SLAC–PUB–4680
August 1988
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CONFORMAL BRANCHING RULES AND MODULAR INVARIANTS

MARK A. WALTON*

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309, USA

ABSTRACT

Using the outer automorphisms of the affine algebra $\hat{SU}(n)$, we show how the branching rules for the conformal subalgebra $\hat{\mathrm{SU}}(pq) \supset \hat{\mathrm{SU}}(p) \times \hat{\mathrm{SU}}(q)$ may be simply calculated. Modular-invariant combinations of $\hat{\mathrm{SU}}(n)$ characters are obtained from the branching rules.

Submitted to Nuclear Physics B [FS]

⋆ The author's research was supported by a postdoctoral fellowship from NSERC of Canada, and by the U.S. Department of Energy under contract DE–AC03– 76SF00515.

0. Introduction

Two-dimensional conformal field theories describe critical points of second order phase transitions and classical vacua of strings. Consequently, the classification of such conformal field theories is an important open problem.

A tractable subclass is the set of rational conformal field theories [1]. Included are the Wess-Zumino-Witten (WZW) models [2], conformal field theories constructed from them by the coset construction [3], and orbifolds of them (see, for example refs. [4] and [5]). WZW models are of central importance in the study of rational conformal field theories.

The symmetry algebras of WZW models are of the affine Kač-Moody type (for reviews, see refs. [6] and [7]). These algebras and their representations are the subject of this paper.

We will show that the branching rules for the affine subalgebra

$$\hat{\mathrm{SU}}(pq) \supset \hat{\mathrm{SU}}(p) \times \hat{\mathrm{SU}}(q) \tag{0.1}$$

can be easily obtained from some simple branching rules for the finite Lie subalgebra

$$\mathrm{SU}(pq) \supset \mathrm{SU}(p) \times \mathrm{SU}(q) \ . \tag{0.2}$$

As far as we know, the branching rules for (0.1) have not been calculated previously.

Modular invariance is a powerful restriction on the spectra of conformal field theories. Besides being intrinsically interesting, the branching rules for (0.1) allow the construction of various modular-invariant combinations of $\hat{\mathrm{SU}}(n)$ characters in a simple way. The modular invariants are possible partition functions for systems with $\hat{\mathrm{SU}}(n)$ symmetry. The classification of such modular invariants has only been carried out for the $\hat{\mathrm{SU}}(2)$ case [8]. Our results aid in generalizing that classification.

The format of this paper is as follows: in the second section we review affine Kač-Moody algebras mainly to establish notation; the third section describes the method of calculation of the branching rules, the fourth illustrates the method with two examples, and the final section is a short conclusion.

1. Review and Notation

Let $\hat{g}$ denote the affine Kač-Moody algebra that is the central extension of the loop algebra of the finite–dimensional Lie algebra $\bar{g}$.* The algebra $\hat{g}$ is

$$[J^a_m,\ J^b_n] \ = \ f^{abc}\ J^c_{m+n} \ + \ k\delta^{ab}\delta_{m+n,o}\ . \tag{1.1}$$

We will sometimes include the value of $k \in Z$ by writing $\hat{g} = \hat{g}^k$. Setting the integral indices m and n in (1.1) to zero reduces this algebra to the finite one $\bar{g} \subset \hat{g}$.

The Sugawara construction

$$L_m = \frac{1}{2(k+h^\vee)} \sum_{n\in Z} K_{ab}\ :\ J^a_{m+n}\ J^b_{-n}\ :\ -\delta_{mo}\frac{c(g)}{24}\ , \tag{1.2}$$

* In general we use the convention that carets denote objects associated with affine algebras and bars their finite algebra counterparts.

associates with $\hat{g}$ the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c(g)}{12}(m-1)m(m+1)\,\delta_{m+n,o}\ , \tag{1.3}$$

with central charge

$$c(g) = \frac{k \cdot dim\bar{g}}{k+h^\vee}\ . \tag{1.4}$$

Here $K_{ab}, h^\vee$ are the Killing form and dual Coxeter number of $\bar{g}$, and the normal ordering is defined in the usual way.

The Cartan subalgebra $\hat{h}$ of $\hat{g}$ contains the Cartan subalgebra $\bar{h}$ of $\bar{g}$, with elements $h_i(i = 1,\ldots,r)$ plus the extra element h_0. We denote the elements of $\hat{h}$ by $h_\mu(\mu = 0,1,\ldots,r)$. Dual to these elements, living in the weight space $\hat{h}^*$, are the fundamental weights ω^μ:

$$\omega^\mu(h_\nu) = \delta^\mu_\nu\ . \tag{1.5}$$

Associated with each h_μ is a co-root $\alpha^\vee_\mu$, also living in the weight space $\hat{h}^*$, so that we have

$$\omega^\mu \cdot \alpha^\vee_\nu = \omega^\mu(h_\nu) = \delta^\mu_\nu\ . \tag{1.6}$$

The dot product is determined from the Cartan matrix A by the definition of its elements:

$$A_{\mu\nu} = \alpha_\mu \cdot \alpha^\vee_\nu\ , \tag{1.7}$$

where a root α_μ and its co-root $\alpha^\vee_\mu$ are multiples of each other:

$$\alpha^\vee_\mu = \frac{2\alpha_\mu}{\alpha_\mu \cdot \alpha_\mu}\ .$$

There is an extra operator which commutes with the h_μ; it is L_0 of (1.2).

The Cartan subalgebra can be extended to $\hat{h}^e$ having elements h_μ and $d = -L_0$. We denote the weight dual to d by δ

$$\delta(d) = \delta(-L_0) = 1 , \tag{1.8}$$

and the co-root corresponding to d by Λ_0

$$\delta \cdot \Lambda_0 = \delta(d) = 1 . \tag{1.9}$$

With the usual conventions

$$\Lambda_0(h_i) = \delta_{io} \quad i = 1, \ldots, r ,$$

$$\Lambda_0(d) = \Lambda_0 \, (-L_0) = \Lambda_0 \cdot \Lambda_0 = 0 ,$$

the scalar product on the extended weight space $\hat{h}^{*e}$ is Minkowskian. Let

$$\bar{\psi} = k^i \bar{\alpha}_i = k^{\vee i} \, \bar{\alpha}_i^\vee , \tag{1.10}$$

be the highest root of $\bar{g}$. The $k^\mu, k^{\vee\mu}$ are known as marks and co-marks, respectively with $k^0 \equiv k^{0\vee} \equiv 1$. Then if $B, C \in \hat{h}^{*e}$, we write

$$\begin{aligned} B &= \beta_\mu \omega^\mu + \beta_\delta \delta = (\beta_i \omega^i = \overline{B}, k[B], \beta_\delta) \\ C &= \gamma_\mu \omega^\mu + \gamma_\delta \delta = (\gamma_i \omega^i = \overline{C}, k[C], \gamma_\delta) \end{aligned} \tag{1.11}$$

where $k[B], k[C]$ are the levels of B and C, respectively,

$$k[B] = \beta_\mu k^{\vee\mu} \qquad k[C] = \gamma_\mu k^{\vee\mu} . \tag{1.12}$$

Then the dot product on the expanded weight space $\hat{h}^{*e}$ takes the form

$$B \cdot C = \overline{B} \cdot \overline{C} + k[B]\gamma_\delta + \beta_\delta k[C] . \tag{1.13}$$

With the above notation, the simple roots and fundamental weights of $\hat{g}$ can be written

$$\alpha_i = (\bar{\alpha}_i, 0, 0) \qquad \alpha_0 = (-\bar{\psi}, 0, 1)$$
$$\omega_i = (\bar{\omega}_i, k^{\vee i}, 0) \quad \omega^0 = (0, 1, 0) \ .$$

So the Dynkin diagram of $\hat{g}$ is the extended Dynkin diagram of $\bar{g}$. The Dynkin diagram of $\hat{\mathrm{SU}}(r+1)$ is shown in Fig. 1. Outer automorphisms of $\hat{g}$ are manifested as automorphisms of the Dynkin diagram of $\hat{g}$. From Fig. 1 we see that $\hat{\mathrm{SU}}(r+1)$ has such an automorphism generated by $A\omega^{\mu} = \omega^{\mu+1}$. Note that this is not an automorphism of the extended algebra. Specifically, it doesn't commute with $d = -L_0$. Since L_0 is physically an energy operator, the usefulness of this automorphism is not obvious, but non-zero as we will see.

Of physical interest are the highest weight representations of $\hat{g}$. They are generated from a "vacuum" state $|M>$, labeled by a dominant weight

$$M = M_{\mu}\omega^{\mu} = (\bar{M}, M_{\mu}k^{\vee\mu}, 0) \ . \tag{1.14}$$

The "vacuum" satisfies

$$J_n^a|M> = 0 \ , \ n > 0$$
$$J_0^a|M> = T_{\bar{M}}^a|M> \quad ,$$

where $T_{\bar{M}}^a$ are the matrices representing the generators of $\bar{g}$ in the finite dimensional representation labeled by $\bar{M}$. We will use the notation $M = [M_0M_1\ldots M_r] \equiv [M]$ to denote the weight M and corresponding highest weight representation. Similarly, representations of $\bar{g}$ will be denoted by $\bar{M} = (M_1M_2\ldots M_r) = (M)$. For unitary highest weight representations $[M]$ of (1.1), we must require

$$k[M] \;=\; M_\mu k^{\vee\mu} \;=\; k \; . \tag{1.15}$$

The states in $[M]$ having a fixed eigenvalue of L_0 fall into a finite sum of irreducible representations of $\bar{g}$. For example, the lowest value of L_0 is

$$L_0 \;=\; h[M] - \frac{c(g)}{24} \;=\; \frac{\bar{M}\cdot(\bar{M}+2\bar{\rho})}{2(k+h^\vee)} \;-\; \frac{c(g)}{24} \; , \tag{1.16}$$

where $\bar{\rho}$ is the half-sum of positive roots of $\bar{g}$. The states of $[M]$ with this value of L_0 fill out the representation $\bar{M} = (M)$.

The character of a highest weight representation $M = [M]$ is defined by

$$ch_M(\tau, z) \;=\; tr_{[M]} e^{2\pi i(\tau L_0 + z\cdot \bar{h})} \; . \tag{1.17}$$

The characters at $z = 0$

$$\chi[M] \;=\; \chi[M_0 M_1, \ldots M_r] \;=\; ch_M(\tau, 0) \; , \tag{1.18}$$

are Virasoro characters, since they involve only L_0. The χ appear in the partition function

$$\begin{aligned} Z(\tau) &= tr \; e^{2\pi i(\tau L_0 - \bar{\tau}\bar{L}_0)} \\ &= \sum_{[M],[N]} Z_{MN} \; \chi[M]\chi[N]^* \; . \end{aligned} \tag{1.19}$$

For a physically sensible partition function, $Z_{MN} \in Z^+$, and for $[M] = [10\ldots0], Z_{MM} = 1$. It is a nontrivial and in general unsolved problem to determine all possible modular-invariant matrices Z for a given group $\bar{g}$ and level k.

The modular transformation properties of the characters was found in Ref. [9]. Transforming by $S : \tau \to -1/\tau$ reveals the asymptotic behavior of the characters:

$$\text{as } \tau \to 0, \quad \chi[M] \sim \ell[M] e^{2\pi i c(g)/24\tau}. \tag{1.20}$$

The $\ell[M]$ can be calculated entirely from objects relevant to the finite algebra $\bar{g}$:

$$\ell[M] = \left| \frac{\bar{P}}{\bar{Q}} \right|^{1/2} [k[M] + h^\vee]^{-r} \prod_{\alpha \epsilon \bar{\Delta}_+} 2 \sin \left[\frac{\pi(\overline{M} + \bar{\rho}) \cdot \overline{M}}{k[M] + h^\vee} \right]. \tag{1.21}$$

Here $\bar{\Delta}_+$ denotes the set of positive roots, $\overline{M} = (M_1 \cdots M_r)$, and $\overline{P}$, $\overline{Q}$ are, respectively, the weight and root lattices of $\bar{g}$.

Knowledge of affine subalgebras of affine algebras is nowhere near as extensive as that of finite subalgebras of finite algebras. However, each finite subalgebra $\bar{j}$ of a finite algebra $\bar{g}$ induces an affine subalgebra $\hat{j}$ of $\hat{g}$ (see, for example, Ref. [10]). One identifying feature of a finite subalgebra $\bar{j} \subset \bar{g}$ is the index of embedding σ, equal to the ratio of the length squared of the highest root of $\bar{g}$ to that of $\bar{j}$ embedded in $\bar{g}$. The affine subalgebra induced by $\bar{j}^\sigma \subset \bar{g}$ (obvious notation) is $\hat{j}^{\sigma k} \subset \hat{g}^k$.

All other information concerning a finite subalgebra $\bar{j}^\sigma \subset \bar{g}$ is contained in the so-called projection matrix $\overline{F}$ [11]. $\overline{F}$ is a (rank $\bar{j} = \tilde{r}$) $\times$ (rank $\bar{g} = r$) matrix relating weights of $\bar{g}$ to the weights of $\bar{j}$ onto which they are projected. If $(M) = (M_1 M_2 \cdots M_r)$ is a weight of $\bar{g}$, it is projected onto the weight $(M)\overline{F}^T$.

We can define an affine projection matrix $\hat{F}$, containing $\overline{F}$, so that an affine weight $[M] = [M_0 M_1 \cdots M_r]$ is projected onto the weight $[M]\hat{F}^T$. For our purposes, we can assume that $\bar{j}$ is semi-simple, with f simple terms, $\bar{j} = \sum_{i=1}^{f} \bar{j}_i$, the terms having embedding indices σ_i. Then $\hat{F}$ will be a $(\tilde{r}+f)\times(r+1)$ matrix. The extra column is determined by the projection of the weight $[100\cdots 0]=\omega^0$ of $\hat{g}^1$. Clearly, ω^0 is projected onto $\sum_{i=1}^{f}\sigma_i\omega^0_{(i)}$, where $\omega^0_{(i)}$ is the 0-th fundamental weight of $\hat{j}_i$. The extra rows are easily determined by the values σ_i.

A special class of affine subalgebras is induced by $\bar{j}^\sigma\subset\bar{g}$ when $c(j)=c(g)$. These are the so-called conformal subalgebras and we denote them by $\hat{j}^{\sigma k}\lhd\hat{g}^k$. Now this situation is only possible for $k=1$ [12], so without loss of generality, we write $\hat{j}^\sigma\lhd\hat{g}^1$.

The name conformal subalgebra is appropriate because the Sugawara stress tensors of $\hat{j}^\sigma$ and $\hat{g}^1$ are equivalent [3,7]. Hence the Virasoro characters $ch(\tau,z=0)$ of $\hat{g}$ are finitely reducible in terms of the Virasoro characters of $\hat{j}$, and the relation extends to the full characters $ch(\tau,z\neq 0)$. Conversely, finite reducibility is possible only if the central charge of the embedded algebra equals that of the larger algebra.

Complete lists of conformal subalgebras have been compiled [12], and they include

$$\mathrm{S\hat{U}}(pq)\rhd\mathrm{S\hat{U}}(p)^q\times\mathrm{S\hat{U}}(q)^p \tag{1.22}$$

In the next section, we show that the branching rules for this conformal subalgebra can be easily computed from the well-known branching rules for the finite algebra counterparts $\mathrm{SU}(pq)\supset\mathrm{SU}(p)\times\mathrm{SU}(q)$.

2. Calculation of the Branching Rules

Consider a conformal subalgebra $\hat{j}^k \triangleleft \hat{g}^1$. Suppose $[M]$ is a highest weight representation of $\hat{g}$ satisfying $k[M] = 1$, i.e., it is a basic highest weight representation of $\hat{g}$. Then the branching rule for $[M]$ takes the form

$$[M] \rightarrow \sum_{k[m]=k} N_m \, [m] \tag{2.1}$$

where $N_m \epsilon Z^+$ and the sum is a finite one, over all highest weight representations of $\hat{j}^k$ satisfying $k[m] = k$. Furthermore, there is a branching rule for each level one representation of $\hat{g}$.

Since the Sugawara stress tensors for $\hat{g}^1$ and $\hat{j}^k$ are equivalent, so are their lowest moments L_0. Each state in $[M]$ must be represented by a state in one of the $[m]$ for which $N_m \neq 0$, and having the same eigenvalue of L_0. Every state in $[M]$ has L_0-eigenvalue equal to $h[M] - c/24$ of (1.16) mod an integer; and similarly for $[m]$. Therefore every $[m]$ for which $N_m \neq 0$ in (2.1) must satisfy the level-matching condition [9]:

$$0 \leq h[m] - h[M] \; \epsilon \; Z \quad . \tag{2.2}$$

Clearly, (2.1) implies

$$\chi[M] = \sum_{k[m]=k} N_m \chi[m] \quad . \tag{2.3}$$

By (1.20), as $\tau \rightarrow 0$, this yields the asymptotic constraint

$$\ell[M] = \sum_{k[m]=k} N_m \ell[m] \quad . \tag{2.4}$$

For many conformal subalgebras, asymptotics and level-matching are sufficient to determine the positive integers N_m and therefore the branching rules (2.1) [13,14]. But this is not true for (1.22).

Let V^B, V^b, V^β denote vectors in SU(pq), SU(p) and SU(q), respectively. Then the subgroup SU(p)$\times$ SU(q) $\subset$ SU(pq) is defined by the branching rule

$$V^B = V^b V^\beta \tag{2.5}$$

The projection matrix for the corresponding affine subalgebra can be taken to be

$$\hat{F} = \begin{bmatrix} q & q-1 & q-1 & & q-1 & q & q-1 & q-1 & \\ 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & & 0 & 0 & 0 & 1 & \\ & \vdots & & \ddots & & \vdots & & & \ddots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots \\ - & - & - & & - & - & - & - & \\ p & p-1 & p-2 & & 1 & 0 & 0 & 0 & \\ 0 & 1 & 2 & \cdots & p-1 & p & p-1 & p-2 & \cdots \\ 0 & 0 & 0 & & 0 & 0 & 1 & 2 & \\ 0 & 0 & 0 & & 0 & 0 & 0 & 0 & \\ & \vdots & & & & \vdots & & & \end{bmatrix} \tag{2.6}$$

or a similar matrix, with the roles of SU(p) and SU(q) interchanged.

Now consider the outer automorphism group of $\hat{\mathrm{SU}}(pq)$, generated by $A\omega^\mu = \omega^{\mu+1}$ (here $\omega^\mu = \omega^\nu$ if $\mu = \nu$ mod pq). Clearly, (2.6) reveals there is an image of A^p in the outer automorphism group of $\hat{\mathrm{SU}}(p) \times \hat{\mathrm{SU}}(q)$. If we let a and α be the outer-automorphism-generators for $\hat{\mathrm{SU}}(p)$ and $\hat{\mathrm{SU}}(q)$, respectively, then we have

$$A^p = \alpha \quad . \tag{2.7}$$

Reversing the roles of p and q in (2.6) yields

$$A^q = a \quad . \tag{2.8}$$

If we let these relations act on the pq branching rules of (1.22), we can group them into either p sets of q each that are related by (2.7), or q sets of p each that are related by (2.8).

Suppose we have p sets each containing q related branching rules. We can obtain an entire branching rule, and therefore all q of them, by computing pieces of each of the q branching rules. There are pieces of each of these branching rules that can be computed simply from finite Lie group theory. The states with lowest eigenvalue of L_0 in the level one representations of $\hat{\mathrm{SU}}(pq)$ are the scalar and basic representations of SU(pq). The branching rules of the latter into representations of SU$(p)\times$ SU(q) are easily computed and that for the former is trivial. These can then be translated into part of the appropriate affine branching rule. We repeat this procedure for all q branching rules and use (2.7). The sole remaining question is do we obtain the full branching rule? The answer is yes, as can be verified by the sum rule of (2.4). Remarkably, a few simple finite branching rules suffice to determine the affine ones.

The finite branching rules are easily computed using Young tableaux. The ℓ-th basic representations $\bar{\omega}^\ell$ of SU(pq) can be represented by a tensor with ℓ antisymmetrized indices and therefore by a Young tableau with ℓ boxes for SU(p) and also one of ℓ boxes for SU(q). The SU(p) and SU(q) Young tableaux must be paired to respect the antisymmetry of the original tensor.

As an example, consider the branching of the fourth basic representation $\bar{\omega}^4$ of SU(pq), illustrated in Fig. 2. Note that all Young tableaux made of four boxes are involved, and the SU(q) Young tableaux are obtained from their SU(p) partners by interchanging rows and columns. Furthermore, Fig. 2 gives the $\bar{\omega}^4$ branching rule for all p and q.

The computation would be even simpler if A itself had an image in the outer automorphism groups of $\hat{\mathrm{SU}}(p)$ and $\hat{\mathrm{SU}}(q)$. If this were true, we could write

$$A = a^u \alpha^v \tag{2.9}$$

for some u, v. However, if p is a multiple of q, (or vice-versa) then $A^{Nq} = A^p$ ($A^{Np} = A^q$) for some N. Equations (2.7) and (2.8) then yield $a^N = \alpha (\alpha^N = a)$ which is impossible.

We can, however, express A as in (2.9) when p or q is not a multiple of the other. By substituting (2.9) into (2.7) and (2.8), we obtain

$$\begin{aligned} pv &= 1 \bmod q \\ qu &= 1 \bmod p. \end{aligned} \tag{2.10}$$

These equations are sufficient to determine the integers u and v. Therefore, when p is not a multiple of q and q is not a multiple of p, we can write the conformal branching rules by writing that of one basic representation, since (2.9) acting on it determines the rest.

In the following section we present two examples; one for which q is a multiple of p so that we use the image of A^p, and one for which p and q are not multiples of each other, so that we can use the relation (2.9).

3. Examples

The first example we consider is the conformal subalgebra

$$\hat{SU}(18) \triangleright \hat{SU}(3)^6 \times \hat{SU}(6)^3 \quad , \tag{3.1}$$

i.e., the general case (1.22) with $p = 3$ and $q = 6$. Since q is a multiple of p, the generator A of the outer-automorphism group of $\hat{\mathrm{SU}}(18)$ has no image in the outer-automorphism group of $\hat{\mathrm{SU}}(3) \times \hat{\mathrm{SU}}(6)$, but A^3 does.

Equation (2.7), $A^3 = \alpha$, relates the branching rules for the representations $\omega^0, \omega^3, \omega^6, \omega^9, \omega^{12}, \omega^{15}$. The highest weights for the lowest L_0-eigenvalue subspaces of these representations are $0, \bar{\omega}^3, \bar{\omega}^6, \bar{\omega}^9, \bar{\omega}^{12}, \bar{\omega}^{15}$, respectively. The branching rules for these finite Lie representations may be obtained by the Young tableau method exemplified in Fig. 2. the two "longest" are:

$$\begin{aligned}
\bar{\omega}^6 \rightarrow{} & (00-03000)+(11-11100)+(30-20010)+ \\
& (60-00000)+(41-10001)+(03-00200)+ \\
& (22-01010) \\
\bar{\omega}^9 \rightarrow{} & (00-00300)+(11-01110)+(30-02001)+ \\
& (41-11000)+(03-10020)+ \\
& (14-00011)+(33-00100)+ \\
& (22-10101).
\end{aligned} \tag{3.2}$$

A third, that for $\bar{\omega}^{12}$, is determined from that of $\bar{\omega}^6$ by replacing SU(3) and SU(6) representations by their contragredient representations.

These three finite branching rules provide parts of the affine branching rule we want:

$$
\begin{aligned}
\omega^6 \to\ & [600-003000]+[411-011100]+[330-020010]+\\
& [060-300000]+[141-110001]+[303-100200]+\\
& [222-101010]+\cdots\\
\omega^9 \to\ & [600-000300]+[411-001110]+[330-002001]+\\
& [141-111000]+[303-010020]+\\
& [114-100011]+[033-200100]+\\
& [222-010101]+\cdots\\
\omega^{12} \to\ & [600-000030]+[411-000111]+[330-100200]+\\
& [303-001002]+\\
& [006-300000]+[114-110001]+\\
& [222-101010]+\cdots.
\end{aligned}
\tag{3.3}
$$

With $A^3=\alpha$, applying A^{-6}, A^{-9}, and A^{-12}, respectively, to these branching rules yields the branching rule for ω^0:

$$
\begin{aligned}
\omega^0 \to\ & [600-300000]+[411-110001]+[330-001002]+\\
& [060-000030]+[141-000111]+[303-020010]+\\
& [006-003000]+[114-011100]+[033-100200]+\\
& [222-101010].
\end{aligned}
\tag{3.4}
$$

The branching rules for ω^1 and ω^2 can be derived similarly:

$$\begin{aligned}
\omega^1 \to & [402-020001]+[510-210000]+[213-011010]+ \\
& [240-000102]+[051-000021]+[321-101001]+ \\
& [024-010200]+[105-002100]+[132-100110] \\
\omega^2 \to & [420-201000]+[501-120000]+[231-100101]+ \\
& [042-100020]+[150-000012]+[123-010110]+ \\
& [204-002010]+[015-001200]+[312-011001].
\end{aligned} \tag{3.5}$$

That these branching rules are complete can be verified using the asymptotic sum rule (2.4). Equations (3.4) and (3.5) together with $A^3 = \alpha$ determine the branching rules for all level one representations of $\hat{SU}(18)$.

We now consider an example for which there is an image of A in the outer-automorphism group of the subalgebra. For the case

$$\hat{SU}(12) \triangleright \hat{SU}(3)^4 \times \hat{SU}(4)^3 \tag{3.6}$$

solving (2.8) and (2.7) gives

$$A = a\alpha^3. \tag{3.7}$$

The branching rule for the finite representation $\bar{\omega}^5$ is easily found to be

$$\bar{\omega}^5 \to (31-100)+(12-011)+(20-201)+(01-120) \tag{3.8}$$

which implies for the affine case

$$\omega^5 \to [031-2100]+[112-1011]+[220-0201]+[301-0120]+\cdots \tag{3.9}$$

Acting on (3.9) with $A^{-5} = a\alpha$ does not yield the full branching rule. The complete result is

$$
\begin{aligned}
\omega^0 \rightarrow & [400 - 3000] + [031 - 2100] + [112 - 1011] + \\
& [220 - 0201] + [301 - 0120].
\end{aligned}
\tag{3.10}
$$

The first term in (3.10) is obvious, although it cannot be obtained from (3.9). As in the previous example, that nothing is missing can be verified using the asymptotic sum rule.

4. Modular Invariants

As an application of our results we will derive some modular-invariant combinations of various $\hat{\mathrm{SU}}(n)$ characters of the form (1.19). For every affine algebra $\hat{g}^k$ there exists a trivial "diagonal" modular-invariant Z_d given by replacing Z_{MN} with δ_{MN} in (1.19).

If $\hat{g}$ has a conformal subalgebra $\hat{g} \triangleright \hat{f}$, then a non-trivial modular-invariant for $\hat{f}$ may be obtained by simply substituting the branching rules for the subalgebra in character form [15]. If, furthermore, $\hat{f} = \hat{j} + \hat{k}$, we can obtain a modular invariant for $\hat{j}$ by projecting onto the $\bar{k}$-singlet part of Z_d [13]. That is, if after substituting the branching rules for $\hat{g} \triangleright \hat{j} + \hat{k}$ into Z_d we keep only those terms proportional to $\chi_C \chi^*_{\bar{M}}$, where $\bar{C}$ is the $\bar{k}$ representation contragredient to $\bar{M}$, the result is a modular-invariant combination of $\hat{j}$ characters.

We now apply this procedure to our conformal subalgebra, equation (1.22). First consider the case $p = 2$. Using the methods described above, we can find the branching rules in closed form. For q odd we have

$$\omega^{\ell} \to \sum_{\substack{r=0 \\ r \in 2Z}}^{q} a^{\ell} \alpha^{\ell(q+1)/2} [(q-r)\omega^0 + r\omega^1, \omega^{r/2} + \omega^{q-r/2}] \qquad (q \in 2Z+1). \tag{4.1}$$

Here the ω^{μ} are the fundamental weights of the $\hat{\mathrm{SU}}(n)$ groups involved (i.e., $n = 2q$, 2, or q) and a and α are the outer-automorphism generators of $\hat{\mathrm{SU}}(2)$ and $\hat{\mathrm{SU}}(q)$, respectively. For q even we obtain

$$\omega^{2\ell} \to \sum_{\substack{r=0 \\ r \in 2Z}}^{q} \alpha^{\ell} [r\omega^0 + (q-r)\omega^1, \omega^{(q-r)/2} + \omega^{(q+r)/2}] \qquad (q \in 2Z) \tag{4.2}$$

and

$$\omega^{2\ell+1} \to \sum_{\substack{r=1 \\ r \in 2Z+1}}^{q} \alpha^{\ell} [r\omega^0 + (q-r)\omega^1, \omega^{(q-r+1)/2} + \omega^{(q+r+1)/2}] \qquad (q \in 2Z). \tag{4.3}$$

If we substitute these results into the diagonal $\hat{\mathrm{SU}}(2q)$ modular invariant and then project onto the $\mathrm{SU}(q)$-singlet part as described above we obtain $\hat{\mathrm{SU}}(2)$ modular-invariants. In fact, for q odd we get the trivial diagonal invariants. But for q even we obtain the partition functions for strings on the group manifold $SO(3) \cong \mathrm{SU}(2)/Z_2$, first derived in [5].

Repeating the procedure for $p = 3$ yields the following modular-invariants: for $k \notin 3Z$ we get

$$
\begin{aligned}
Z = & \sum_{a \neq b} \sum_{r=0}^{2} |A^r[abb]|^2 \\
& + \left\{ \sum_{\substack{a \neq b \neq c \neq a \\ b+2c=k \ mod\ 3}} \sum_{r=0}^{2} A^r[abc](A^r R[abc])^* + c.c. \right\}
\end{aligned}
\tag{4.4}
$$

where

$$
A[abc] = [cab] \qquad R[abc] = [cba] \tag{4.5}
$$

For $k = 3t$ the modular invariants are

$$
\begin{aligned}
Z = & \sum_{a \neq b} |\sum_{r=0}^{2} A^r[abb]|^2 + 3|[ttt]|^2 \\
& + \left\{ \sum_{\substack{a \neq b \neq c \neq a \\ b+2c=k \ mod\ 3}} \sum_{r=0}^{2} \sum_{s=0}^{2} A^{r+s}[abc](A^r R A^s[abc])^* + c.c. \right\}
\end{aligned}
\tag{4.6}
$$

Equations (4.4) and (4.6) can be used to complete a possibly exhaustive list of $\hat{\mathrm{SU}}(3)$ modular-invariants [16].

The first few in these series agree with the partition functions for strings on the group manifold $\mathrm{SU}(3)/Z_3$ found in [5]. So (4.4) and (4.6) probably constitute the full infinite series of such partition functions. Since it seems to hold for $p = 2$ and 3, we conjecture that the partition functions for strings on the group manifolds $\mathrm{SU}(p)/Z_p$ can be calculated in the same manner.

We must mention that $\mathrm{SU}(p)/Z_p$ partition functions and many more modular-invariants were found previously by Bernard [17]. It is not surprising that our results agree with his since the main tools in his work, as in ours, were the outer-automorphisms of affine Lie algebras. However, Bernard did not work out the branching rules for (1.22) as we did, and these contain much more information than just the partition functions he derived. As a simple illustration of this, we will work out another modular-invariant from a branching rule used above to obtain an SO(3) partition function.

It is possible to generalize the procedure for obtaining modular-invariants from $\hat{g} \triangleright \hat{j} + \hat{k}$ in the following way [18]. Suppose after substituting the branching rules into the diagonal invariant for $\hat{g}$ we obtain the modular-invariant:

$$\sum_{\substack{\mu,\nu \\ m,n}} Z_{\mu m,\nu n} \chi[\mu] \chi[m] \chi^*[\nu] \chi^*[n] \quad , \tag{4.7}$$

where $[\mu]$, $[\nu]$, denote $\hat{j}$ representations and $[m]$, $[n]$ denote $\hat{k}$ representations. Equivalently, we can say that $Z_{\mu m,\nu n}$ is a modular-invariant tensor. But we can obtain another such tensor by contraction with an invariant tensor given by a $\hat{k}$ modular-invariant:

$$Z'_{mn} = \sum_{\mu,\nu} Z_{\mu m,\nu n} \tilde{Z}_{\mu\nu} \quad . \tag{4.8}$$

The procedure we have used above is just a special choice of the tensors $\tilde{Z}$ and Z.

If we use the branching rules for

$$\hat{\mathrm{SU}}(20) \triangleright \hat{\mathrm{SU}}(2)^{10} \times \hat{\mathrm{SU}}(10)^2 \tag{4.9}$$

in the trivial invariant of $\hat{\mathrm{SU}}(20)$ and contract with the exceptional $\hat{\mathrm{SU}}(2)$ modular-invariant [8]

$$Z(E_6) = |[10,0] + [46]|^2 + |[73] + [37]|^2 + |[64] + [0,10]|^2 \tag{4.10}$$

we obtain a new $\hat{\mathrm{SU}}(10)^2$ modular-invariant:

$$\begin{aligned} Z = \sum_{\ell=0}^{4} \Bigg\{ & \left| \left[2\omega^{\ell}\right] + \left[\omega^{3+\ell} + \omega^{7+\ell}\right] \right|^2 + \\ & \left| \left[\omega^{2+\ell} + \omega^{9+\ell}\right] + \left[\omega^{4+\ell} + \omega^{7+\ell}\right] \right|^2 + \\ & \left| \left[\omega^{1+\ell} + \omega^{8+\ell}\right] + \left[2\omega^{5+\ell}\right] \right|^2 \Bigg\}. \end{aligned} \tag{4.11}$$

We also mention that there are uses for conformal branching rules other than the derivation of modular-invariants. For example, in ref. [19] it was shown how they can provide information about certain conformal field theories defined by the GKO coset construction [3].

5. Conclusion

In summary, we have shown how to use the outer-automorphism group of $\hat{\mathrm{SU}}(n)$ to calculate branching rules for the conformal subalgebra $\hat{\mathrm{SU}}(pq) \triangleright \hat{\mathrm{SU}}(p) \times \hat{\mathrm{SU}}(q)$, given the branching rules for the basic representations of $\mathrm{SU}(pq)$ in the subalgebra $\mathrm{SU}(pq) \supset \mathrm{SU}(p) \times \mathrm{SU}(q)$.

We have also constructed modular-invariant combinations of $\hat{\mathrm{SU}}(n)$ characters from our results. We found explicitly the partition functions for strings on the group manifold $SU(n)/Z_n$ for $n = 2$ and 3, and showed how the results may be obtained for arbitrary n. Generically, our branching rules yield modular-invariants of the type found previously by Bernard [17], who also used the outer-automorphism group. However, other modular-invariants can also be obtained, as we illustrated.

The method used here may be extended to other conformal subalgebras [20], and possibly also to nonconformal affine subalgebras. We believe that our method, in conjunction with previously known ones, should make possible the calculation of the branching rules for any conformal subalgebra.

It is a pleasure to acknowledge useful conversations with C. Ahn, P. Griffin, C. S. Lam, D. Lewellen and J. Patera.

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FIGURE CAPTIONS

1) The Dynkin diagram of $\hat{SU}(r+1)$, the extended Dynkin diagram of $SU(r+1)$.

2) Branching rule for the fourth basic representation of $SU(pq)$ into $SU(p) \times SU(q)$, using Young tableaux.

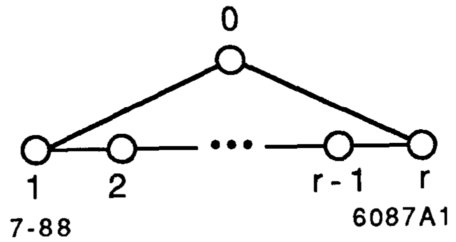

Fig. 1

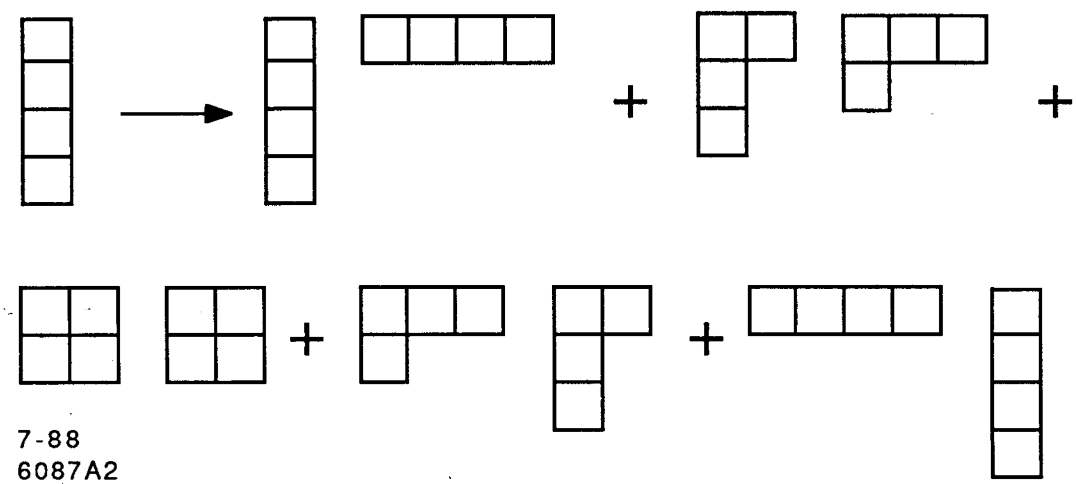

Fig. 2