is large, i.e., when beamstrahlung is in the so-called quantum regime, n_{cl} is not equal to the actual average number of photons n_γ. The latter has to be calculated using the quantum theory. The two quantities are related by $n_\gamma = n_{cl} U_0(\xi)$, where $U_0(\xi)$, to be defined in the next section, is a slowly decreasing function of ξ and $U_0(0) = 1$ (see Fig. 1). Therefore, if one uses n_γ instead of n_{cl}, Eqs. (25) and (26) will give an under estimation of depolarization. Since, however, the variation of U_0 is very slow, the difference is only by a factor 0.7 even for $\xi \sim 0.5$.

Second, we have treated the precession angular frequency $d\phi/dt$ as if it is a positive definite scalar quantity, but it is actually a vector. Our integration which led to Eq. (22) was

$$\phi(t) = \int^{t} \left| \frac{d\vec{\phi}}{dt} \right| dt \quad , \tag{28}$$

which is not equal to $|\vec{\phi}(t)|$. In this sense, therefore, Eq. (25) gives an overestimation of the depolarization, and it is correct only if (a) the orbit is confined in a plane and (b) $d\phi/dt$ is positive (or negative) definite. Actually, condition (a) is generally satisfied. As for (b), when the disruption parameter, defined as

$$D_{x,y} = \frac{2r_e N\sigma_z}{\gamma\sigma_{x,y}(\sigma_x + \sigma_y)} \quad , \tag{29}$$

is of the order unity, a particle can in general cross the axis within the oncoming bunch and be bent backwards, causing a change of sign in $d\phi/dt$. For $D \gg 1$, particles will oscillate about the axis by the strong focusing force of the oncoming beam, and $d\phi/dt$ will change sign frequently. In that situation the problem has to be treated vectorially, and becomes rather intricate.

Third, for very flat beams, i.e., $\sigma_x \gg \sigma_y$, we expect $D_y \gg D_x$ and $D_x \ll 1$. Computer simulations on the *rms* deflecting angles using the code ABEL[5] show that, for Gaussian beams,

$$\theta_{j,rms} = \frac{1}{2}\frac{\sigma_j}{\sigma_z}\frac{D_j}{[1 + (D_j/2)^5]^{1/6}} \quad , \qquad (j = x, y) \tag{30}$$

where the overall coefficient corresponds to the analytic expression for small disruptions. In practice, the denominator in the above equation can be removed for $j = x$ since typically $D_x \ll 1$.

The final depolarization is then given by

$$\langle \Delta P \rangle = \frac{1}{2}(\gamma a)^2 [\theta_{x,rms}{}^2 + \theta_{y,rms}{}^2] \quad , \tag{31}$$

which approximately agrees with Eq. (9) when both $D_x, D_y \ll 1$. The small discrepancy is due to the form factor $f(R)$ for flat beams. In the case where $D_y \gg 1$, we have $\theta_{x,rms} \gg \theta_{y,rms}$, thus the contribution from the vertical dimension can be ignored. Furthermore, for flat beams the relation between $[\Delta P]$ and $\langle \Delta P \rangle$ in Eq. (16) always holds as long as $D_x \ll 1$, regardless of the value of D_y. But when both D_x and D_y are large, the relation has to be modified.

As examples, we consider the newly built Stanford Linear Collider (SLC) and the design studies for a TeV linear collider (TLC) by Palmer.[6] The SLC is designed to have beam energy at 50 GeV, $N = 5 \times 10^{10}$ particles per bunch,with bunch size $\sigma_x = \sigma_y = 1.6\mu$m, and $\sigma_z = 1$ mm, at the interaction point. This corresponds to a disruption parameter $D_x = D_y = 0.67$. Thus the formulas derived in this section is directly applicable.

Since the local deflecting field is not constant during collision, it is in principle very complicated to carry out any calculation that involves the beam-beam field. It has been observed, however, that an effective beamstrahlung parameter[3] can be introduced in terms of the initial beam parameters only, to represent the entire beam as if all particles are seeing a constant effective field:

$$\Upsilon_0 = \frac{5}{12} \frac{\gamma r_e^2 N}{\alpha \sigma_z \sqrt{\sigma_x \sigma_y}} f(R) \quad , \tag{32}$$

where by definition $\Upsilon = \gamma B/B_c = 2\xi/3$ ($B_c = m^2c^3/eh \sim 4.4 \times 10^{13}$ gauss), and $f(R)$ is defined in Eq. (10). Notice that the coefficient 5/12 in the above expression is some what arbitrary. With the above parameters for SLC, we find $\Upsilon_0 = 0.0014$, or $\xi_0 = 0.0021$. One can easily find that n_{cl} is of order unity in this case, and the depolarization is negligible according to Eq. (25) and Eq. (26).

As for the TLC, the beam energy is 0.5 TeV, and in one version of the studies $N = 8 \times 10^9$, $\sigma_x = 190$ nm, $\sigma_y = 1$ nm, and $\sigma_z = 26\mu$m. This corresponds to $D_x = 0.033$ and $D_y = 6.27$. Plugging numbers into Eq. (30) give $\theta_{x,rms} = 0.13 \times 10^{-3} > \theta_{y,rms} = 0.047 \times 10^{-3}$. Therefore the depolarization through precession is actually dominated by the horizontal disruption in this case. Since $D_x \ll 1$, the formulas in this section are again applicable. The effective beamstrahlung parameter for a TLC with the above beam parameters turns out to be $\Upsilon_0 = 1.54$, or $\xi_0 = 2.3$. Computer simulation further shows that the average number of photons radiated per electron is $n_\gamma = 1.33$. From Fig. 1 and with the estimated ξ_0, we find n_{cl} to be around 2. Thus we expect that $\langle \Delta P \rangle \sim 0.024$ and $[\Delta P] \sim 0.007$, which are also reasonably small.

SPIN-FLIP RADIATION

As is well-known, the electron (positron) beam in storage rings tend to polarize anti-parallel (parallel) to the guiding magnetic field by the spin-flip radiation, which is called the Sokolov-Ternov effect. The spin-flip transition rate of an unpolarized electron, i.e., the average of the up-down and the down-up transition rates, is given by

$$w_T = \frac{5\sqrt{3}}{16} \frac{\lambda_e r_e \gamma^5}{\rho^3} \quad , \tag{33}$$

where λ_e is the Compton wavelength of electron. The polarization time ranges from several minutes to several hours for storage rings. For linear colliders the characteristic time is much shorter because of the generally strong beam-beam field and the high beam energy. In the case of TLC, the above expression gives the polarization time of the order of picoseconds, which is not negligible when compared with the bunch length. Because the beam-beam field is perpendicular to the particle orbit and because we are interested in the longitudinal polarization, the Sokolov–Ternov effect leads to a depolarization in this case.

The above expression, again, is not directly applicable to the TeV linear colliders if the beamstrahlung is in the quantum regime. When the critical energy u_c of the synchrotron radiation spectrum is comparable to or larger than the beam energy γmc^2, we have to employ the quantum theory of radiation. Define the parameter ξ as

$$\xi = \frac{u_c}{\gamma mc^2} = \frac{3}{2}\frac{\lambda_e \gamma^2}{\rho} \quad . \tag{34}$$

In fact, the expression (33) corresponds to the first non-trivial order in the expansion in terms of ξ:

$$w_T = \frac{1}{6}\frac{dn_{cl}}{dt}\xi^2 \quad , \tag{35}$$

where dn_{cl}/dt, defined in Eq. (21), is the rate of (spin non-flip) radiation by the classical theory.

The spectrum formula of radiation in the full quantum theory was first derived also by Sokolov and Ternov.[7] When the field is perpendicular to the orbit and the electron is polarized longitudinally, the spectrum of photons is given by

$$\frac{d^2 n_\gamma}{dtdy} = \frac{dn_{cl}}{dt}\left[\frac{1+\zeta\zeta'}{2}F_{nf} + \frac{1-\zeta\zeta'}{2}F_f\right] \quad . \tag{36}$$

Here, y is a dimensionless variable related to the photon energy u as

$$y = \frac{u/u_c}{1-u/E} \quad , \qquad (E = \gamma mc^2) \tag{37}$$

ζ and ζ' are the helicities of the initial and final electron ($= \pm 1$), and the functions F_{nf} and F_f, corresponding to the spin non-flip and flip radiation respectively, are given by

$$F_{nf} = \frac{3}{5\pi}\frac{1+\xi y + \frac{1}{2}\xi^2 y^2}{(1+\xi y)^3}\int_y^\infty K_{5/3}(x)\,dx \quad , \tag{38}$$

and

$$F_f = \frac{3}{5\pi}\frac{\frac{1}{2}\xi^2 y^2}{(1+\xi y)^3}\int_y^\infty K_{1/3}(x)\,dx \quad , \tag{39}$$

K's being the modified Bessel functions. By integrating these expressions over the photon energy by using the relation $K_{1/3} = -2K'_{2/3} - K_{5/3}$, we get the total number of photons and spin-flip photons per unit time:

$$\frac{dn_\gamma}{dt} = \frac{dn_{cl}}{dt} U_0(\xi) \quad , \tag{40}$$

and

$$\frac{dn_f}{dt} = \frac{dn_{cl}}{dt} U_f(\xi) \quad , \tag{41}$$

with

$$U_0(\xi) = \int_0^\infty (F_{nf} + F_f)\, dy = \frac{3}{5\pi} \int_0^\infty dy \frac{K_{2/3}(y)}{1+\xi y} \left[\frac{2}{3} + \frac{1}{1+\xi y} + \frac{\xi^2 y^2}{(1+\xi y)^2} \right] \quad , \tag{42}$$

and

$$U_f(\xi) = \int_0^\infty F_f\, dy = \frac{3}{10\pi} \frac{1}{\xi} \int_0^\infty dy \left[\log(1+\xi y) - \frac{\xi y}{1+\xi y} - \frac{1}{2} \frac{\xi^2 y^2}{(1+\xi y)^2} \right] K_{1/3}(y) \quad . \tag{43}$$

(Caution: the integrands of these formulae of U_0 and U_f do not give the spectrum since we have used partial integration.) The function U_0 is normalized such that $U_0(0) = 1$. It is a very slowly decreasing function of ξ, not far from unity in the region for linear colliders in the near future. The functions U_f and U_f/U_0 are plotted in Fig. 1. Explicit and approximate expressions of these functions are given in Appendix A. The symptotic form of U_f for small ξ gives the spin-flip transition rate

$$w_L = \frac{7}{54} \frac{dn_{cl}}{dt} \xi^2 \quad , \qquad (\xi \ll 1) \tag{44}$$

which differs from Eq. (35) by a factor 7/9 because this is the transition rate of longitudinal spin. As ξ becomes larger, the spin-flip rate increases to a broad maximum around $\xi = 4$ and then decreases as $\log \xi / \xi$. The ratio of the spin-flip rate to the total photon emission rate reaches a maximum of 0.0200 around $\xi = 11$.

In order to get the depolarization, we have to integrate w_L over the time and average it over the transverse distribution. Again, as in the case of classical precession, the calculation can be approximately performed if one replaces ξ by the effective ξ_0 of the entire beam, then

$$\langle \Delta P \rangle = 2\left\langle \int_{-\infty}^{\infty} w_L \, dt \right\rangle \simeq 2n_{cl} U_f(\xi_0) \simeq 2n_\gamma U_f(\xi_0)/U_0(\xi_0) \quad . \tag{45}$$

For very small ξ_0, the asymptotic forms of U_f and U_0 gives

$$\langle \Delta P \rangle = \frac{7}{27} n_\gamma \xi_0^2 \quad , \qquad (\xi_0 \ll 1) \tag{46}$$

and the inequality $U_f/U_0 < 0.02$ gives

$$\langle \Delta P \rangle < 0.04 \, n_\gamma \quad . \tag{47}$$

As can be seen from Fig. 1, the maximum of U_f/U_0 is actually very broad, thus the above relation is true for a very large range of ξ_0: $2 \lesssim \xi_0 \lesssim 100$. As for the relation between $[\Delta P]$ and $\langle \Delta P \rangle$, Eq. (16) is still approximately valid in this case.

For SLC, $\xi_0 = 0.0021$ and $n_\gamma \sim 1$, as we discussed earlier. So $\langle \Delta P \rangle \sim 1.1 \times 10^{-6} \ll 1$. On the other hand, $\xi_0 = 2.3 > 1$ in the case of TLC, so Eq. (46) does not apply and we need to use Eq. (45) directly. From Fig. 1 we see that $U_f(\xi_0 = 2.3)/U_0(\xi_0 = 2.3) \sim 0.015$. Since $n_\gamma = 1.33$, Eq. (45) gives $\langle \Delta P \rangle \sim 0.04$, which is about twice as large as the contribution from the classical spin precession. Putting the two effects together, we estimate the depolarization in TLC to be $\langle \Delta P \rangle \sim 0.064$.

APPENDIX A

In this appendix we shall give some formulae concerning the functions U_0 and U_f defined by Eqs. (42) and (43). For completeness we shall also give formulae for the function U_1 defined by

$$U_1(\xi) = \frac{9\sqrt{3}}{8\pi} \int_0^\infty dy \left[\frac{y}{3(1+\xi y)^2} + \frac{y}{(1+\xi y)^3} + \frac{\xi^2 y^3}{(1+\xi y)^4} \right] K_{2/3}(y) \quad , \tag{48}$$

which is related to the radiation power as

$$P = P_{cl}\, U_1(\xi) \quad , \tag{49}$$

where P_{cl} is the radiation power given by the classical formula:

$$P_{cl} = \frac{2}{3} \frac{r_e m c^3 \gamma^4}{\rho^2} \quad . \tag{50}$$

One can easily get the asymptotic form for $\xi \to 0$ by expanding the integrands of (42), (43) and (48) into power series of ξ and by using the formula

$$\int_0^\infty x^\mu K_\nu(x) dx = 2^{\mu-1} \Gamma\left(\frac{\mu+\nu+1}{2}\right) \Gamma\left(\frac{\mu-\nu+1}{2}\right) \qquad (\Re\mu - \Re\nu + 1 > 0) \quad . \tag{51}$$

Thus, we find

$$U_0(\xi) = \frac{1}{20\pi} \sum_{n=0}^\infty (3n^2 + 3n + 10) \Gamma\left(\frac{n}{2} + \frac{1}{6}\right) \Gamma\left(\frac{n}{2} + \frac{5}{6}\right) (-2\xi)^n \quad , \tag{52}$$

$$U_1(\xi) = \frac{3\sqrt{3}}{16\pi} \sum_{n=0}^\infty (n+1)(n^2 + 2n + 8) \Gamma\left(\frac{n}{2} + \frac{2}{3}\right) \Gamma\left(\frac{n}{2} + \frac{4}{3}\right) (-2\xi)^n \quad , \tag{53}$$

$$U_f(\xi) = \frac{3}{20\pi} \sum_{n=2}^\infty \frac{n(n-1)}{n+1} \Gamma\left(\frac{n}{2} + \frac{5}{6}\right) \Gamma\left(\frac{n}{2} + \frac{7}{6}\right) (-2\xi)^n \quad . \tag{54}$$

These expansions do not converge for any finite ξ. They are merely asymptotic expansions, but still useful for very small ξ, say $\xi < 0.03$. The first few terms are

$$U_0 = 1 - \frac{16}{15\sqrt{3}}\xi + \frac{14}{9}\xi^2 - \frac{1472}{135\sqrt{3}}\xi^3 + O(\xi^4) \quad , \tag{55}$$

$$U_1 = 1 - \frac{55}{8\sqrt{3}}\xi + \frac{64}{3}\xi^2 - \frac{8855}{36\sqrt{3}}\xi^3 + O(\xi^4) \quad , \tag{56}$$

$$U_f = \frac{7}{54}\xi^2 - \frac{16}{3\sqrt{3}}\xi^3 + \frac{1001}{135}\xi^4 + O(\xi^5) \quad . \tag{57}$$

In order to get the expansion at $\xi = \infty$, we replace $1/(1+\xi y)^m$ in the integrands using

$$\frac{1}{1+x} = -\frac{1}{2i}\int\limits_{c-i\infty}^{c+i\infty} \frac{x^s}{\sin \pi s}\, ds \qquad (-1 < c < 0) \quad , \tag{58}$$

and its derivatives with respect to x, integrate over y using Eq. (51) and close the integration contour along the left hemicircle in the complex s-plane. The results are

$$U_0(\xi) = \frac{\pi}{45}\sum_{n=1}^{\infty} \frac{(n^2-3n+30)\epsilon_n}{\Gamma(\frac{n+1}{6})\Gamma(\frac{n+5}{6})}(2\xi)^{-n/3} \quad , \tag{59}$$

$$U_1(\xi) = \frac{\pi}{36\sqrt{3}}\sum_{n=1}^{\infty} \frac{n(n^2+63)\epsilon_n}{\Gamma(\frac{n+1}{6})\Gamma(\frac{n+5}{6})}(2\xi)^{-n/3-1} \quad , \tag{60}$$

$$\begin{aligned} U_f(\xi) = \frac{3}{10\xi}\Bigg[& \frac{1}{\sqrt{3}}\left(\log 2\xi - \gamma_E - \frac{3}{2}\log 3 - \frac{3}{2}\right) \\ & + \pi \sum_{n=2}^{\infty} \frac{(n/3+1)(n/3+2)\epsilon_{n+3}}{n\Gamma(\frac{n}{6}+\frac{1}{3})\Gamma(\frac{n}{6}+\frac{2}{3})}(2\xi)^{-n/3}\Bigg] \quad , \end{aligned} \tag{61}$$

where γ_E is Euler's constant (=0.57721...) and

$$\epsilon_n = -(-1)^n - 2\cos\frac{2n\pi}{3} \quad . \tag{62}$$

Actually, $\epsilon_n = 0$ for $n = 2$ and 4 (mod 6). These formulae converge for any positive ξ but, of course, cannot be used for very small ξ because of loss of digits. The first few terms are

$$U_0 = 1.15830\xi^{-1/3} - 0.86603\xi^{-1} + 1.94870\xi^{-5/3} - 2.88000\xi^{-2} + O(\xi^{-7/3}) \quad , \tag{63}$$

$$U_1 = 0.95535\xi^{-4/3} - 2.25000\xi^{-2} + 7.73495\xi^{-8/3} - 12.8605\xi^{-3} + O(\xi^{-10/3}) \quad , \tag{64}$$

$$U_f = 0.173205\log\xi/\xi - 0.52516\xi^{-1} + 1.94870\xi^{-5/3} - 2.70000\xi^{-2} + O(\xi^{-7/3}) \quad . \tag{65}$$

The maximum of U_f and U_f/U_0 are

$$U_f = 0.010505 \quad , \qquad \text{at} \quad \xi = 4.14 \quad ; \tag{66}$$

$$U_f/U_0 = 0.019980 \quad , \qquad \text{at} \quad \xi = 11.35 \quad . \tag{67}$$

The functions U_0, U_1 and U_f are plotted in Fig. 1 together with U_f/U_0.

The following approximate formulae are useful for practical purposes. (ϵ is the maximum relative error in the range $0 \leq \xi < \infty$.)

$$U_0(\xi) = \frac{1 - 0.59797\xi + 1.06082\xi^{5/3}}{1 + 0.92176\xi^2} \quad , \qquad \epsilon = 0.0064 \quad ; \tag{68}$$

$$U_1(\xi) = \left\{ \frac{1 + 18.91145\xi}{1 + 19.58981\xi + 19.48734\xi^{5/3}} \right\}^2 , \qquad \epsilon = 0.014 \quad . \tag{69}$$

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