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CROSS–PLANE COUPLING AND ITS EFFECT ON PROJECTED EMITTANCE*

Karl L. Brown and Roger V. Servranckx

Stanford Linear Accelerator Center

Stanford University, Stanford, CA 94309

ABSTRACT

We have developed a general first-order theory of coupled motion between the transverse phase planes in single-pass static magnetic beam transport systems. The results are expressed in terms of the projected emittances in the x and y transverse planes at any point s along the optical axis of the system.

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I. INTRODUCTION

One of the possible consequences of coupled motion in a beam transport system is an increase in the emittance projected onto the transverse planes at points downstream. To evaluate the magnitude of this effect we use the symplectic condition imposed upon the system by Hamiltonian mechanics to minimize the number of free parameters needed to describe the coupled motion. We then calculate a four-dimensional monoenergetic beam envelope (σ matrix) at any position s along the optic axis of the transport line and from this determine the emittance projected onto the x and y transverse phase planes. The results are expressed in terms of a 4×4 linear transformation matrix R and its 2×2 submatrices A, B, C, D. An important result is that if the determinant of any one of the 2×2 submatrices becomes negative, then the projected emittance can become large, depending upon the magnitudes of the absolute values of the submatrix determinants.

The calculations are applicable to any single-pass charged particle beam transport system constructed from static-magnetic optical elements.

II. NOTATION

For this report we use the definitions and notation of the TRANSPORT[1,2] program where the monoenergetic linear transformation in a beam transport system is expressed in the following matrix form:

$$\vec{X}_1 = R\vec{X}_0 \quad ; \qquad \vec{X} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \quad . \tag{1}$$

R is a 4×4 linear transformation matrix and, to first-order, the coordinates x, x', y, and y' form a set of canonical variables.

For static magnetic systems, Liouville's theorem of phase space conservation requires that $\det R = 1$. If R has the form

$$R = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}_{4\times4} , \qquad (2)$$

where A and D are 2×2 submatrices, then the x– and y–plane optics are decoupled and are independent of each other, in which case

$$\det A = \det D = 1 . \qquad (3)$$

Thus, six independent parameters are needed to determine the uncoupled R matrix, three for the x plane and three for the y plane.

In the more general case, where the motion between the x and y phase planes is coupled, the R–matrix has the form:

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{4\times4} , \qquad (4)$$

where R is a 4×4 matrix and A, B, C and D are 2×2 submatrices. B is a matrix coupling the y–plane to the x–plane and C is a matrix coupling the x–plane to the y–plane.

As will be shown in Section III, there are six independent symplectic conditions imposed upon R from Hamiltonian mechanics. Therefore there are (sixteen minus six =) ten independent parameters needed to uniquely determine the matrix elements of the coupled R matrix.

III. THE SYMPLECTIC CONDITION AND ITS CONSEQUENCES

The role of the symplectic condition for coupled motion in particle accelerators has been previously addressed by Courant and Snyder[3] and Teng[5] among others. For our purpose we begin with a derivation of the symplectic condition and develop from this the implications to our particular problem.

Let us consider motion in three dimensional space and denote the coordinates by $q = (q_1, q_2, q_3)$. As is well known, the motion of particles can be described by the Hamiltonian equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad , \tag{5}$$

where p_i are the conjugate momenta of the variables q_i, and H is the Hamiltonian describing the system.

The transformation $Q = Q(q,p,t)$, $P = P(q,p,t)$ is canonical when there exists a function $K(Q,P,t)$ (a new Hamiltonian) such that the variables Q, P satisfy the equations:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \quad , \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \quad . \tag{6}$$

According to Hamiltonian mechanics the functions Q and P satisfy the following conditions:

$$[Q_i, Q_j] = 0 \quad , \qquad [P_i, P_j] = 0 \quad , \qquad [Q_i, P_j] = \delta_{ij} \quad ,$$

where $[F,G]$ is the Poisson bracket of F, G defined as:

$$[F,G] = \sum_j \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right) \quad .$$

Given initial conditions for the variables q_i and the momenta p_i, a solution exists which can be written in the form:

$$q_i = q_i(q_{i0}, p_{i0}, t) \quad , \qquad p_i = p_i(q_{i0}, p_{i0}, t) \quad . \tag{7}$$

This transformation, from the initial conditions to the values at time t, is a canonical transformation. Thus the functions q_i and p_i satisfy the relations:

$$[q_i, q_j] = 0 \quad , \qquad [p_i, p_j] = 0 \quad , \qquad [q_i, p_j] = \delta_{ij} \quad ,$$

or explicitly:

$$\sum_{j0} \left(\frac{\partial q_i}{\partial q_{j0}} \frac{\partial q_k}{\partial p_{j0}} - \frac{\partial q_i}{\partial p_{j0}} \frac{\partial q_k}{\partial q_{j0}} \right) = 0 \quad ,$$

$$\sum_{j0} \left(\frac{\partial p_i}{\partial q_{j0}} \frac{\partial p_k}{\partial p_{j0}} - \frac{\partial p_i}{\partial p_{j0}} \frac{\partial p_k}{\partial q_{j0}} \right) = 0 \quad , \tag{8}$$

$$\sum_{j0} \left(\frac{\partial q_i}{\partial q_{j0}} \frac{\partial p_k}{\partial p_{j0}} - \frac{\partial q_i}{\partial p_{j0}} \frac{\partial p_k}{\partial q_{j0}} \right) = \delta_{ik} \quad .$$

Let us now consider the six-dimensional vector

$$v = (q_1, p_1, q_2, p_2, q_3, p_3)$$

and the Jacobian of the transformation (7) defined as:

$$M = \begin{pmatrix} \frac{\partial q_1}{\partial q_{10}} & \cdots & \frac{\partial q_1}{\partial p_{30}} \\ \cdots & \cdots & \cdots \\ \frac{\partial p_3}{\partial q_{10}} & \cdots & \frac{\partial p_3}{\partial p_{30}} \end{pmatrix} \quad .$$

Then the conditions (8) can be expressed in terms of the matrix M as follows:

$$\widetilde{M} S M \; = \; S \tag{9}$$

where S is the matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

and $\widetilde{M}$ is the transpose of M. Equation (9) is called the *symplectic condition. All solutions to Hamiltonian equations must satisfy this condition.* Conversely, if a transform (7) satisfies the conditions (9), then there exists locally a Hamiltonian and a set of associated Hamiltonian equations to which (7) is a solution.

As shown in Appendix B, to first-order, the coordinates x, x', y, and y' form a set of canonical variables. Therefore the symplectic condition also applies to the 4×4 R matrix, defined in Eq. (1), as follows:

$$\widetilde{R} S R = S \,, \tag{10}$$

where $\widetilde{R}$ means the transpose of R and

$$\widetilde{R} = \begin{pmatrix} \widetilde{A} & \widetilde{C} \\ \widetilde{B} & \widetilde{D} \end{pmatrix} \quad . \tag{11}$$

S is now a 4×4 symplectic matrix defined as

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}_{4\times 4} \quad . \tag{12}$$

Note that $S\,S = -I$ (where I is the unity matrix).

We now define the *symplectic conjugate*, $\overline{A}$, of the 2×2 submatrix, A, as:

$$\overline{A} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \widetilde{A} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \quad ; \tag{13}$$

from which it follows that

$$A\overline{A} = \overline{A}A = \begin{pmatrix} \det A & 0 \\ 0 & \det A \end{pmatrix} \quad . \tag{14}$$

Similarly, the symplectic conjugate of the 4×4 R–matrix is defined as:

$$\overline{R} = -S\widetilde{R}S = \begin{pmatrix} \overline{A} & \overline{C} \\ \overline{B} & \overline{D} \end{pmatrix}_{4\times 4} \quad . \tag{15}$$

Such that

$$\overline{R}R = -S\widetilde{R}SR = -SS = I \quad . \tag{16}$$

As a consequence, it follows that:

$$\overline{R} = R^{-1} \quad . \tag{17}$$

We now expand the matrix products $R\overline{R}$ and $\overline{R}R$ in terms of their 2×2 submatrices.

$$R\overline{R} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \overline{A} & \overline{C} \\ \overline{B} & \overline{D} \end{pmatrix} \quad , \tag{18}$$

from which

$$R\overline{R} = \begin{pmatrix} A\overline{A} + B\overline{B} & A\overline{C} + B\overline{D} \\ C\overline{A} + D\overline{B} & C\overline{C} + D\overline{D} \end{pmatrix}_{4\times 4} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}_{4\times 4} \quad . \tag{19}$$

Similarly,

$$\overline{R}R = \begin{pmatrix} \overline{A}A+\overline{C}C & \overline{A}B+\overline{C}D \\ \overline{B}A+\overline{D}C & \overline{B}B+\overline{D}D \end{pmatrix}_{4\times4} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}_{4\times4} \quad . \tag{20}$$

From Eqs. (14)and (19), it follows that

$$\overline{A}A+\overline{B}B = \begin{pmatrix} \det A+\det B & 0 \\ 0 & \det A+\det B \end{pmatrix}_{2\times2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times2} \tag{21}$$

and

$$\overline{C}C+\overline{D}D = \begin{pmatrix} \det C+\det D & 0 \\ 0 & \det C+\det D \end{pmatrix}_{2\times2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times2} \quad . \tag{22}$$

Similar results may be obtained from Eq. (20).

From Eq. (19) we conclude that the symplectic condition imposes the following constraints upon the coupled R–matrix.

$$\begin{aligned} \det A+\det B &= 1 \ , \qquad & \det C+\det D &= 1 \ , \\ A\overline{C}+B\overline{D} &= 0 \ , \qquad & C\overline{A}+D\overline{B} &= 0 \ ; \end{aligned} \tag{23}$$

and from Eq. (20) we have the alternate form

$$\begin{aligned} \det B+\det D &= 1 \ , \qquad & \det A+\det C &= 1 \ , \\ \overline{A}B+\overline{C}D &= 0 \ , \qquad & \overline{B}A+\overline{D}C &= 0 \ . \end{aligned} \tag{24}$$

It can be shown that Eqs. (23) and (24) are two different, but equivalent, ways of expressing the symplectic constraint. We will use both forms in this report, but first we show that the set of Eqs. (24) represent *only* six independent conditions. The first two are:

$$\begin{aligned} \det A + \det C &= 1 \quad , \\ \det B + \det D &= 1 \quad . \end{aligned} \tag{25}$$

Now we show that the last of Eqs. (24) produces four additional conditions.

Given:

$$\overline{B}A + \overline{D}C = 0 \quad , \tag{26}$$

$$\overline{B}A = \begin{pmatrix} B_{22} & -B_{12} \\ -B_{21} & B_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad , \tag{27}$$

or

$$\overline{B}A = \begin{pmatrix} \begin{vmatrix} A_{11} & B_{12} \\ A_{21} & B_{22} \end{vmatrix} & \begin{vmatrix} A_{12} & B_{12} \\ A_{22} & B_{22} \end{vmatrix} \\ -\begin{vmatrix} A_{11} & B_{11} \\ A_{21} & B_{21} \end{vmatrix} & -\begin{vmatrix} A_{12} & B_{11} \\ A_{22} & B_{21} \end{vmatrix} \end{pmatrix}_{2\times 2} \quad , \tag{28}$$

where we define

$$\begin{vmatrix} A_{11} & B_{12} \\ A_{21} & B_{22} \end{vmatrix} = \det \begin{pmatrix} A_{11} & B_{12} \\ A_{21} & B_{22} \end{pmatrix} \quad \cdots \quad . \tag{29}$$

Similarly,

$$\overline{D}C = \begin{pmatrix} D_{22} & -D_{12} \\ -D_{21} & D_{11} \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \quad , \tag{30}$$

or

$$\overline{D}C = \begin{pmatrix} \begin{vmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{vmatrix} & \begin{vmatrix} C_{12} & D_{12} \\ C_{22} & D_{22} \end{vmatrix} \\ -\begin{vmatrix} C_{11} & D_{11} \\ C_{21} & D_{21} \end{vmatrix} & -\begin{vmatrix} C_{12} & D_{11} \\ C_{22} & D_{21} \end{vmatrix} \end{pmatrix}_{2\times 2} . \tag{31}$$

Adding Eqs. (28) and (31), we conclude that Eq. (26) is equivalent to the following set of four equations:

$$\begin{aligned} &\begin{vmatrix} A_{11} & B_{11} \\ A_{21} & B_{21} \end{vmatrix} + \begin{vmatrix} C_{11} & D_{11} \\ C_{21} & D_{21} \end{vmatrix} = 0 \ , \\ &\begin{vmatrix} A_{11} & B_{12} \\ A_{21} & B_{22} \end{vmatrix} + \begin{vmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{vmatrix} = 0 \ , \\ &\begin{vmatrix} A_{12} & B_{11} \\ A_{22} & B_{21} \end{vmatrix} + \begin{vmatrix} C_{12} & D_{11} \\ C_{22} & D_{21} \end{vmatrix} = 0 \ , \\ &\begin{vmatrix} A_{12} & B_{12} \\ A_{22} & B_{22} \end{vmatrix} + \begin{vmatrix} C_{12} & D_{12} \\ C_{22} & D_{22} \end{vmatrix} = 0 \ . \end{aligned} \tag{32}$$

It can be shown that expanding the matrix equation

$$\overline{A}B + \overline{C}D = 0 ,$$

from Eqs. (24) yields the same set of four Eqs. (32). Thus there are only six independent relations imposed by the symplectic condition. These results may be summarized as follows:

Define the "sum of determinants" of any two columns of the matrix R as the sum of the two determinants formed by rows 1 and 2 and rows 3 and 4. There are six such combinations. The above six symplectic conditions represent all of the combinations of the columns (i,j). If $(i,j) = (1,2)$ or (3,4), then the sum of the determinants is equal to 1, which is equivalent to Eqs. (25). For the remaining combinations of columns, such as (1,3), the sum of the determinants is equal to zero, which is equivalent to Eqs. (32). It is readily deduced from Eqs. (25) and (32) that if all of the matrix elements of B are identically equal to zero, then the same is true for the matrix elements of C.

If one interchanges the words, rows, and columns in the above paragraph, then the resulting statement applies to the set of Eqs. (23). From this one can show that Eqs. (23) and (24) are equivalent ways of expressing the six symplectic constraints.

IV. CALCULATION OF THE PROJECTED BEAM EMITTANCE

We are now in a position to define and calculate the projected emittances in the x and y transverse planes of a beam as it passes through a coupled system.

Let σ be a 4×4 symmetric, positive definite, *beam envelope matrix* as defined and used in TRANSPORT.[2]

$$\sigma = \begin{pmatrix} \sigma_x & t \\ \tilde{t} & \sigma_y \end{pmatrix} , \tag{33}$$

where σ_x and σ_y are 2×2 symmetric, positive definite, matrices representing the x and y projections of the beam, and the 2×2 matrix t describes the x–y coupling present in the beam.

The relation between the beam matrices at a position 0 and at a position 1 is given by the equation:

$$\sigma_1 = R\,\sigma_0\,\widetilde{R} \quad , \tag{34}$$

or explicitly

$$\begin{pmatrix} \sigma_{x1} & t_1 \\ \widetilde{t}_1 & \sigma_{y1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \sigma_{x0} & t_0 \\ \widetilde{t}_0 & \sigma_{y0} \end{pmatrix} \begin{pmatrix} \widetilde{A} & \widetilde{C} \\ \widetilde{B} & \widetilde{D} \end{pmatrix} \tag{35}$$

$$= \begin{pmatrix} A\sigma_{x_0}\widetilde{A} + At_0\widetilde{B} + B\widetilde{t}_0\widetilde{A} + B\sigma_{y_0}\widetilde{B} & A\sigma_{x_0}\widetilde{C} + At_0\widetilde{D} + B\widetilde{t}_0\widetilde{C} + B\sigma_{y_0}\widetilde{D} \\ C\sigma_{x_0}\widetilde{A} + Ct_0\widetilde{B} + D\widetilde{t}_0\widetilde{A} + D\sigma_{y_0}\widetilde{B} & C\sigma_{x_0}\widetilde{C} + Ct_0\widetilde{D} + D\widetilde{t}_0\widetilde{C} + D\sigma_{y_0}\widetilde{D} \end{pmatrix} . \tag{36}$$

Thus

$$\begin{aligned} \sigma_{x1} &= A\sigma_{x_0}\,\widetilde{A} + At_0\,\widetilde{B} + B\,\widetilde{t}_0\,\widetilde{A} + B\sigma_{y_0}\,\widetilde{B} \quad , \\ \sigma_{y1} &= C\sigma_{x_0}\,\widetilde{C} + Ct_0\,\widetilde{D} + D\,\widetilde{t}_0\,\widetilde{C} + D\sigma_{y_0}\,\widetilde{D} \quad , \\ t_1 &= A\sigma_{x_0}\,\widetilde{C} + At_0\,\widetilde{D} + B\,\widetilde{t}_0\,\widetilde{C} + B\sigma_{y_0}\,\widetilde{D} \quad . \end{aligned} \tag{37}$$

If we assume that the initial beam is uncoupled, i.e., $t_0 = 0$, then

$$\begin{aligned} \sigma_{x1} &= A\sigma_{x_0}\,\widetilde{A} + B\sigma_{y_0}\,\widetilde{B} \quad , \\ \sigma_{y1} &= C\sigma_{x_0}\,\widetilde{C} + D\sigma_{y_0}\,\widetilde{D} \quad . \end{aligned} \tag{38}$$

The *projected emittances* at position 0 are defined[1,2] as:

$$\begin{aligned} \varepsilon_{x0}^2 &= \det\left(\sigma_{x0}\right)_{2\times 2} \quad , \\ \varepsilon_{y0}^2 &= \det\left(\sigma_{y0}\right)_{2\times 2} \quad , \end{aligned} \tag{39}$$

and at position 1 as:

$$\begin{aligned} \varepsilon_{x_1}^2 &= \det\left(\sigma_{x_1}\right)_{2\times 2} \quad , \\ \varepsilon_{y_1}^2 &= \det\left(\sigma_{y_1}\right)_{2\times 2} \quad , \end{aligned} \tag{40}$$

from which:

$$\begin{aligned} \varepsilon_{x_1}^2 &= \det\left(A\sigma_{x_0}\widetilde{A} + B\sigma_{y_0}\widetilde{B}\right)_{2\times 2} \quad , \\ \varepsilon_{y_1}^2 &= \det\left(C\sigma_{x_0}\widetilde{C} + D\sigma_{y_0}\widetilde{D}\right)_{2\times 2} \quad . \end{aligned} \tag{41}$$

In order to evaluate the above expressions, we use the following matrix identity for the determinant of the sum of two 2×2 matrices M and N:

$$\det(M+N) = \det M + \det N + \det\begin{pmatrix} M_{11} & M_{12} \\ N_{21} & N_{22} \end{pmatrix} + \det\begin{pmatrix} N_{11} & N_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad . \tag{42}$$

To further simplify the calculation, we assume that the initial uncoupled beam is represented, in both the x and y planes, by upright beam ellipses, i.e., that

$$\sigma_{x_0} = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix} \quad , \quad \sigma_{y_0} = \begin{pmatrix} \sigma_{33} & 0 \\ 0 & \sigma_{44} \end{pmatrix} \quad . \tag{43}$$

Then it follows from Eq. (39) that

$$\begin{aligned} \varepsilon_{x_0}^2 &= \sigma_{11}\,\sigma_{22} \quad , \\ \varepsilon_{y_0}^2 &= \sigma_{33}\,\sigma_{44} \quad . \end{aligned} \tag{44}$$

This simplification does not restrict the generality of the result since it is easily demonstrated that, given any uncoupled beam, there always exists at least one symplectic, uncoupled transformation which transforms the beam to an uncoupled, upright ellipse in both the x and y phase planes. The proof of this statement is readily obtained from Eq. (3.7) in Ref. 4.

After some algebraic manipulation, the following expressions evolve from Eq. (41):

$$
\begin{aligned}
\varepsilon_{x_1}^2 = \det\sigma_{x_1} &= \det\left(A\,\sigma_{x_0}\,\widetilde{A}\right) + \det\left(B\,\sigma_{y_0}\,\widetilde{B}\right) \\
&+ \sigma_{11}\,\sigma_{33}\begin{vmatrix} A_{11} & B_{11} \\ A_{21} & B_{21}\end{vmatrix}^2 + \sigma_{11}\,\sigma_{44}\begin{vmatrix} A_{11} & B_{12} \\ A_{21} & B_{22}\end{vmatrix}^2 \\
&+ \sigma_{22}\,\sigma_{33}\begin{vmatrix} A_{12} & B_{11} \\ A_{22} & B_{21}\end{vmatrix}^2 + \sigma_{22}\,\sigma_{44}\begin{vmatrix} A_{12} & B_{12} \\ A_{22} & B_{22}\end{vmatrix}^2 ,
\end{aligned}
\tag{45}
$$

and

$$
\begin{aligned}
\varepsilon_{y_1}^2 = \det\sigma_{y_1} &= \det\left(C\,\sigma_{x_0}\,\widetilde{C}\right) + \det\left(D\,\sigma_{y_0}\,\widetilde{D}\right) \\
&+ \sigma_{11}\,\sigma_{33}\begin{vmatrix} C_{11} & D_{11} \\ C_{21} & D_{21}\end{vmatrix}^2 + \sigma_{11}\,\sigma_{44}\begin{vmatrix} C_{11} & D_{12} \\ C_{21} & D_{22}\end{vmatrix}^2 \\
&+ \sigma_{22}\,\sigma_{33}\begin{vmatrix} C_{12} & D_{11} \\ C_{22} & D_{21}\end{vmatrix}^2 + \sigma_{22}\,\sigma_{44}\begin{vmatrix} C_{12} & D_{12} \\ C_{22} & D_{22}\end{vmatrix}^2 ,
\end{aligned}
\tag{46}
$$

where $|\;|$ means the determinant of the enclosed matrix.

Using the expanded form Eqs. (32) of the symplectic condition (26), the expression $(\varepsilon_{x_1}^2 - \varepsilon_{y_1}^2)$ can be written as follows:

$$
\left(\varepsilon_{x_1}^2 - \varepsilon_{y_1}^2\right) = \det\left(A\,\sigma_{x_0}\,\widetilde{A}\right) + \det\left(B\,\sigma_{y_0}\,\widetilde{B}\right) - \det\left(C\,\sigma_{x_0}\,\widetilde{C}\right) - \det\left(D\,\sigma_{y_0}\,\widetilde{D}\right) .
\tag{47}
$$

Now using the symplectic conditions:

$$
\det A = \det D\,, \quad \det B = \det C\,, \quad \text{and} \quad \det A + \det C = \det D + \det B = 1\,,
\tag{48}
$$

we conclude that:

$$
\begin{aligned}
\left(\varepsilon_{x_1}^2 - \varepsilon_{y_1}^2\right) &= \left(\varepsilon_{x_0}^2 - \varepsilon_{y_0}^2\right) \; (\det A - \det C) \quad , \\
&= \left(\varepsilon_{x_0}^2 - \varepsilon_{y_0}^2\right) \; (1 - 2 \det C) \quad .
\end{aligned}
\tag{49}
$$

Two important conclusions may be extracted from Eq. (49).

1) If a beam is uncoupled at the beginning of a system, and the initial x and y emittances are equal, i.e., if $\varepsilon_{x_0} = \varepsilon_{y_0}$, then at *all* points downstream, the projected emittances ε_{x_1} and ε_{y_1} will also be equal to each other.

2) If at any point downstream the determinant of any one of the submatrices A, B, C, or D is equal to 1/2, then it follows from the symplectic conditions, (48), that *all* of the submatrix determinants will be equal to 1/2; i.e., $\det A = \det B = \det C = \det D = 1/2$. At such a location the projected emittances in the x and y planes will be equal, i.e., $\varepsilon_{x_1} = \varepsilon_{y_1}$, *independent* of the ratio, $\varepsilon_{x_0}/\varepsilon_{y_0}$, of the initial uncoupled x and y emittances at the starting point.

We now wish to explore the magnitude of the projected emittances under various circumstances. First of all, we consider the special case where, at the beginning of a system, the x emittance is finite but the y emittance is set equal to zero. This, for example, is the situation when one is launching *betatron oscillations* in the x plane and observing the resultant coupling into the y plane. This corresponds to the following:

$$
\varepsilon_{x_0} \neq 0 \quad \text{but} \quad \varepsilon_{y_0} = 0 \quad . \tag{50}
$$

From Eqs. (41), it follows that

$$\begin{aligned} \varepsilon_{x_1}^2 &= \det \sigma_{x_1} = \det\left(A\,\sigma_{x_0}\,\widetilde{A}\right) = \varepsilon_{x_0}^2(\det A)^2 \quad , \\ \varepsilon_{y_1}^2 &= \det \sigma_{y_1} = \det\left(C\,\sigma_{x_0}\,\widetilde{C}\right) = \varepsilon_{x_0}^2(\det C)^2 \quad . \end{aligned} \tag{51}$$

or

$$\begin{aligned} \varepsilon_{x_1} &= \varepsilon_{x_0}\,|\det A| = \varepsilon_{x_0}\,|1-\det C| \quad , \\ \varepsilon_{y_1} &= \varepsilon_{x_0}\,|\det C| \quad . \end{aligned} \tag{52}$$

where now | ... | means the *"absolute value of."*

At this point, it should be noted that the determinants of the submatrices A and D or B and C can become *negative* !!

We now explore the implications of Eqs. (52) for both positive and negative values of the 2×2 submatrices. If all of the submatrix determinants are *positive,* which is equivalent to

$$0 \le \det C \le 1 \quad , \tag{53}$$

then we readily conclude from Eqs. (52) that

$$\varepsilon_{x_1} + \varepsilon_{y_1} = \varepsilon_{x_0} \quad ; \tag{54}$$

that is, the sum of the projected x and y emittances at position 1 is *bounded* by the magnitude of the initial emittance ε_{x_0}.

However, if either the $\det A = \det D$ is *negative* or the $\det B = \det C$ is *negative,* which is equivalent to

$$\det C > 1 \qquad \text{or} \qquad \det C < 0 \quad , \tag{55}$$

then it follows from Eqs. (52) that

$$|\varepsilon_{x_1} - \varepsilon_{y_1}| = \varepsilon_{x_0} \quad ; \tag{56}$$

that is, the *difference* between the projected x and y emittances at position 1 is equal to the initial x emittance at the beginning of the system. Under these circumstances the projected emittances, ε_{x_1} and ε_{y_1}, are *unbounded* and as a consequence can become *very large* depending upon the magnitude of the absolute values of the submatrix determinants. The consequences of both Eqs. (54) and (56) are illustrated in Fig. 1.

Next we consider the case of a beam propagating through a single-pass transport system where *both* of the initial, uncoupled x and y emittances have finite values; i.e., when

$$\varepsilon_{x_0} \neq 0 \quad \text{and} \quad \varepsilon_{y_0} \neq 0 \quad . \tag{57}$$

We may then write the following expressions for the projected emittances:

$$\begin{aligned} \varepsilon_{x_1} &\geq \varepsilon_{x_0} \, |\det A| + \varepsilon_{y_0} \, |\det B| \quad , \\ \varepsilon_{y_1} &\geq \varepsilon_{x_0} \, |\det C| + \varepsilon_{y_0} \, |\det D| \quad . \end{aligned} \tag{58}$$

A formal proof of the validity of these inequalities is given in Appendix A.

Using the symplectic conditions, as stated in Eq. (48), the above inequalities may be written as a function of the determinant of any one of the submatrices A, B, C or D. For example, if we choose the $\det C$ as the variable, then:

$$\begin{aligned} \varepsilon_{x_1} &\geq \varepsilon_{x_0} \, |1 - \det C| + \varepsilon_{y_0} \, |\det C| \quad , \\ \varepsilon_{y_1} &\geq \varepsilon_{x_0} \, |\det C| + \varepsilon_{y_0} \, |1 - \det C| \quad . \end{aligned} \tag{59}$$

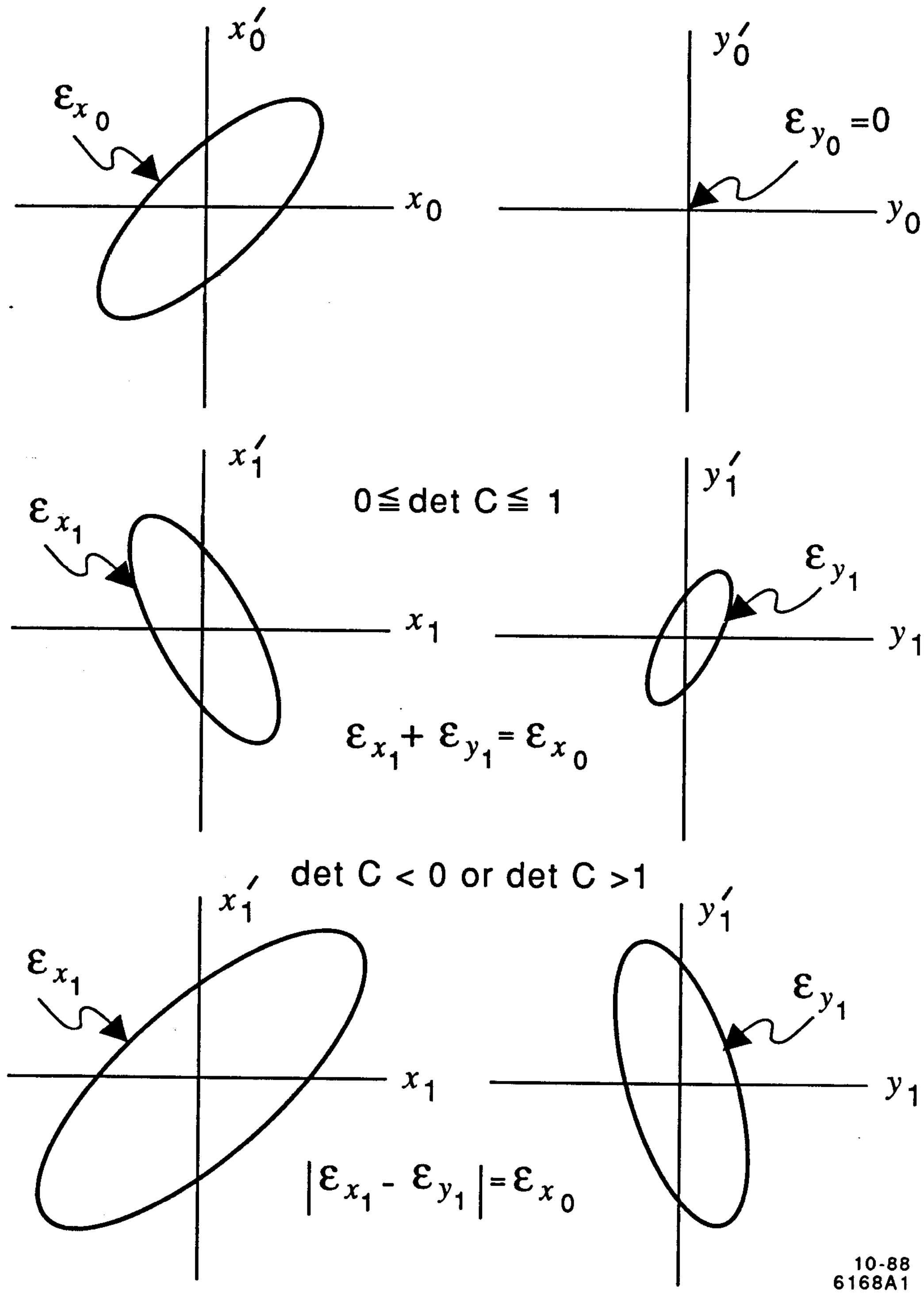

Fig 1. An illustration of the effect on projected emittances of positive and negative values of the submatrix determinants when ε_{x0} is finite and $\varepsilon_{y0} = 0$.

From Eq. (59) it is observed that at any position, s, where x–y coupling is present, the sum of the projected x and y emittances will always be equal to or greater than the sum of the initial, uncoupled x and y emittances.

We now wish to explore the physical meanings of the equal sign and of the inequality sign in the above expressions. It is sufficient to study the implications in the x–plane as the interpretation will be the same for the y–plane.

Suppose we rewrite the first of Eqs. (38) as follows:

$$\sigma_{x_1} = \left(A\,\sigma_{x_0}\,\widetilde{A}\right) + \left(B\,\sigma_{y_0}\,\widetilde{B}\right) = \sigma_A + \sigma_B \quad , \tag{60}$$

where σ_A and σ_B are 2×2 beam matrices, σ_A is that portion of the projected beam matrix σ_{x_1} that originates from the initial phase space in the x–plane and σ_B is that portion which arises from the y–plane, but is projected onto the x–plane at position 1. σ_A and σ_B each represent *beam phase ellipses,* and σ_{x_1} is the *matrix sum* of these two ellipses, both of which lie in the $(x,\ x')$ phase plane at position 1.

There are two distinct possibilities regarding the relationship between σ_A and σ_B. We first consider the case where σ_B represents an ellipse that is *similar to and has the same orientation* as the ellipse represented by σ_A, as illustrated in Fig. 2(a). This situation is expressed mathematically by the following matrix equation:

$$\sigma_B = K\sigma_A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}_A \quad , \tag{61}$$

where K is a scaling matrix relating σ_A to σ_B. Then

$$\sigma_{x_1} = \sigma_A + \sigma_B = (I + K)\sigma_A = \begin{pmatrix} 1+k & 0 \\ 0 & 1+k \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}_A \quad . \tag{62}$$

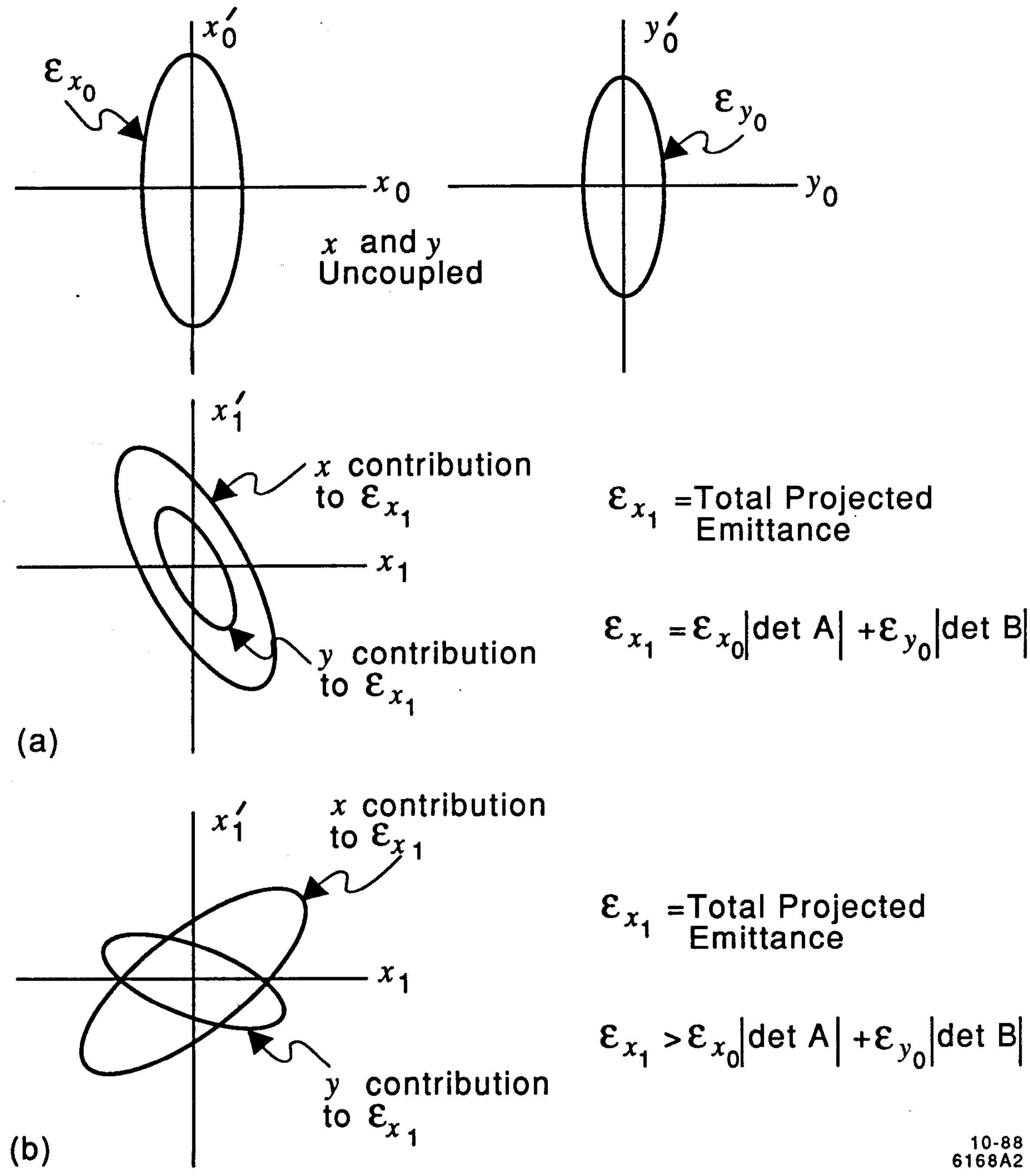

Fig 2. An illustration of the projected emittance in the x plane as a function of the relative orientation and similarity of the x and y contributions.

If we now calculate the projected emittance at position 1, we have

$$\varepsilon_{x_1}^2 = \det \sigma_{x_1} = \det(\sigma_A + \sigma_B) = \det[(I+K)\sigma_A] \quad , \tag{63}$$

from which

$$\begin{aligned} \varepsilon_{x_1} &= (1+k)(\det \sigma_A)^{1/2} = (\det \sigma_A)^{1/2} + k(\det \sigma_A)^{1/2} \quad , \\ &= (\det \sigma_A)^{1/2} + (\det \sigma_B)^{1/2} \quad , \end{aligned} \tag{64}$$

or

$$\varepsilon_{x_1} = \varepsilon_{x_0} \, |\det A| + \varepsilon_{y_0} \, |\det B| \quad . \tag{65}$$

Thus, the equal sign in expressions (58) and (59) applies when the two projected ellipses σ_A and σ_B are similar to each other and have the same orientation.

Next we consider the case when σ_A and σ_B are not similar and/or do not have the same orientation, as is illustrated in Fig. 2(b). In this case, the inequality sign applies, i.e.,

$$\varepsilon_{x_1} > \varepsilon_{x_0} \, |\det A| + \varepsilon_{y_0} \, |\det B| \quad . \tag{66}$$

V. EXAMPLES

We now wish to explore the implications of Eq. (49) and of the inequalities (58) and (59) for both *positive and negative* values of the submatrix determinants. We first consider the case where the initial, uncoupled emittances are equal in the x and y planes, i.e., when

$$\varepsilon_{x_0} = \varepsilon_{y_0} = \varepsilon_0 \quad ;$$

then from Eq. (49), it follows that, at all points downstream, the projected x and y emittances are equal, i.e.,

$$\varepsilon_{x_1} = \varepsilon_{y_1} = \varepsilon_1 \quad .$$

From (58) and (59), we conclude that

$$\varepsilon_1 / \varepsilon_0 \geq |\det C| + |\det D| \quad ,$$

or

$$\varepsilon_1 / \varepsilon_0 \geq |\det C| + |1 - \det C| \quad . \tag{67}$$

Consider now the case when all of the submatrix determinants are *positive;* i.e., when

$$0 \leq \det C \leq 1 \quad .$$

From Eq. (67) we conclude that the projected emittance at any position downstream will be *equal to or greater than* the initial, uncoupled emittance at the beginning of the system; i.e.,

$$\varepsilon_1 \geq \varepsilon_0 \quad . \tag{68}$$

Note that the inequality sign *may still be applicable* when the $\det C = 0$ unless *all* of the matrix elements of C are zero, in which case there is no cross-plane coupling present in the system.

However, if any of the submatrix determinants become *negative,* which is equivalent to

$$\det C \; > \; 1 \qquad \text{or} \qquad \det C \; < \; 0 \quad , \tag{69}$$

then the projected emittances will definitely grow in magnitude. For example, if

$$\det C \; = \; -1 \quad \text{or} \quad +2 \quad ; \tag{70}$$

then

$$\varepsilon_1 \; \geq \; 3\varepsilon_0 \quad .$$

As another example, if

$$\det C \; = \; -2 \quad \text{or} \quad +3 \quad , \tag{71}$$

then

$$\varepsilon_1 \; \geq \; 5\varepsilon_0 \quad . \tag{72}$$

Thus we see that when any of the submatrix determinants become *negative and large,* the *projected* emittances at that position will be *larger* than the initial, uncoupled emittance at the beginning of the transport system.

The situation can become even more serious if the incoming x and y emittances are not equal, as can been seen from substitution of typical values into the inequalities (58) or (59).

VI. CONCLUSIONS AND SUMMARY

A general first-order theory has been derived for coupled motion in a single-pass static magnetic beam transport system. The emphasis has been on determining the projected emittances onto the x and y transverse planes at any position s along the beam line. The theory is expressed in terms of a monoenergetic 4×4 linear TRANSPORT matrix R and its 2×2 submatrices A, B, C, D. Some conclusions that can be extracted from the theory are the following:

1. For linear optics, x–y cross-plane coupling may occur whenever the submatrices B and C possess *any nonzero* matrix elements. However if, at any point in the system, all of the matrix elements of the submatrix B are equal to zero, then the symplectic condition requires that the same be true for the matrix C. At such a location an initially uncoupled beam will have no x–y coupling.

2. If the initial emittances in the x and y planes at the beginning of a system are uncoupled and are equal to each other then at any point downstream the projected x and y emittances will always be equal to each other, independent of the magnitude of the x–y coupling at that point. Their magnitudes may, however, be equal to or greater than the values of the initial, uncoupled emittances.

3. If the initial emittances are equal and uncoupled, and if at any point downstream B and C have some nonzero matrix elements, and if all of the determinants of the submatrices A, B, C, and D are *positive,* i.e., in the range of 0 to +1, then the projected x and y emittances will be equal and coupled and *may become larger* than the initial emittance. Under these circumstances,

the emittance growth will usually not become excessively large. If, on the other hand, any of the determinants of the submatrices A, B, C, D become *negative* at a position s, then the projected x and y emittances at s will be equal and coupled but, in addition, *will become larger* than the initial, uncoupled emittances. The magnitude of the growth is determined by the absolute values of the submatrix determinants.

4. If the initial x and y emittances are uncoupled but are *unequal in magnitude* then, in general, the projected x and y emittances at any position s downstream will also be unequal. The one exception to this is when the determinant of one of the submatrices is equal to $1/2$. Then all of the submatrix determinants will equal $1/2$ and the projected x and y emittances will, although coupled, have the same value. At such a position, the projected x and y emittance will be equal to or greater than the average of the initial uncoupled x and y emittances.

5. At any position, s, where x–y coupling is present the sum of the projected x and y emittances will always be equal to or greater than the sum of the initial, uncoupled x and y emittances.

APPENDIX A

The purpose of this appendix is to derive a formal proof of the inequality (58). As we show in detail in the appendix, this amounts to proving the following statements:

1. Given two positive definite matrices, their sum is also a positive definite matrix. See auxiliary inequality $A4$ below.

2. Given two positive definite matrices, the square root of the determinant of the sum matrix is always larger than the sum of the square roots of the determinants of the given matrices. See inequality $A1$ below.

3. Equality in the above statement occurs if and only if the two given matrices are a scalar multiple of each other.

1) If

$$\sigma_0 = \begin{pmatrix} \sigma_{x0} & 0 \\ 0 & 0 \end{pmatrix} ,$$

then

$$\sigma_{x_1} = A\sigma_{x_0}\widetilde{A} , \qquad \varepsilon^2_{x_1} = \varepsilon^2_{x_0}(\det A)^2 ,$$

$$\sigma_{y_1} = C\sigma_{x_0}\widetilde{C} , \qquad \varepsilon^2_{y_1} = \varepsilon^2_{x_0}(\det C)^2 .$$

2) If

$$\sigma_0 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{y0} \end{pmatrix} ,$$

then

$$\sigma_{x2} = B\sigma_{y0}\widetilde{B} \quad , \qquad \varepsilon_{x2}^2 = \varepsilon_{y0}^2 (\det B)^2 \quad ,$$

$$\sigma_{y2} = D\sigma_{y0}\widetilde{D} \quad , \qquad \varepsilon_{y2}^2 = \varepsilon_{y0}^2 (\det D)^2 \quad .$$

3) If

$$\sigma_0 = \begin{pmatrix} \sigma_{x0} & 0 \\ 0 & \sigma_{y0} \end{pmatrix} \quad ,$$

then

$$\sigma_{x3} = A\sigma_{x0}\widetilde{A} + B\sigma_{y0}\widetilde{B} \quad , \qquad \varepsilon_{x3}^2 = \det(\sigma_{x3}) \quad ,$$

$$\sigma_{y3} = C\sigma_{x0}\widetilde{C} + D\sigma_{y0}\widetilde{D} \quad , \qquad \varepsilon_{y3}^2 = \det(\sigma_{y3}) \quad .$$

Observe that the indices 1, 2 and 3 do not refer to positions along the beamline, but characterize the three cases analyzed above. We are now going to prove that

$$\varepsilon_{x3} \geq \varepsilon_{x1} + \varepsilon_{x2} \qquad \text{and} \qquad \varepsilon_{y3} \geq \varepsilon_{y1} + \varepsilon_{y2} \quad .$$

To make the proof more readable, we shall use a somewhat simpler notation. $A\sigma_{x0}\widetilde{A}$ is a 2×2 symmetric positive definite matrix. Let us denote it as

$$\begin{pmatrix} a & m \\ m & b \end{pmatrix}$$

with

$$a > 0 \; , \quad b > 0 \quad ab - m^2 > 0 \qquad \text{and} \qquad \varepsilon_1 = \sqrt{ab - m^2} \quad .$$

$B\sigma_{y_0}\widetilde{B}$ is also a 2×2 symmetric positive definite matrix which we denote as

$$\begin{pmatrix} c & n \\ n & d \end{pmatrix}$$

with

$$c > 0\ , \quad d > 0 \quad cd - n^2 > 0 \quad \text{and} \quad \varepsilon_2 \ = \ \sqrt{cd - n^2}\ .$$

Then σ_{x_3} can be written as

$$\sigma_{x_3} \ = \ \begin{pmatrix} a+c & m+n \\ m+n & b+d \end{pmatrix}\ , \qquad \varepsilon_3 \ = \ \sqrt{(a+c)(b+d) - (m+n)^2} \quad .$$

We want to prove that

$$\sqrt{(a+c)(b+d) - (m+n)^2} \ \geq \ \sqrt{ab - m^2} \ + \ \sqrt{cd - n^2} \quad . \qquad (A1)$$

AI. Auxiliary Inequalities

$$-\sqrt{ab} \leq m \leq \sqrt{ab} \quad , \qquad -cd \leq n \leq \sqrt{cd} \quad ,$$

so

$$\begin{aligned} ad + bc - 2mn &\geq \left(\sqrt{ad}\right)^2 + \left(\sqrt{bc}\right)^2 - 2\sqrt{abcd} \quad , \\ &\geq \left(\sqrt{ad} - \sqrt{bc}\right)^2 \geq 0 \quad , \end{aligned} \tag{A2}$$

and

$$\begin{aligned} \left(\sqrt{ad} + \sqrt{bc}\right)^2 - 4mn &\geq \left(\sqrt{ad}\right)^2 + \left(\sqrt{bc}\right)^2 - 2\sqrt{abcd} \quad , \\ &\geq \left(\sqrt{ad} - \sqrt{bc}\right)^2 \geq 0 \quad . \end{aligned} \tag{A3}$$

Note also that σ_{x_3} is positive definite.

$$\begin{aligned} (a+c)(b+d) - (m+n)^2 &= ab - m^2 + cd - n^2 + ad + bc - 2mn \quad , \\ &= \varepsilon_1^2 + \varepsilon_2^2 + ad + bc - 2mn > 0 \end{aligned} \tag{A4}$$

as a consequence of the first auxiliary inequality A2.

Inequality (A4) also proves that

$$\varepsilon_3^2 \geq \varepsilon_1^2 + \varepsilon_2^2 \quad . \tag{A5}$$

Note that inequality (A5) is weaker than inequality (A1).

AII. Main Proof

$$\varepsilon_3 \ \geq \ \varepsilon_1 + \varepsilon_3 \iff \quad \text{(is equivalent to)}$$

$$\iff \varepsilon_3^2 \ \geq \ \varepsilon_1^2 + \varepsilon_2^2 + 2\varepsilon_1\varepsilon_2 \quad ,$$

or

$$\iff \varepsilon_3^2 - \varepsilon_1^2 - \varepsilon_2^2 \ \geq \ 2\varepsilon_1\varepsilon_2 \quad ;$$

$$\iff (a+c)\,(b+d) - (m+n)^2 \ - ab + m^2 - cd + n^2 \ \geq \ 2\sqrt{(ab-m^2)\,(cd-n^2)} \quad ,$$

$$\iff ad + bc - 2mn \ \geq \ 2\sqrt{(ab-m^2)\,(cd-n^2)} \quad .$$

Both sides are > 0 because of (A2), so we may square:

$$\iff a^2d^2 + b^2c^2 + 4m^2n^2 + 2adbc - 4mn(ad+bc) \ \geq \ 4\left(abcd - abn^2 - cdm^2 + m^2n^2\right) \quad ;$$

$$\iff (ad-bc)^2 - 4mn(ad+bc) + 4(abn^2 + cdm^2) \ \geq \ 0 \quad ;$$

$$\iff (ad-bc)^2 - 4mn(ad+bc) + 4(\sqrt{ab}\,n - \sqrt{cd}\,m)^2 + 8\sqrt{abcd}\,mn \ \geq \ 0 \quad ;$$

$$\iff (ad-bc)^2 - 4mn(\sqrt{ad} - \sqrt{bc})^2 + 4(\sqrt{ab}\,n - \sqrt{cd}\,m)^2 \ \geq \ 0 \quad ;$$

$$\iff (\sqrt{ad}-\sqrt{bc})^2(\sqrt{ad}+\sqrt{bc})^2 - 4mn(\sqrt{ad}-\sqrt{bc})^2 + 4(\sqrt{ab}\,n - \sqrt{cd}\,m)^2 \ \geq \ 0 \quad ;$$

$$\iff (\sqrt{ad}-\sqrt{bc})^2[(\sqrt{ad}+\sqrt{bc})^2 - 4mn] + 4(\sqrt{ab}\,n - \sqrt{cd}\,m)^2 \ \geq \ 0 \quad .$$

The last inequality is true by virtue of (A3). This proves the second statement contained in the introduction of this appendix.

AIII. Conditions for Equality

Equality in (A1) would hold only if

$$\sqrt{ad} = \sqrt{bc} \quad \text{and} \quad n\sqrt{ab} = m\sqrt{cd} \quad , \qquad (A6)$$

or if

$$4mn = (\sqrt{ad} + \sqrt{cd})^2 \qquad \text{and} \qquad n\sqrt{ab} = m\sqrt{cd} \quad . \qquad (A7)$$

Note that (A6) implies

$$c = ka \; , \quad d = kb \quad , \qquad n = km \quad ;$$

that is, the two ellipses σ_{x_1} and σ_{x_2} are similar.

We show now that (A7) never occurs. Consider conditions (A7) and let a, b, c, d be chosen arbitrarily; n and m are obtained as follows:

$$n = m\,\frac{\sqrt{cd}}{\sqrt{ab}} \qquad \text{and} \qquad 4m^2 = \frac{(\sqrt{ad} + \sqrt{bc})^2\,\sqrt{ab}}{\sqrt{cd}} \quad ,$$

or

$$m = \left(\frac{(\sqrt{ad} + \sqrt{bc})}{2}\right)\,\sqrt[4]{\frac{ab}{cd}} \quad ,$$

$$n = \left(\frac{(\sqrt{ad} + \sqrt{bc})}{2}\right)\,\sqrt[4]{\frac{cd}{ab}} \quad .$$

We now prove that $ab - m^2$ and $cd - n^2$ are both negative, which contradicts our assumption that both beam matrices are positive definite.

$$
\begin{aligned}
ab - m^2 &= ab - \frac{(\sqrt{ad} + \sqrt{bc})^2}{4} \sqrt{\frac{ab}{cd}} \quad , \\
&= \frac{1}{4}\sqrt{\frac{ab}{cd}} \left[4\sqrt{abcd} - \left(\sqrt{ad}\right)^2 - \left(\sqrt{bc}\right)^2 - 2\sqrt{abcd} \right] \quad , \\
&= -\frac{1}{4}\sqrt{\frac{ab}{cd}} \left(\sqrt{ad} - \sqrt{bc}\right)^2 < 0 \quad .
\end{aligned}
$$

Similarly,

$$
cd - n^2 = -\frac{1}{4}\sqrt{\frac{cd}{ab}} \left(\sqrt{ad} - \sqrt{bc}\right)^2 < 0 \quad .
$$

So, equality in (A1) holds *if and only if* the two ellipses are similar. This proves statement 3 of the introduction to this appendix.

APPENDIX B

The purpose of this appendix is to establish that, to first-order, the TRANSPORT coordinates x, x', y, and y' form a set of canonical variables.

Let x, p_x, y, p_y, z and p_z be a set of canonical coordinates. If the system is conservative, we may define the relative momenta as $x' = p_x/p$, $y' = p_y/p$ and $z' = p_z/p$, where

$$p = \sqrt{p_x^2 + p_y^2 + p_z^2} \quad ,$$

then the set x, x', y, y', z, and z' is also a canonical set of coordinates.

We now need to establish the physical interpretation of x' and y'. If the transverse momenta p_x and p_y are small compared to p and p_z then

$$x' = \frac{(p_x/p_z)}{\sqrt{1 + (p_x/p_z)^2 + (p_y/p_z)^2}} \simeq (p_x/p_z)\left[1 - 1/2[(p_x/p_z)^2 + (p_y/p_z)^2]\right] \quad .$$

So, to second-order we conclude that

$$x' = p_x/p \simeq p_x/p_z = \tan\theta \quad ,$$

where θ is the angle between the central trajectory and the projection of the arbitrary trajectory onto the local x–z coordinate plane of the transport system. A similar result holds for y'. This corresponds to the definitions of x' and y' in the TRANSPORT[1,2] program. So when the computations are limited to first-order, the TRANSPORT coordinates x, x', y, and y' form a set of canonical coordinates and the *symplectic condition* represented by Eq. (10) is valid.

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