SLAC–PUB–4683
July 1988
T/AS

Consistency of Quantum Cosmology for Models of Plane Symmetry*

TAKAYUKI MATSUKI†

Stanford Linear Accelerator Center

Stanford University, Stanford, California 94309

and

BEVERLY K. BERGER

Physics Department

Oakland University

Rochester, MI 48309

Submitted to *Physical Review D*

* Work supported by the Department of Energy, contract DE–AC03–76SF00515.

† Permanent address is Physics Dept., Oakland Univ., Rochester, MI 48309

ABSTRACT

The consistency of D-dimensional quantum cosmology is studied. It is shown that the effective Lagrangian in a minisuperspace for the D-dimensional Einstein Lagrangian plus scalar terms with (D-2)– dimensional plane symmetry has exactly the same form as that for the bosonic string theory. From this it is derived that the maximum number of space-time dimensions for this kind of quantum cosmology must be $\underline{D = 29}$. A local scale invariance is also found in this model, which is reduced to the 2-dimensional Weyl invariance and a local translation invariance in string theory.

Superstring theories which automatically unify gravitational and matter fields are currently under intensive investigation as serious candidates for a completely unified description of Nature.[1] Since the interactions of all fields at the microscopic level were important in the very early universe, the cosmological implications of string theories have been sought. For example, one may consider the relationship between the restrictions imposed on matter fields by the string framework and the value of the cosmological constant. An alternative route toward the unification of quantum mechanical phenomena and gravity has been the direct application of quantum field theoretic methods to classical general relativity. An approximation to this procedure which has experienced a revival of interest in the past few years has been the quantization of the degrees of freedom of cosmological models in the hope that features of the full theory would survive this simplification.[2,3] Here we apply string methodology to matter-free cosmological models to determine if, as in string theories, restrictions exist on the space-time dimension of the quantized system.

We study a class of cosmologies which lead to an Einstein-Hilbert Lagrangian which can be put into close analogy with the Polyakov action for the bosonic string.[4] These models are described in a coordinate basis by

$$g_{\mu\nu} = \exp(2\alpha) \begin{pmatrix} g_{ab}(\sigma) & 0 \\ 0 & \exp[2\phi_i(\sigma)]\,\delta_{ij} \end{pmatrix}, \tag{1}$$

where α is some functional which is determined later in terms of $\phi_i(\sigma)$ so that a resultant expression becomes a simplified one, $g_{ab}(\sigma)$ stands for a general N-dimensional curved metric with signature (-,+,+, ...), indices i and j run from 1 to D - N, and a and b from 0 to N - 1. Both $g_{ab}(\sigma)$ and $\phi_i(\sigma)$ are functions only of σ^a (a = 0, ..., N-1). Two well-known pure gravity (vacuum) cosmological solutions to Einstein's equations are members of this class for D = 4. They are the spatially homogeneous Kasner model (N = 1)[5] and the plane symmetric single polarization Gowdy T^3 model (N = 2).[6] The metric form (1) has been chosen to emphasize field theoretic and string analogies in the final reduced Lagrangian. If we define a D-dimensional scalar curvature R in terms of the Ricci and Riemann-Christoffel tensors as $R = R^{\mu}{}_{\mu} = R^{\mu\nu}{}_{\mu\nu}$, then the D-dimensional Einstein-Hilbert Lagrangian is given by

$$L = \sqrt{{}^{(D)}g}\;{}^{(D)}R\ , \tag{2}$$

where a D-dimensional gravitational coupling $1/(16\pi G)$ is set equal to 1 and D-dimensional quantities are attached by a pre-superscript (D) to clearly distinguish them from N- and (D-N)-

dimensional ones. The D - N dimensional space with a metric $\exp[2\phi_i(\sigma)]\,\delta_{ij}$ is assumed to be compact with its space volume set equal to 1.

As a first example we study the D-dimensional Kasner model. It will be shown that in the Kasner model, the effective first-quantized Lagrangian in a minisuperspace (the space of all 3-geometries with the prescribed symmetry) is that which describes a trajectory for a massless particle and that the second-quantized Lagrangian corresponds to that for a massless scalar particle in D - 1 dimensions. On the other hand it has been noted that Einstein's equations for the single polarization Gowdy T^3 (plane symmetric) model in four dimensions bear a striking similarity to those for the relativistic bosonic string theory[7] and it will be actually shown that in a generalization of this model to D dimensions, the effective first-quantized Lagrangian in the appropriate minisuperspace is given by that for a bosonic string described by D - 3 massless scalar fields embedded in a two-dimensional curved space-time. Therefore to obtain the second quantization of this model which becomes the wave equation in quantum cosmology,[2] we have to second-quantize a bosonic string theory,[1] which has not yet been completed although the properties of the D = 4 quantized model have long since been studied by other methods.[8] Here we develop and exploit a method which is similar to that used in compactification of higher dimensional theories to obtain the resultant effective Lagrangian in a minisuperspace. We also develop in Appendices A and B formulae to diagonalize the symmetric, bilinear kinetic terms, which appear in the final reduced Lagrangian. This diagonalization is required to allow the counting of the number of degrees of freedom.

Now we derive a N-dimensional effective Lagrangian for the fields $g_{ab}(\sigma)$ and $\phi_i(\sigma)$. Calculation is tedious but straightforward and is given by

$$\sqrt{{}^{(D)}g}\;{}^{(D)}R = {}^{(D)}g^{ab}\,{}^{(D)}g^{cd}\,{}^{(D)}R_{acbd} + {}^{(D)}g^{ab}\,{}^{(D)}g^{ij}\,{}^{(D)}R_{aibj} + {}^{(D)}g^{ij}\,{}^{(D)}g^{kl}\,{}^{(D)}R_{ikjl}\ , \qquad (3)$$

where a sum over repeated indices is tacitly assumed. Each term in (3) is given as follows:

$$\text{first term} = \sqrt{g}\,R - (N-2)(N-1)\sqrt{g}\,g^{ab}\,\partial_a\alpha\,\partial_b\alpha - 2\,(N-1)\,\partial_a(\sqrt{g}\,g^{ab}\,\partial_b\,\alpha\,)\ , \qquad (4)$$

$$\text{second term} = -\,2\sqrt{g}\,g^{ab}\sum_{i=1}^{D-N}\Big[\partial_a(\alpha+\phi_i)\partial_b(\alpha+\phi_i) + (N-2)\,\partial_a\alpha\partial_b(\alpha+\phi_i)\Big]$$
$$-\,\frac{2(N-2)}{D-2}\,\partial_a(\sqrt{g}\,g^{ab}\sum_i\partial_b\phi_i)\ , \qquad (5)$$

$$\text{third term} = -2\sqrt{g}\, g^{ab} \sum_{i<j} \partial_a(\alpha + \phi_i)\, \partial_b(\alpha + \phi_j) \ , \tag{6}$$

where the coefficient of the N-dimensional $\sqrt{g}\,R$ in (4) is determined to be 1 when

$$\alpha = -\frac{1}{D-2} \sum_i \phi_i \ . \tag{7}$$

The final result is given by

$$L = \sqrt{g}\,R - \sqrt{g}\, g^{ab} \sum_i \partial_a\phi_i\, \partial_b\phi_i + \frac{1}{D-2}\sqrt{g}\, g^{ab}\left(\sum_i \partial_a\phi_i\right)\left(\sum_j \partial_b\phi_j\right)$$
$$+ \frac{2}{D-2}\partial_a(\sqrt{g}\, g^{ab} \sum_i \partial_b\phi_i\,) \ . \tag{8}$$

Note that coefficients of each term depend only on a total dimension D even though the number of fields $\phi_i(\sigma)$ is D - N and the space-time dimension of σ is N. We can check our results by comparison with other theoretical models. For instance, (4) with D = N and $\alpha = -\frac{1}{D-2}\ln\phi$ has the same form as D-dimensional Brans-Dicke theory, $\phi\sqrt{{}^{(D)}g_1}\,{}^{(D)}R$, with the coupling parameter $\omega = 0$.[9] The final result (8) with N = 4, D = 5 and $\phi_1 = \frac{1}{2}\ln\phi$ is nothing but a Kaluza-Klein theory in five dimensions and coincides with a known expression.[10] Now let us apply (8) to the two cosmological models.

1) D-dimensional Kasner model[5]

In this model, spatial homogeneity is assumed, hence the metric for this model corresponds to the one with N=1 and an arbitrary D and is given by

$$g_{\mu\nu} = \exp(2\alpha)\begin{pmatrix} -e(t) & 0 \\ 0 & \exp[2\phi_i(t)]\,\delta_{ij} \end{pmatrix} , \tag{9}$$

where α is given by (7) and e(t) may be considered a one-dimensional "metric". The effective Lagrangian in the minisuperspace is given by

$$L = e^{-1/2}(t)\left[\sum_i (\partial_0\phi_i)^2 - \frac{1}{D-2}\left(\sum_i \partial_0\phi_i\right)\left(\sum_j \partial_0\phi_j\right)\right]$$

$$-\frac{2}{D-2}\partial_0\left[e^{-1/2}(t)\sum_i \partial_0\phi_i\right] \qquad (10)$$

We need to diagonalize the bilinear terms in ϕ_i to find the minisuperspace metric. The space spanned by $\phi_i(t)$ is the minisuperspace for this model. In Appendix A we derive two different recursion formulae depending on the oddness or evenness of D - N. Applying this to a special case of our model, i.e., D = 4 and N = 1, a normalized set of fields $\{\psi_i\}$ is given as follows.

$$\psi = \sqrt{2}\, C\, R\, \phi \qquad (11)$$

where the matrices C and R are given by (B2) in Appendix B in case of D = 4 dimensions. Inserting this expression (11) into (10) gives us

$$\begin{aligned} L &= \frac{1}{2}\, e^{-1/2}(t)\, \eta^{ij}\, \partial_0\psi_i(t)\partial_0\psi_j(t) \\ &= p^i(t)\, \partial_0\psi_i(t) - \frac{1}{2}\, e^{1/2}(t)\, \eta_{ij}\, p^i(t)\, p^j(t)\ , \end{aligned} \qquad (12)$$

where η^{ij} =diag(-,+,+), $p^i = \delta L/\delta(\partial_0\psi_i) = \eta^{ij}\, e^{-1/2}(t)\, \partial_0\psi_j(t)$ and a total divergence has been neglected. This is nothing but a Lagrangian describing a trajectory for a massless particle with one-parameter (t) reparametrization invariance in three dimensions (i, j = 1, 2, and 3) and reproduces the standard Hamiltonian formulation fo the Kasner model. The function e(t) plays a role of a one-dimensional metric, which may be gauge-fixed to be 1. As is well-known, this first quantized Lagrangian yields upon second-quantization a Lagrangian for a massless scalar particle $\Phi(\psi_i)$ which satisfies the Wheeler-DeWitt equation and is thus the wave function of this universe.

2) D-dimensional plane-symmetric model (generalized Gowdy model)[6]

In this model, the metric components depend on one space direction as well as time hence the metric for this case is given by (1) with N = 2 and an arbitrary D. The two dimensional metric $g_{ab}(\sigma)$ must be non-diagonal in general to allow the (now two-dimensional) reparametrization invariance. Therefore the effective Lagrangian in a minisuperspace is already given by (8) with N = 2 and an arbitrary D, where $g_{ab}(\sigma)$ is a 2-dimensional metric and $\sigma^0 = t$ and $\sigma^1 = x$. In N = 2 dimensions, the first term $\sqrt{g}\, R$ becomes a total divergence and is neglected as is the last term in (8). This assumes of no contribution from the boundary and requires that x be compact. Since a coefficient matrix of remaining bilinear terms in $\phi_i(\sigma)$ is symmetric and real, it can be in principle diagonalized by an orthogonal matrix. The actual diagonalization is found by using recursion formulae in Appendix A. After appropriately normalizing the diagonalized bilinear terms and neglecting total divergent terms, we obtain

$$L = -\frac{1}{2}\sqrt{g(\sigma)}\, g^{ab}(\sigma)\, \eta^{ij}\, \partial_a \psi_i(\sigma) \partial_b \psi_j(\sigma) \ , \tag{13}$$

where the $\psi_i(\sigma)$ are linear combinations of the $\phi_i(\sigma)$. In Appendix B, it is proven that there is one zero and (D - 3) +1 eigenvalues for a coefficient matrix of $\phi_i(\sigma)\phi_j(\sigma)$ in (8) in the case of N = 2 and $D \geq 4$. Hence the metric in (13) is given by $\eta^{ij} = \mathrm{diag}(+,...,+,0)$, where the last component $\psi_{D-2}(\sigma)$ is chosen to be that with zero eigenvalue. This is consistent with the results[9] obtained for the D = 4 Gowdy T^3 model[7] with the ADM method,[11] in which just one polarization of gravitational waves is obtained although N = 2. The final effective Lagrangian is nothing but that for string theory, i.e., a massless scalar Lagrangian embedded in a two-dimensional curved space-time. In the minisuperspace $\{\phi_i(\sigma)\}$, the first quantization of this model with the reduced Lagrangian (13) in contrast to the Kasner model with (12) has a conformal or BRST anomaly due to the conformal invariance in 2 dimensions. The closure of the BRST global symmetry[12] is necessary to second-quantize this model. If the exact BRST symmetry for this Lagrangian (13) is required to have a consistent second-quantized model or a consistent wave equation, the conformal anomaly should vanish. This is achieved by setting the well-known coefficient of the anomaly {(D-3) - 26} equal to zero,[4] i.e.,

$$D = 29. \tag{14}$$

A normalized set $\{\psi_i(\sigma)\}$ in D = 29 is given by

$$\psi = \sqrt{2}\, C_4\, R_4\, R_3\, C_2\, R_2\, C_1\, R_1\, \phi \ , \tag{15}$$

where matrices R_i and C_i are given in Appendix D, a vector ψ includes the component $\psi_{27}(\sigma)$ which does not appear in (13) and an appropriate matrix multiplies the right hand side of this equation to ensure that the missing component is in the 27th position. The complete second quantization of this model, which is equivalent to the solution of the Wheeler-DeWitt equation in quantum cosmology, is not known.

We may consider the following:

i) As one may easily notice, D = 29 is the maximum value for the dimension to have a consistent quantum cosmology. If the addition of matters to the Einstein-Hilbert action is allowed, one may construct a similar Lagrangian to (13) in D less than 29 dimensions but with more explicit scalar terms of the form

$$\frac{1}{2}\sqrt{{}^{(D)}g}\ {}^{(D)}g^{\mu\nu}\,\partial_\mu\psi_0(\sigma)\partial_\nu\psi_0(\sigma) = \frac{1}{2}\sqrt{g(\sigma)}\ g^{ab}(\sigma)\ \partial_a\psi_0(\sigma)\partial_b\psi_0(\sigma)\ , \tag{16}$$

where the $\psi_0(\sigma)$ has the same space-time dependence as the $g_{ab}(\sigma)$ and $\phi_i(\sigma)$. The total number of scalar terms must be 26 to make the conformal anomaly vanish.

ii) The final effective Lagrangian corresponds to a free bosonic string theory. One way to include interaction terms may be to consider a two-polarization model of the Gowdy type,[13] which is characterized by nonlinear coupling between the two polarizations. With interaction terms we may consider creation and annihilation of universes of this particular type just like those of strings in string theory.

iii) We must discuss the meaning of the disappearance of one degree of freedom in the resultant effective Lagrangian for $N = 2$ and $D \geq 4$. In (13), we have assumed that the component $\psi_{D-2}(\sigma)$ disappeared from the result after a suitable renumbering of indices. Let us locally translate all the $\phi_i(\sigma)$ ($i = 1,\ldots, D-2$) by an amount $\phi_i'(\sigma)$. The $\psi_i(\sigma)$ are transformed by

$$\psi \;\rightarrow\; \psi + \psi' = R\,(\phi + \phi') = R\,\phi + R\,\phi'\ , \tag{17}$$

where R is a matrix calculated by using (A3) and (A7) in Appendix A with an appropriate normalization matrix. The Lagrangian (13) is invariant under this transformation if

$$\psi' = R\,\phi' = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \psi_{D-2}' \end{pmatrix} \quad \text{or} \quad \phi' = R^{-1}\,\psi'\ , \tag{18}$$

which since the Lagrangian is independent of the (D-2)-th component of the vector $\psi(\sigma)$. In the D-dimensional language, the transformation invariance can be reexpressed through the metric, containing Weyl invariance in $N = 2$ dimensions, as

$$g_{\mu\nu} \;\rightarrow\; g_{\mu\nu}' = \begin{pmatrix} \exp[2\phi_W(\sigma)]\ I_{2\times 2} & 0 \\ 0 & \exp[2\phi_i'(\sigma)]\ \delta_{ij} \end{pmatrix} g_{\mu\nu}\ , \tag{19}$$

where $\phi_W(\sigma)$ is a gauge function for the Weyl transformation and D-2 $\phi_i^{'}(\sigma)$ satisfy the condition given by (19). Notice the condition (19) includes only one gauge function $\psi_{D-2}^{'}(\sigma)$, i.e., all D-2 $\phi_i^{'}(\sigma)$ are expressed only by $\psi_{D-2}^{'}(\sigma)$. Our argument here shows that when it is expressed by (2), our Lagrangian (13) has a local scale invariance which becomes two-dimensional Weyl invariance and local translational invariance in the bosonic string theory. Since it becomes trivial when the Lagrangian is expressed in terms of completed square terms, the local translational invariance in string theory does not exist so that the field associated with it must vanish in the Lagrangian.

iv) Even though the Lagrangian (17) is exactly the same form as that for string theory, the general Lagrangian given by (2) even with additional scalar terms is not completely equivalent to a general string theory. This can be readily seen when one notices that the mass term $\sqrt{g}\mu^2$ which is allowed in the usual formulation of string theory in two dimensions does not appear in our formulation in D dimensions. The term $\sqrt{^{(D)}g}$ does not include the term $\sqrt{g}\mu^2$ at all and there is no other way to produce such a term from D-dimensional gravity.

v) Our analysis in this paper may be used in the opposite sense. That is, one can embed string theory in a D = 29 dimensional Kaluza-Klein type metric. This computation may lead to interesting features of string theory.

vi) We have shown the equivalence between a bosonic string theory and a particular quantum cosmology with (D-2)-dimensional plane symmetry. Since the Lagrangian in quantum cosmology is written in a minisuperspace while the one for string theory is written in real space-time coordinates, there is no physical connection between these two models. The match of the two different models may, however, suggest an alternative framework for the study of both cosmological models and superstring theories. For this purpose, extension of the analogy to a more general class of cosmological models is required. In particular, the relation between a supersymmetric quantum cosmology[14] and a (super-) string theory should be studied. There must be a restriction on the maximum value of the dimension D for a supersymmetric cosmology as was found here. Imposing a supersymmetry on a cosmology might also yield a restriction on the form of the superstring theory.

Acknowledgements

This research is supported in part by NSF Grant PHY-82-13411. B.B wishes to thank the Institute for Geophysics and Planetary Physics at the Lawrence Livermore National Laboratory for hospitality. T.M. wishes to thank Prof. R. Blankenbeckler for warm hospitality at Stanford Linear Accelerator Laboratory. T.M. would like to thank people in SLAC for pointing out a simple way to find out eigenvalues of a coefficient matrix.

Appendix A

In this appendix we derive recursion formulae to diagonalize the sum of the following bilinear terms:

$$I_n[\{x_i^{(0)}\}, a] = \sum_{i=1}^{n} \left(x_i^{(0)}\right)^2 - a\left(\sum_{i=1}^{n} x_i^{(0)}\right)^2 . \tag{A1}$$

i) even $n = 2m$

We define a new basis $\{x^{(1)}\}$ by

$$x_1^{(0)} = \frac{x_1^{(1)} + x_2^{(1)}}{\sqrt{2}} , \; x_2^{(0)} = \frac{x_1^{(1)} - x_2^{(1)}}{\sqrt{2}} , \; \ldots, \; x_{n-1}^{(0)} = \frac{x_{n-1}^{(1)} + x_n^{(1)}}{\sqrt{2}} , \; x_n^{(0)} = \frac{x_{n-1}^{(1)} - x_n^{(1)}}{\sqrt{2}} , \tag{A2}$$

or

$$x^{(1)} = R\, x^{(0)} , \quad R = R_0\, I_{m\times m} , \quad \text{and } R_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} , \tag{A3}$$

where $I_{m\times m}$ is a m by m unit matrix. Since the Jacobian for this transformation is det $R = 1$, there exists an inverse matrix R^{-1}, which means both $\{x_i^{(0)}\}$ and $\{x_i^{(1)}\}$ form a complete basis. Then (A1) becomes

$$\begin{aligned} I_n[\{x_i^{(0)}\}, a] &= \sum_{i=1}^{m} \left(x_{2i}^{(1)}\right)^2 + \sum_{i=1}^{m} \left(x_{2i-1}^{(1)}\right)^2 - 2a\left(\sum_{i=1}^{m} x_{2i-1}^{(1)}\right)^2 \\ &= \sum_{i=1}^{m} \left(x_{2i}^{(1)}\right)^2 + I_m[\{x_{2i-1}^{(1)}\}, 2a] . \end{aligned} \tag{A4}$$

Hence we only need to diagonalize the second term by using either recursion formula in this subsection when m = even or one in the next subsection when m = odd.

ii) odd $n = 2m + 1$

In this case we define a new basis $\{x^{(1)}\}$ by

$$x_1^{(0)} = \frac{x_1^{(1)} + x_2^{(1)}}{\sqrt{2}} , \quad x_2^{(0)} = \frac{x_1^{(1)} - x_2^{(1)}}{\sqrt{2}} , \quad \ldots, \quad x_{n-2}^{(0)} = \frac{x_{n-2}^{(1)} + x_{n-1}^{(1)}}{\sqrt{2}} , \quad x_{n-1}^{(0)} = \frac{x_{n-2}^{(1)} - x_{n-1}^{(1)}}{\sqrt{2}} ,$$

$$x_n^{(0)} = x_n^{(1)} + b \sum_{i=1}^{m} x_{2i-1}^{(1)} , \tag{A5}$$

where b is determined so that the final expression is separated into terms with even indices and those with odd indices,

$$b = \frac{\sqrt{2}a}{1-a} , \tag{A6}$$

or

$$x^{(1)} = R\, x^{(0)} , \quad R = \begin{pmatrix} R_0\, I_{m \times m} & & 0 \\ -\frac{a}{1-a} & \cdots -\frac{a}{1-a} & 1 \end{pmatrix} . \tag{A7}$$

Since the Jacobian for this transformation is det $R = 1$ again, there exists an inverse matrix R^{-1}, which means both $\{x_i^{(0)}\}$ and $\{x_i^{(1)}\}$ form a complete basis. Then (A1) becomes

$$\begin{aligned} I_n[\{x_i^{(0)}\}, a] &= \sum_{i=1}^{m} \left(x_{2i}^{(1)}\right)^2 + (1-a) \left(x_{2m+1}^{(1)}\right)^2 + \sum_{i=1}^{m} \left(x_{2i-1}^{(1)}\right)^2 - \frac{2a}{1-a}\left(\sum_{i=1}^{m} x_{2i-1}^{(1)}\right)^2 \\ &= \sum_{i=1}^{m} \left(x_{2i}^{(1)}\right)^2 + (1-a) \left(x_{2m+1}^{(1)}\right)^2 + I_m[\{x_{2i-1}^{(1)}\}, \frac{2a}{1-a}] . \end{aligned} \tag{A8}$$

Appendix B

In this appendix we prove that one eigenvalue is zero and others are 1 for the coefficient matrix $M^{(n)}$ given by (A1) with $a = \frac{1}{D-2}$ and $N = 2$ (which corresponds to $n = D - 2$);

$$M^{(n)} = I_{nxn} - a\,E^{(n)}\ ,\quad \left(E^{(n)}\right)_{ij} = 1 \text{ for } i, j = 1, \ldots, n\ . \tag{B1}$$

Eigenvectors for $E^{(n)}$ are readily constructed as

$$v_1 = N_1\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix},\ v_2 = N_2\begin{pmatrix}1\\ -1\\ 0\\ \vdots\\ 0\end{pmatrix}, \ldots, v_n = N_n\begin{pmatrix}1\\ \vdots\\ 1\\ -n+1\end{pmatrix}, \tag{B2}$$

where the eigenvectors v_i are orthogonal to each other and are normalized to be 1 with appropriate normalizations N_i. The eigenvalues of v_i are given by

$$E^{(n)}\, v_1 = n\, v_1\ ,\quad E^{(n)}\, v_i = 0 \ \text{ for } i \neq 1\ . \tag{B3}$$

That is all eigenvalues are zero except for n for v_1. Since the v_i are also eigenvectors of a unit matrix I_{nxn} with eigenvalues 1, they are eigenvectors of the coefficient matrix $M^{(n)}$. Therefore

$$M^{(n)}\, v_1 = (1 - an)\, v_1\ ,\quad M^{(n)}\, v_i = v_i \ \text{ for } i \neq 1\ . \tag{B4}$$

Since $a = \frac{1}{D-2}$ and $n = D - N$ in general, the eigenvalue of v_1 is given by

$$1 - an = \frac{N-2}{D-2}\ , \tag{B5}$$

which vanishes if $N = 2$ irrespective of a value of D. Using normalized eigenvectors given by (B2), one can directly diagonalize $M^{(n)}$ by an orthogonal matrix O as

$$O\, M^{(n)}\, O^T = \mathrm{diag}(1, \ldots, 1, 0)\ ,\quad O = \begin{pmatrix} v_2^T\\ \vdots\\ v_n^T\\ v_1^T\end{pmatrix}, \tag{B6}$$

where T is a transpose operator and we change the order of vectors so that a zero eigenvalue appears as the last component in the diagonalized form.

Appendix C

In this appendix we apply recursion the formulae derived in Appendix A to diagonalize the sum of the bilinear terms for n = 3 and $a = \frac{1}{2}$ which describes the kinetic term in (10) for D = 4 and N = 1. Only the recursion formula for n = odd is necessary with the final result given by

$$\sum_{i=1}^{3} (x_i)^2 - \frac{1}{2}\left(\sum_{i=1}^{3} x_i\right)^2 = -(y_1)^2 + (y_2)^2 + (y_3)^2 \ , \qquad \text{(C1)}$$

where

$$y \equiv \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = CR\,x \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \qquad \text{(C2)}$$

Here C is a normalization matrix and R is a rotational matrix.

Appendix D

In this appendix we apply recursion formulae derived in Appendix A to diagonalize the sum of the bilinear terms for n = 27 and $a = \frac{1}{27}$ which is used to diagonalize the kinetic term leading to (12) for D = 29 and N = 2. Both of the recursion formulae for n = even and odd are necessary with the final result given by (for the decomposition 27 = 2x13 + 1 ; 13 = 2x6 + 1 ; 6 = 2x3 ; 3 = 2 + 1)

$$y = C_4\, x^{(4)} = C_4\, R_4\, R_3\, C_2\, R_2\, C_1\, R_1\, x^{(0)} \ . \qquad \text{(D1)}$$

First since 27 is odd, we apply (A7) to $x^{(0)}$, i.e., $C_1\, x^{(1)} = C_1\, R_1\, x^{(0)}$,

$$R_1 = \begin{pmatrix} R_0\, I_{13\times 13} & 0 \\ -\frac{1}{26} \cdots -\frac{1}{26} & 1 \end{pmatrix} , \quad C_1 = \begin{pmatrix} I_{26\times 26} & 0 \\ 0 & \sqrt{26/27} \end{pmatrix} , \qquad \text{(D2)}$$

where R_0 is given by (A3); next since 13 is odd, we apply (A7) to $x^{(1)}$, i.e., $C_2\, x^{(2)} = C_2\, R_2\, \tilde{x}^{(1)}$ with $\tilde{x}^{(1)}_i \equiv \left(C_1\, x^{(1)}\right)_{2i-1} = x^{(1)}_{2i-1}$ (i = 1,..., 13) and

$$R_2 = \begin{pmatrix} R_0\, I_{6x6} & 0 \\ -\frac{1}{12} \cdots -\frac{1}{12} & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} I_{12x12} & 0 \\ 0 & \sqrt{12/13} \end{pmatrix}; \tag{D3}$$

since 6 is even, now we apply (A3) to $x^{(2)}$, i.e., $x^{(3)} = R_3\, \tilde{x}^{(2)}$ with $\tilde{x}_i^{(2)} \equiv \left(C_2\, x^{(2)}\right)_{2i-1} = x_{2i-1}^{(2)}$ $(i = 1,\dots, 6)$ and

$$R_3 = R_0\, I_{3x3}; \tag{D4}$$

and finally since 3 is odd, we apply (A7) again to $x^{(3)}$, i.e., $C_4\, x^{(4)} = C_4\, R_4\, \tilde{x}^{(3)}$ with $\tilde{x}_i^{(3)} \equiv x_{2i-1}^{(3)}$ $(i = 1,\dots, 3)$,

$$R_4 = \begin{pmatrix} R_0 & 0 \\ -1\ -1 & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} I_{2x2} & 0 \\ 0 & \sqrt{2/3} \end{pmatrix}, \tag{D5}$$

and

$$\begin{aligned} L_3[\{\tilde{x}_i^{(3)}\}, \tfrac{1}{3}] &= \sum_{i=1}^{3}\left(\tilde{x}_i^{(3)}\right)^2 - \frac{1}{3}\left(\sum_{i=1}^{3} \tilde{x}_i^{(3)}\right)^2 \\ &= \left(x_2^{(4)}\right)^2 + \frac{2}{3}\left(x_3^{(4)}\right)^2 . \end{aligned} \tag{D6}$$

As seen from the final step given by (D6), one degree of freedom , $x_1^{(4)}$, drops out of the final expression as we have expected from a general argument in Appendix B. In (D2) ~ (D6), matrices R_1, R_2, R_3, R_4 , C_1, C_2, and C_4 operate only on appropriate components of each vector.

References

1. M. B. Green J. H. Schwartz, and E. Witten, *Superstring Theory* (Cambridge University, Cambridge, 1987).
2. B. S. DeWitt, Phys. Rev. **160**, 113 (1967); J. A. Wheeler, in *Batelle Rencontres*, edited by C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).
3. K. V. Kuchar and M. P. Ryan Jr., p.451 in *Proceedings of Yamada Conference XIV on Gravitational Collapse and Relativity*, edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1986).
4. A. M. Polyakov, Phys. Lett., **103B**, 207 (1981).
5. E. Kasner, Am. J. Math. **43**, 217 (1921); C. W. Misner, in *Relativity*, edited by M. Carmelin S. Fickler, and L. Witten (Plenum, New York, 1970).
6. R. H. Gowdy, Phys. Rev. Lett. **27**, 827 (1971); Ann. Phys. (N. Y.) **83**, 203 (1974).
7. B. K. Berger, Ann. Phys. (N. Y.) **83**, 458 (1974); Phys. Rev. **D11**, 2770 (1975); C. Misner, Phys. Rev. **D8**, 3271 (1973).
8. B. K. Berger, Ann. Phys. (N. Y.) **156**, 155 (1984).
9. P. G. O. Freund, Nucl. Phys. **B209**, 146 (1982).
10. For instance, see *Modern Kaluza-Klein Theories*, edited by T. Appelquist, A. Chodos, and P. G. O. Freund (Addison-Wesley, Menlo Park, 1987).
11. R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. **113**, 745 (1959); **116**, 1322 (1959); **117**, 1595 (1960).
12. C. Becchi, A. Rouet, and R. Stora, Ann. Phys. (N. Y.) 98, 28 (1976); I. V. Tyutin, Lebedev preprint, FIAN No.39.
13. V. Moncrief, Ann. Phys. (N. Y.) **132**, (1981) 87.
14. A. Macías, O. Obregón, and M. P. Ryan Jr., Class. Q. Grav. **4**, 1477 (1987).