SLAC–PUB–4681
November, 1988
(T)

Conformal Branching Rules from Kač–Moody Automorphisms

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ABSTRACT

We calculate the branching rules for the conformal subalgebra $\hat{SU}(2)^N \times \hat{SO}(N)^4 \subset \hat{Sp}(2N)^1$. The method used relies on the outer automorphisms of affine Kač–Moody algebras, and was first applied by the author to the conformal subalgebra $\hat{SU}(p)^q \times \hat{SU}(q)^p \subset \hat{SU}(pq)^1$. The results presented here demonstrate the general applicability of the method.

Submitted to *J. Math. Phys.*

* The author's research was supported by a postdoctoral fellowship from NSERC of Canada, and by the U.S. Department of Energy under contract DE-AC03-76SF00515.

1. Introduction

Affine Kač–Moody algebras[1] have made a mark in theoretical physics (for a review see reference 2). They are realized in many two-dimensional conformal field theories. These theories describe both the critical points of second order phase transitions and the basic building blocks of classical string theories. In fact, a string theory can only have a local spacetime gauge symmetry if there is a corresponding Kač–Moody algebra realized in the conformal field theory on its world sheet.

The subalgebras of affine Kač–Moody algebras are therefore important. Even a small subclass of affine subalgebras, the so-called conformal subalgebras, are remarkably useful[3–8].Conformal subalgebras are subalgebras having central charge equal to that of the algebra in which they are embedded. Lists of these subalgebras have been compiled[9] but there is no universally applicable method for calculating their branching rules. Hence all of the conformal branching rules have not been worked out.

In this paper we show there does exist a quite general procedure for calculating conformal branching rules. It makes use of the outer automorphisms of affine Kač–Moody algebras. The method was first applied to the subalgebras $\hat{SU}(p) \times \hat{SU}(q) \subset \hat{SU}(pq)$ in reference 8. Here we calculate the branching rules for $\hat{SU}(2) \times \hat{SO}(N) \subset \hat{Sp}(2N)$, demonstrating the general applicability of the outer automorphism method.

The layout of this paper is as follows. The second section contains a short review of affine Kač–Moody algebras and conformal subalgebras (serving mainly to establish notation) and a general description of the outer automorphism method. Section 3 contains the explicit calculation of the branching rules for the conformal embedding $\hat{SU}(2)\times\hat{SO}(N)\subset\hat{Sp}(2N)$; part (a) treats N odd and part (b) N even. Finally, section 4 is a short conclusion.

2. Review and Notation

Let $\hat{g}$ denote the affine Kač-Moody algebra that is the central extension of the loop algebra of the finite–dimensional Lie algebra $\overline{g}$.#1 The algebra $\hat{g}$ is

$$[J^a_m,\ J^b_n] \ =\ f^{abc}\ J^c_{m+n}\ +\ k\delta^{ab}\delta_{m+n,0}\ . \tag{2.1}$$

We will often include the value of $k\in Z$ by writing $\hat{g}=\hat{g}^k$. Setting the integral indices m and n in (2.1) to zero reduces this algebra to the finite one $\overline{g}\subset\hat{g}$.

The Sugawara construction

$$L_m=\frac{1}{2(k+h^\vee)}\sum_{n\in Z} K_{ab}\ :\ J^a_{m+n}\ J^b_{-n}\ :\ -\delta_{m0}\frac{c(g)}{24}\ , \tag{2.2}$$

associates with $\hat{g}$ the Virasoro algebra

$$[L_m,L_n]\ =\ (m-n)L_{m+n}\ +\ \frac{c(g)}{12}\ (m-1)m(m+1)\ \delta_{m+n,0}\ , \tag{2.3}$$

with central charge

$$c(g)\ =\ \frac{k\cdot dim\overline{g}}{k+h^\vee}\ . \tag{2.4}$$

Here $K_{ab},h^\vee$ are the Killing form and dual Coxeter number of $\overline{g}$, and the normal ordering is defined in the usual way.

#1 In general we use the convention that carets and square brackets denote objects associated with affine algebras and bars and round brackets their finite algebra counterparts.

The Cartan subalgebra $\hat{h}$ of $\hat{g}$ contains the Cartan subalgebra $\overline{h}$ of $\overline{g}$, with elements h_i $(i = 1, \ldots, r)$ plus the extra element h_0. We denote the elements of $\hat{h}$ by h_μ $(\mu = 0, 1, \ldots, r)$. Dual to these elements, living in the weight space $\hat{h}^*$, are the fundamental weights ω^μ:

$$\omega^\mu(h_\nu) = \delta^\mu_\nu \ . \tag{2.5}$$

Associated with each h_μ is a co-root $\alpha^\vee_\mu$, also living in the weight space $\hat{h}^*$, so that we have

$$\omega^\mu \cdot \alpha^\vee_\nu = \omega^\mu(h_\nu) = \delta^\mu_\nu \quad . \tag{2.6}$$

The dot product is determined from the Cartan matrix A by the definition of its elements:

$$A_{\mu\nu} \ = \ \alpha_\mu \cdot \alpha^\vee_\nu \ , \tag{2.7}$$

where a root α_μ and its co-root $\alpha^\vee_\mu$ are multiples of each other:

$$\alpha^\vee_\mu = \frac{2\alpha_\mu}{\alpha_\mu \cdot \alpha_\mu} \quad .$$

There is an extra operator which commutes with the h_μ; it is L_0 of (2.3). The Cartan subalgebra can be extended to $\hat{h}^e$ having elements h_μ and $d = -L_0$. We denote the weight dual to d by δ

$$\delta(d) \ = \ \delta(-L_0) \ = \ 1 \ , \tag{2.8}$$

and the co-root corresponding to d by Λ_0

$$\delta \cdot \Lambda_0 \ = \ \delta(d) \ = \ 1 \ . \tag{2.9}$$

With the usual conventions

$$\Lambda_0(h_i) = \delta_{i0} \quad i = 1, \ldots, r \ ,$$

$$\Lambda_0(d) \ \doteq \ \Lambda_0 \ (-L_0) \ = \ \Lambda_0 \cdot \Lambda_0 \ = \ 0 \ ,$$

the scalar product on the extended weight space $\hat{h}^{*e}$ is Minkowskian. Let

$$\overline{\psi} \ = \ k^i \overline{\alpha}_i \ = \ k^{\vee i} \, \overline{\alpha}_i^{\vee} \ , \tag{2.10}$$

be the highest root of $\overline{g}$. The $k^\mu, k^{\vee\mu}$ are known as marks and co-marks, respectively, with $k^0 \equiv k^{0\vee} \equiv 1$. Then if $B, C \in \ \hat{h}^{*e}$, we write

$$\begin{aligned} B &= \beta_\mu \omega^\mu + \beta_\delta \delta \ = \left[\beta_i \omega^i = \overline{B}, k[B], \beta_\delta\right] \\ C &= \gamma_\mu \omega^\mu + \gamma_\delta \delta \ = \left[\gamma_i \omega^i = \overline{C}, k[C], \gamma_\delta\right] \end{aligned} \tag{2.11}$$

where $k[B], k[C]$ are the levels of B and C, respectively,

$$k[B] \ = \ \beta_\mu k^{\vee\mu} \qquad k[C] \ = \ \gamma_\mu k^{\vee\mu} \ . \tag{2.12}$$

Then the dot product on the expanded weight space $\hat{h}^{*e}$ takes the form

$$B \cdot C \ = \ \overline{B} \cdot \overline{C} \ + \ k[B]\gamma_\delta \ + \ \beta_\delta k[C] \ . \tag{2.13}$$

With the above notation, the simple roots and fundamental weights of $\hat{g}$ can be written

$$\begin{aligned} \alpha_i \ &= \ [\overline{\alpha}_i, 0, 0] \qquad \alpha_0 \ = \ \left[-\overline{\psi}, 0, 1\right] \\ \omega^i \ &= \ \left[\overline{\omega}^i, k^{\vee i}, 0\right] \quad \omega^0 \ = \ [0, 1, 0] \ . \end{aligned}$$

So the Dynkin diagram of $\hat{g}$ is the extended Dynkin diagram of $\overline{g}$. Outer automorphisms act as symmetries of the Dynkin diagram of $\hat{g}$. The Dynkin diagrams of $\hat{Sp}(2N)$, $\hat{SU}(2)$, and $\hat{SO}(N)$ are shown in Figure 1; their outer automorphisms will be discussed later.

The highest weight representations of $\hat{g}$ are generated from a "vacuum" state $|M>$, labeled by a dominant weight

$$M = M_\mu \omega^\mu = [\overline{M}, M_\mu k^{\vee\mu}, 0] \ . \tag{2.14}$$

The "vacuum" satisfies

$$J_n^a |M> = 0 \ , \ n > 0$$
$$J_0^a |M> = T_{\overline{M}}^a |M> \quad ,$$

where $T_{\overline{M}}^a$ are the matrices representing the generators of $\overline{g}$ in the finite dimensional representation labeled by $\overline{M}$. We will use the notation $M = [M_0 M_1 \ldots M_r] \equiv [M]$ to denote the weight M and corresponding highest weight representation. Similarly, representations of $\overline{g}$ will be denoted by $\overline{M} = (M_1 M_2 \ldots M_r) = (M)$. For unitary highest weight representations $[M]$ of (2.1), we must require

$$k[M] = M_\mu k^{\vee\mu} = k \ . \tag{2.15}$$

The states in $[M]$ having a fixed eigenvalue of L_0 fall into a finite sum of irreducible representations of $\overline{g}$. For example, the lowest value of L_0 is

$$L_0 = h[M] - \frac{c(g)}{24} = \frac{\overline{M} \cdot (\overline{M} + 2\overline{\rho})}{2(k + h^\vee)} - \frac{c(g)}{24} \ , \tag{2.16}$$

where $\overline{\rho}$ is the half-sum of positive roots of $\overline{g}$. The states of $[M]$ with this value of L_0 fill out the representation $\overline{M} = (M)$.

The character of a highest weight representation $M = [M]$ is defined by

$$ch_M(\tau, z) = tr_{[M]} e^{2\pi i(\tau L_0 + z \cdot \overline{h})} \ . \tag{2.17}$$

The characters at $z = 0$

$$\chi[M] \;=\; \chi[M_0 M_1, \ldots M_r] \;=\; ch_M(\tau, 0) \;, \tag{2.18}$$

are called restricted characters. The modular transformation properties of the characters were found in reference 10. Transforming by $S : \tau \rightarrow -1/\tau$ reveals the asymptotic behavior of the characters:

$$\text{as } \tau \rightarrow \; 0, \;\; \chi[M] \sim \ell[M] e^{2\pi i c(g)/24\tau}. \tag{2.19}$$

The $\ell[M]$ can be calculated entirely from objects relevant to the finite algebra $\overline{g}$:

$$\ell[M] = \left| \frac{\overline{P}}{\overline{Q}} \right|^{1/2} \left[k[M] + h^\vee \right]^{-\frac{r}{2}} \prod_{\alpha \in \overline{\Delta}_+} 2 \, \sin \left[\frac{\pi(\overline{M} + \overline{\rho}) \cdot \alpha}{k[M] + h^\vee} \right] . \tag{2.20}$$

Here $\overline{\Delta}_+$ denotes the set of positive roots, $\overline{M} = (M_1 \cdots M_r)$, and $\overline{P}$, $\overline{Q}$ are, respectively, the weight and root lattices of $\overline{g}$.

Knowledge of affine subalgebras of affine algebras is nowhere near as extensive as that of finite subalgebras of finite algebras. However, each subalgebra $\overline{j}$ of a Lie finite algebra $\overline{g}$ induces a subalgebra $\hat{j}$ of $\hat{g}$ (see, for example, Ref. 3.One identifying feature of a finite subalgebra $\overline{j} \subset \overline{g}$ is the index of embedding σ, equal to the ratio of the length squared of the highest root of $\overline{g}$ to that of $\overline{j}$ embedded in $\overline{g}$. The affine subalgebra induced by $\overline{j}^\sigma \subset \overline{g}$ (obvious notation) is $\hat{j}^{\sigma k} \subset \hat{g}^k$.

All other information concerning a finite subalgebra $\overline{j}^\sigma \subset \overline{g}$ is contained in the so-called projection matrix $\overline{F}$[11]. $\overline{F}$ is a (rank $\overline{j} = \tilde{r}$) $\times$ (rank $\overline{g} = r$) matrix relating weights of $\overline{g}$ to the weights of $\overline{j}$ onto which they are projected. If $(M) = (M_1 M_2 \cdots M_r)$ is a weight of $\overline{g}$, it is projected onto the weight $(M)\overline{F}^T$.

We can define an affine projection matrix $\hat{F}$, containing $\overline{F}$, so that an affine weight $[M] = [M_0 M_1 \cdots M_r]$ is projected onto the weight $[M]\hat{F}^T$. For our purposes, we can assume that $\overline{j}$ is semi-simple, with f simple terms, $\overline{j} = \sum_{i=1}^{f} \overline{j}_i$, the terms having embedding indices σ_i. Then $\hat{F}$ will be a $(\tilde{r}+f) \times (r+1)$ matrix. The extra column is determined by the projection of the weight $[100\cdots 0] = \omega^0$ of $\hat{g}^1$. Clearly, ω^0 is projected onto $\sum_{i=1}^{f} \sigma_i \omega^0_{(i)}$, where $\omega^0_{(i)}$ is the 0-th fundamental weight of $\hat{j}_i$. The extra rows are easily determined by the values σ_i.

A special class of affine subalgebras is induced by $\overline{j}^{\sigma} \subset \overline{g}$ when $c(j) = c(g)$. These are the so-called conformal subalgebras and we denote them by $\hat{j}^{\sigma k} \triangleleft \hat{g}^k$. Now this situation is only possible for $k = 1$[9], so without loss of generality, we write $\hat{j}^{\sigma} \triangleleft \hat{g}^1$. The name conformal subalgebra is appropriate because the Sugawara stress tensors of $\hat{j}^{\sigma}$ and $\hat{g}^1$ are equivalent[2]. Complete lists of conformal subalgebras have been compiled[9].

Consider a conformal subalgebra $\hat{j}^k \triangleleft \hat{g}^1$. Suppose $[M]$ is a highest weight representation of $\hat{g}$ satisfying $k[M] = 1$, i.e., it is a level one representation. Then the branching rule for $[M]$ takes the form

$$[M] \rightarrow \sum_{k[m]=k} N_m \; [m] \tag{2.21}$$

where $0 \leq N_m \in Z$ and the sum is a finite one[6], over all highest weight representations of $\hat{j}^k$ satisfying $k[m] = k$. Furthermore, there is a branching rule for each level one representation of $\hat{g}$.

Since the Sugawara stress tensors for $\hat{g}^1$ and $\hat{j}^k$ are equivalent, so are their lowest moments L_0. Each state in $[M]$ must be represented by a state in one of the $[m]$ for which $N_m \neq 0$, and having the same eigenvalue of L_0. Every state in

$[M]$ has L_0-eigenvalue equal to $h[M]-c/24$ of (2.16) mod an integer; and similarly for $[m]$. Therefore every $[m]$ for which $N_m \neq 0$ in (2.21) must satisfy the level matching condition:

$$0 \leq h[m] - h[M] \in Z \quad . \tag{2.22}$$

Clearly, (2.21) implies

$$\chi[M] = \sum_{k[m]=k} N_m \chi[m] \quad . \tag{2.23}$$

By (2.21), as $\tau \to 0$, this yields the asymptotic constraint

$$\ell[M] = \sum_{k[m]=k} N_m \ell[m] \quad . \tag{2.24}$$

For many conformal subalgebras, asymptotics and level matching are sufficient to determine the positive integers N_m and therefore the branching rules (2.21)[5,12]. But there are many others for which this is not true.

Note that the outer automorphisms of $\hat{g}$ map level one representations into each other. One can hope to obtain from the branching rule of one representation of $\hat{g}^1$ into $\hat{j}^k$ the branching rule for another level one representation. This will be possible when there is an image of the outer automorphism in the outer automorphism of the subalgebra $\hat{j}$. In that case there exists a projection matrix $\hat{F}$ manifesting the relation between the algebra and subalgebra automorphism.

Pieces of each branching rule can be computed simply from finite Lie algebra theory. The states with lowest eigenvalue of L_0 in the level one representation $[M]$ of $\hat{g}^1$ fill out the representation (M) of $\overline{g}$. (M) branches into several representations

(m) of the subalgebra $\bar{j}$:

$$(M) \rightarrow \sum_m N_m \ (m) \tag{2.25}$$

These latter retain the lowest eigenvalue of L_0, so they must fill out the lowest L_0–level of highest weight representations of $\hat{j}$. The finite branching rules (2.25) provide part of the full affine branching rule (2.21).

The outer automorphisms may be sufficient to generate the full affine branching rule from the parts of them just mentioned. This is the case for the infinite series of conformal subalgebras $\hat{SU}(p) \times \hat{SU}(q) \triangleleft \hat{SU}(pq)$[8]. In the next section we show that this conformal subalgebra is not special, by calculating in the same manner the branching rules for $\hat{SU}(2)^N \times \hat{SO}(N)^4 \triangleleft \hat{Sp}(2N)^1$.

3. Branching Rules for $\hat{Sp}(2N) \triangleright \hat{SU}(2)^N \times \hat{SO}(N)^4$

The finite subalgebra

$$Sp(2N) \supset SU(2)^N \times SO(N)^4 \tag{3.1}$$

is defined by the branching rule

$$\varpi^1 \ \rightarrow \ (\varpi^1, \, \varpi^1) \, . \tag{3.2}$$

The first entry in the brackets is the first fundamental weight of $SU(2)$ and the second that of $SO(N)$. Equation (3.2) says that the fundamental 2N-dimensional representation of $Sp(2N)$ branches into the direct product of an $SU(2)$ doublet with a vector of $SO(N)$.

The $Sp(2N)$ representations at the lowest eigenvalues of L_0 in the highest weight representation ω^μ ($\mu = 0, 1, ...N$) of $\hat{S}p(2N)$ are the scalar and basic representations, $0, \varpi^i$ ($i = 1, ...N$). The branching rules for the latter into $SU(2)\times SO(N)$ are

$$\varpi^i \quad \rightarrow \quad \sum_{s=0}^{[\frac{i}{2}]} \left(\, (i-2s)\varpi^1, \; \overline{\nu}^s + \overline{\nu}^{i-s} \right). \tag{3.3}$$

Here the ϖ^1 on the right hand side is the fundamental weight of $SU(2)$, and the $\overline{\nu}^i$ are weights of $SO(N)$. The definitions of the $\overline{\nu}^i$ differ for N odd and N even. For $N = 2Q + 1$,

$$\begin{aligned} \overline{\nu}^j \; &= \; \varpi^j \qquad (1 \leq j \leq Q-1) \\ \overline{\nu}^Q \; &= \; 2\varpi^Q \qquad \overline{\nu}^0 \; = \; 0. \end{aligned} \tag{3.4}$$

The weights with indices larger than the rank Q in (3.3) are handled by duality:

$$\overline{\nu}^{2Q+1-j} \; = \; \overline{\nu}^j. \tag{3.5}$$

For $N = 2Q$, the definitions are

$$\begin{aligned} \overline{\nu}^j \; &= \; \varpi^j \qquad (1 \leq j \leq Q-2) \\ \overline{\nu}^0 \; &= \; 0 \qquad \overline{\nu}^{Q-1} \; = \; \varpi^Q + \varpi^{Q-1} \\ \overline{\nu}^Q \; &= 2\varpi^Q \oplus 2\varpi^{Q-1} \, . \end{aligned} \tag{3.6}$$

The symbol $\oplus$ indicates there are *two* separate representations in $\overline{\nu}^Q$. Weights with indices larger than Q are again handled by duality:

$$\overline{\nu}^{2Q-j} \; = \; \overline{\nu}^j, \;\; j > Q. \tag{3.7}$$

Note that (3.3) also gives correctly the branching rule for the scalar representation when $i = 0$. For the reader's convenience we sketch the derivation of the finite branching rules in an appendix.

The outer automorphisms of $\hat{Sp}(2N), \hat{SU}(2)$ and $\hat{SO}(N)$ can be explained with the help of Figure 1. $\hat{Sp}(2N) = \hat{C}_N$ has a Z_2 outer automorphism group generated by A:

$$\hat{C}_N : \quad A\omega^s = \omega^{N-s}, \ 0 \leq s \leq N. \tag{3.8}$$

The action of A is illustrated in Figure 1(a). $\hat{SU}(2) = \hat{A}_1$ also has a Z_2 outer automorphism group. Its generator a is depicted in Figure 1(b):

$$\hat{A}_1 : \quad a\omega^0 = \omega^1; \ a\omega^1 = \omega^0. \tag{3.9}$$

Since the root structure of $SO(N)$ differs for odd $N = 2Q+1$, and even $N = 2Q$, we will discuss them separately.

(a) $\hat{C}^1_{2Q+1} \triangleright \hat{A}_1^{2Q+1} \times \hat{B}^4_Q$

With $N = 2Q+1$, the subalgebra is that shown above. The outer automorphism group of $\hat{B}_Q$ is Z_2, generated by α:

$$\alpha\omega^0 = \omega^1 \ ; \ \alpha\omega^1 = \omega^0 \ ; \ \alpha\omega^s = \omega^s, \ s \neq 0, 1. \tag{3.10}$$

Figure 1(c) illustrates the action of α.

There is considerable freedom in the choice of affine projection matrix $\hat{F}$. One restriction is that it should contain an acceptable projection matrix $\overline{F}$, for the finite Lie algebra embedding. $\overline{F}$ is obtained from $\hat{F}$ by deleting the rows and columns associated with the 0th fundamental weights of the "large" algebra and all embedded algebras. For $\overline{F}$ to be satisfactory for (3.1), it is necessary and sufficient that it reproduce (3.2).

An acceptable projection matrix $\hat{F}$ is

$$\left[\begin{array}{cccccccccccccc}
N & N-1 & N-2 & & Q+2 & Q+1 & Q & Q-1 & & 2 & 1 & 0 \\
0 & 1 & 2 & \cdots & Q-1 & Q & Q+1 & Q+2 & \cdots & N-2 & N-1 & N \\
- & - & - & & - & - & - & - & & - & - & - \\
4 & 3 & 3 & & 3 & 2 & 2 & 3 & & 3 & 3 & 4 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 1 & & 0 & 0 & 0 & 0 & & 1 & 0 & 0 \\
 & & & \ddots & & & & & \iddots & & & \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & & 0 & 2 & 2 & 0 & & 0 & 0 & 0
\end{array}\right] . \tag{3.11}$$

This matrix makes manifest the relation

$$A \ = \ a \times 1. \tag{3.12}$$

The finite branching rules give us part of the full affine branching rules. Equation (3.3) implies

$$\omega^\mu \ \rightarrow \ \sum_{s=0}^{[\frac{\mu}{2}]} \ \left[\ (N-\mu+2s)\omega^0 + (\mu-2s)\omega^1, \nu^s + \nu^{\mu-s} \right] + \ldots \ . \tag{3.13}$$

where we define the weights ν^s such that $[\nu^\rho + \nu^\lambda]$ is always a level four representation of $\hat{B}_Q$:

$$\begin{aligned}
\nu^\mu \ &= \ \omega^\mu, \qquad 2 \le \mu \le Q-1 \\
\nu^Q \ &= \ 2\omega^Q \qquad \nu^1 \ = \ \omega^0 + \omega^1
\end{aligned} \tag{3.14}$$

and also $\nu^0 \ = \ 2\omega^0$. Weights with negative indices in (3.13) are defined by

$$\nu^\mu \ = \ \nu^{-\mu}. \tag{3.15}$$

Equation (3.13) also implies

$$\omega^{N-\mu} \;\rightarrow\; \sum_{s=0}^{Q-[\frac{\mu}{2}]} \left[(\mu+2s)\omega^0 + (N-\mu-2s)\omega^1,\ \nu^s - \nu^{N-\mu-s}\right] + \dots \quad (3.16)$$

Using duality

$$\nu^{N-\mu-s} \;=\; \nu^{\mu+s} \quad (3.17)$$

and flipping the sign of s we get

$$\omega^{N-\mu} \;\rightarrow\; \sum_{s=[\frac{\mu}{2}]-Q}^{0} \left[\,(\mu-2s)\omega^0 + (N-\mu+2s)\omega^1,\ \nu^s + \nu^{\mu-s}\,\right] + \dots\,.$$

Now applying (3.12) to this last equation and adding the result to (3.13) yields

$$\omega^{\mu} \;\rightarrow\; \sum_{s=[\frac{\mu}{2}]-Q}^{[\frac{\mu}{2}]} \left[\,(N-\mu+2s)\omega^0 + (\mu-2s)\omega^1,\ \nu^s + \nu^{\mu-s}\,\right] + \dots\,. \quad (3.18)$$

To see whether or not (3.18) is the complete affine branching rule one can check numerically to see if the asymptotic sum rule (2.24) is satisfied. We find that it is not; there are representations missing from (3.18). But we find the unique way to satisfy the asymptotic sum rule by adding representations obeying the level matching condition (2.22) is to modify the definition of ν^0 to

$$\nu^0 \;=\; 2\omega^0 \oplus 2\omega^1. \quad (3.19)$$

Again, the symbol $\oplus$ indicates ν^0 consists of *two* representations. Then the following are the complete branching rules for the conformal embedding $\hat{C}_{2Q+1} \triangleright \hat{A}_1^{2Q+1} \times \hat{B}_Q^4$:

$$\omega^{\mu} \;\rightarrow\; \sum_{s=[\frac{\mu}{2}]-Q}^{[\frac{\mu}{2}]} \left[\,(N-\mu+2s)\omega^0 + (\mu-2s)\omega^1,\ \nu^s + \nu^{\mu-s}\,\right] \quad (3.20)$$

with the definitions (3.14) and (3.19).

To make the procedure perfectly clear, let us go through a specific example: the branching rule of $\omega^2 = [0010000000]$ of $\hat{C}_9$ into $\hat{A}_1^9 \times \hat{B}_4^4$. (3.3) tells us the branching rules for the representations ϖ^2, ϖ^7 of the finite Lie algebra C_9 into representations of $A_1 \times B_4$:

$$\begin{aligned} \varpi^2 &\rightarrow (2-0100)+(0-2000) \\ \varpi^7 &\rightarrow (7-0100)+(5-1010)+(3-0102)+(1-0012). \end{aligned}$$

Each of these tells us part of the corresponding affine branching rule:

$$\begin{aligned} \omega^2 &\rightarrow [72-20100]+[90-22000]+\dots \\ \omega^7 &\rightarrow [27-20100]+[45-11010]+[63-00102]+[81-00012]+\dots\ . \end{aligned}$$

Applying the automorphism (3.12) to the second equation and combining the result with the first gives

$$\omega^2 \rightarrow [90-22000]+[72-20100]+[54-11010]+[36-00102]+[18-00012]+\dots\ .$$

The unique way to satisfy (2.24) by adding representations obeying (2.22) is to include $[72-02100]$. Then the result is that dictated by (3.20).

(b) $\hat{C}_{2Q} \triangleright \hat{A}_1^{2Q} \times \hat{D}_Q^4$

For even $N = 2Q$ in $\hat{Sp}(2N) \triangleright \hat{SU}(2)^N \times \hat{SO}(N)^4$, the embedding is $\hat{C}_{2Q} \triangleright \hat{A}_1^{2Q} \times \hat{D}_Q^4$. The outer automorphisms of $\hat{D}_Q$ differ for Q odd and even, and can be explained using the Dynkin diagram symmetries of Figure 1(d). The symmetries e_b, e_f and μ are defined by

$$\begin{aligned} e_b\omega^0 &= \omega^1 \ ; \ e_b\omega^1 = \omega^0 \ ; \ e_b\omega^\lambda = \omega^\lambda, \ \lambda \neq 0,1 \\ e_f\omega^{Q-1} &= \omega^Q \ ; \ e_f\omega^Q = \omega^{Q-1} \ ; \ e_f\omega^\lambda = \omega^\lambda, \ \lambda \neq Q-1, Q \\ \mu\omega^\lambda &= \omega^{Q-\lambda}, \quad 0 \leq \lambda \leq Q. \end{aligned} \tag{3.21}$$

For Q odd there is an outer automorphism

$$\mu' \;=\; \mu e_f \;=\; e_b \mu \tag{3.22}$$

of period four. For Q even the outer automorphism group is $Z_2 \times Z_2$, with generators μ and

$$\sigma \;=\; e_b e_f \tag{3.23}$$

Notice σ is also an outer automorphism for odd Q, since $(\mu')^2 = \sigma$.

We have found three different affine projection matrices manifesting the outer automorphisms of the embedding $\hat{A}_1^{2Q} \times \hat{D}_Q^4 \triangleleft \hat{C}_{2Q}^1$. One is

$$\hat{F} = \begin{bmatrix} N & N{-}1 & N{-}2 & & Q{+}1 & Q & Q{-}1 & & 2 & 1 & 0 \\ 0 & 1 & 2 & \cdots & Q{-}1 & Q & Q{+}1 & \cdots & N{-}2 & N{-}1 & N \\ - & - & - & & - & - & - & & - & - & - \\ 4 & 3 & 2 & & 2 & 2 & 2 & & 2 & 3 & 4 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 & & 1 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & & \iddots & & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 2 & 0 & & 0 & 0 & 0 \end{bmatrix} . \tag{3.24}$$

This matrix shows the branching rules obey (3.12), as in the case when N is odd, discussed above. A second matrix is

$$
\hat{F} = \left[\begin{array}{ccccccccccc}
N & N-1 & N & N-1 & N & & N & N-1 & N & N-1 & N \\
0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\
- & - & - & - & - & & - & - & - & - & - \\
4 & 3 & 2 & 1 & 0 & & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & & 0 & 0 & 0 & 0 & 0 \\
 & & \vdots & & & \ddots & & & \vdots & & \\
0 & 0 & 0 & 0 & 0 & & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & & 0 & 1 & 2 & 3 & 4
\end{array}\right] . \tag{3.25}
$$

For Q odd, this manifests

$$A \; = \; 1 \times \mu' \tag{3.26}$$

while for Q even it realizes

$$A \; = \; 1 \times \mu . \tag{3.27}$$

The third matrix is obtained by interchanging the last two rows of (3.25). So we also have

$$A \; = \; 1 \times \sigma\mu' \tag{3.28}$$

for Q odd and

$$A \; = \; 1 \times \sigma\mu \tag{3.29}$$

for Q even, when acting on the affine branching rules.

As in section (a), we have $A = a \times 1$. Also, the finite branching rules (3.3) take the same form for N odd or even. So we arrive immediately at the result (3.18).

However, the $\overline{\nu}^i$ of (3.3) are defined differently for $N = 2Q$ (compare (3.6) and (3.4)), so the corresponding ν^μ must also be defined differently.

In fact, comparing (3.26) with (3.28) for Q odd and (3.27) with (3.29) for Q even tells us that the affine branching rules must be invariant under σ. This in turn means that the affine weights ν^μ in (3.18) must be redefined to be σ invariant. The appropriate ν^μ are therefore:

$$\begin{aligned} \nu^\mu &= \omega^\mu, \quad 2 \leq \mu \leq Q-2 \\ \nu^0 &= 2\omega^0 \oplus 2\omega^1, \quad \nu^1 = \omega^0 + \omega^1 \\ \nu^{Q-1} &= \omega^{Q-1} + \omega^Q, \quad \nu^Q = 2\omega^{Q-1} \oplus 2\omega^Q. \end{aligned} \tag{3.30}$$

We have verified numerically (for a large number of cases) that the asymptotic sum rule (2.24) is also satisfied by the branching rule (3.20) for N even. Again, weights with negative indices are defined by $\nu^\mu = \nu^{-\mu}$, and the ν^μ satisfy the duality relation (3.17).

4. Conclusion

For ease of reference, let us first state that the branching rules for the conformal embedding $\hat{SU}(2)^N \times \hat{SO}(N)^4 \triangleleft \hat{Sp}(2N)$ are

$$\omega^\mu \rightarrow \sum_{s=[\frac{\mu}{2}]-Q}^{[\frac{\mu}{2}]} \left[\ (N-\mu+2s)\omega^0 + (\mu-2s)\omega^1,\ \nu^s + \nu^{\mu-s}\ \right].$$

The $\hat{SO}(N)$ weights ν^μ are defined differently for N odd or even; for $N = 2Q+1$ the definitions are given in (3.14) and (3.19), and for $N = 2Q$ they are those written in (3.30). Weights ν^μ with indices μ too small or too large are to be understood using $\nu^{-\mu} = \nu^\mu$ and $\nu^\mu = \nu^{N-\mu}$, respectively.

Our results demonstrate the general utility of outer automorphisms in the calculation of conformal branching rules.

They also complete the calculation of the conformal branching rules for infinite series of nonsimple higher level embeddings. There are four such infinite series of conformal subalgebras[9]. $\hat{SU}(2)^N \times \hat{SO}(N)^4 \triangleleft \hat{Sp}(2N)$ was treated here and $\hat{SU}(p)^q \times \hat{SU}(q)^p \triangleleft \hat{SU}(pq)$ in reference 8. The remaining two are $\hat{SO}(p)^q \times \hat{SO}(q)^p \triangleleft \hat{SO}(pq)$ and $\hat{Sp}(2p)^q \times \hat{Sp}(2q)^p \triangleleft \hat{SO}(4pq)$. But their branching rules may be calculated using a theorem[13] applicable to conformal embeddings $\hat{j} \triangleleft \hat{SO}(D)$, when there exists a symmetric space $\overline{g}/\overline{j}$ of dimension D.

We believe the use of Kač–Moody outer automorphisms will allow the calculation of all conformal branching rules.

Acknowledgements:

This research was supported in part by a postdoctoral fellowship from NSERC of Canada and by the U.S. Department of Energy, under contract DE–AC03–76SF00515.

APPENDIX

In this appendix we derive formula (3.3) for the branching of the basic representation of $Sp(2N)$ into $SU(2) \times SO(N)$ representations.

$Sp(2N)$ is the group of transformations of 2N-dimensional vectors that leaves invariant the antisymmetric tensor T with nonzero comonents $T_{M,M+1} = 1 = -T_{M+1,M}$. The basic representations $\overline{\omega}^i$ can be represented by traceless antisymmetric tensors of rank i, the traces taken using the invariant 2-tensor T. As such, they can be represented by Young tableaux consisting of one column of i boxes.

Representations of $SU(2)$ can also be symbolized by Young tableaux. A row of m boxes realizes the representation (m). Since $SU(2)$ leaves invariant the ϵ tensor of rank 2, a column of two boxes is equivalent by duality to a scalar (0). A column of more than two boxes is impossible and must be excluded.

Totally antisymmetric tensors again correspond to irreducible representations of $SO(N)$. The representations are not the basic ones, however. In fact, an antisymmetric tensor of rank i transforms as the representation with highest weight $\overline{\nu}^i$ of (3.4) or (3.6), according to whether N is odd or even.

Since $SO(N)$ transformations preserve determinants, they leave invariant the ϵ tensor of rank N. So the concept of dual tensors applies to $SO(N)$ as well. This is the origin of the relations (3.5) and (3.7). In the language of Young tableaux, it says that a column of j boxes is dual to a column of $N-j$ boxes.

The embedding $Sp(2N) \supset SU(2) \times SO(N)$ is defined by the branching rule

$$\overline{\omega}^1 \rightarrow (\overline{\omega}^1, \overline{\omega}^1).$$

This is depicted in Figure 2(a), and allows us to use Young tableaux to find the branching rules for (3.3).

Consider a basic representation $\overline{\omega}^i$ of $Sp(2N)$. The i boxes of the column that is the corresonding Young tableau break up into $2i$ boxes, i for $SU(2)$ and the other i for $SO(N)$. Each of the two sets of i boxes can form a Young tableau of any type, but the antisymmetry of the original rank i tensor must be respected. This is done by pairing $SU(2)$ and $SO(N)$ representations such that their Young tableaux can be obtained from each other by interchanging rows and columns.

As an example, the branching of the fourth basic representation $\overline{\omega}^4$ of $Sp(2N)$ is shown in Figure 2(b). If the first Young tableau in each pair on the right hand side is that of $SU(2)$, the first two pairs are superfluous, since the tableaux do not correspond to $SU(2)$ representations. It is easy to convince oneself that this procedure for any basic representation leads to (3.3).

For completeness we mention that Young tableaux can be used to find the branchings of other $Sp(2N)$ representations, but some care is needed. Consider the adjoint representation. It can be represented by a symmetric rank 2 tensor. Because of its symmetry, its branching is represented by identical Young tableaux for $SU(2)$ and $SO(N)$, as shown in Figure 2(c). But the original $Sp(2N)$ tensor is not traceless, since there is no $Sp(2N)$ invariant symmetric tensor with which to contract. Therefore, the row of two $SO(N)$ boxes on the right hand side of Figure 2(c) represents two representations; $2\varpi^1$ and a scalar that is essentially the trace of the $SO(N)$ tensor. The adjoint branching rule is therefore

$$2\varpi^1 \rightarrow (2\varpi^1, 2\varpi^1) + (2\varpi^1, 0) + (0, \varpi^2).$$

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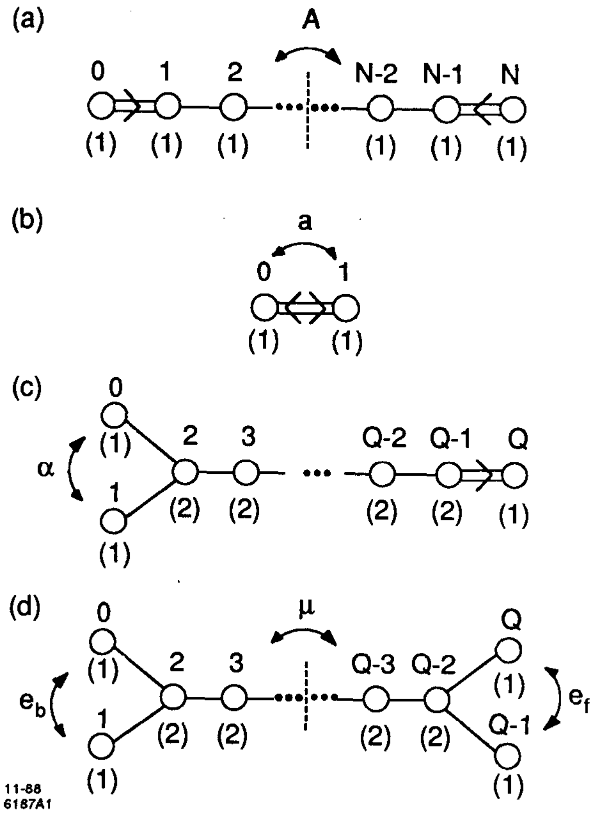
FIGURE CAPTIONS

1) The Dynkin diagrams of (a) $\hat{Sp}(2N) = \hat{C}_N$, (b) $\hat{SU}(2) = \hat{A}_1$, (c) $\hat{SO}(2Q+1) = \hat{B}_Q$, and (d) $\hat{SO}(2Q) = \hat{D}_Q$ are shown. The nodes are labelled above and the corresponding co-mark $k^{\vee\mu}$ is written below in brackets. Depicted are symmetries of the Dynkin diagrams. As explained in the text, these are either outer automorphisms themselves or can be used to define them.

2) $Sp(2N) \supset SU(2) \times SO(N)$ branching rules using Young tableaux. (a) Depicted is the defining branching rule of the fundamental representation $\overline{\omega}^1 \to (\overline{\omega}^1, \overline{\omega}^1)$. Branchings of the fourth basic representation and the adjoint representation of $Sp(2N)$ are shown in (b) and (c), respectively.


(a)
A
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1
2
N-2
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a
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1
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(c)
0
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α
1
(1)
2
3
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Q-2
Q-1
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Fig. 1

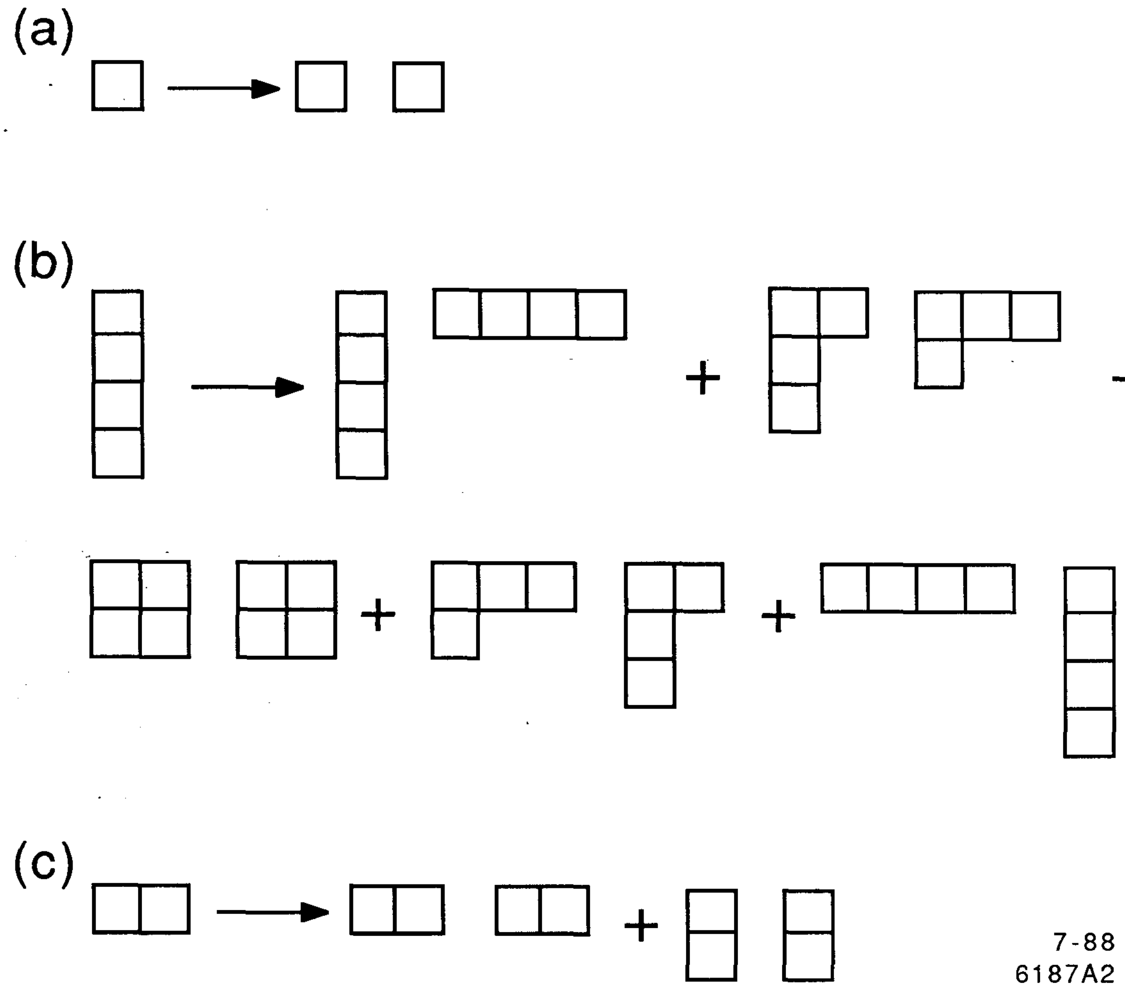

Fig. 2