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Topological Features of the Nonlinear Sigma Model with Hopf Term Using Chiral Lagrangians*

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ABSTRACT

We rewrite the O(3) nonlinear sigma model in 2+1 dimensions in terms of SU(2) matrices, thereby solving the constraint. The Lagrangian has the symmetry $SU(2)_{Global} \times U(1)_{Local}$. Static soliton solutions to this Lagrangian have energy $4\pi N$ as usual. We then show that the Hopf instantons, in the formalism of principle chiral fields, are just the skyrmions of QCD in 3+1 dimensions.

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I. Motivation

Recently, it has been suggested that the novelties of the 2+1 nonlinear sigma model with Hopf term could account for the strange properties of the new high-Tc superconductors.[1] The Lagrangian density for the nonlinear sigma model is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu n^a \partial^\mu n^a + \lambda[n^a n^a - 1] \tag{1}$$

where $\mu = 0, 1, 2$ and $a = 1, 2, 3$. The nonpropogating constraint field $\lambda(x)$ insures that the modulus of the n-fields is one. The above Lagrangian admits soliton solutions of energy 4πN,[4] with N an integer, for configurations satisfying $\partial_i n^a = \pm e^{abc}\varepsilon_{ij} n_b \partial_j n_c$ where i , $j = 1$, 2 only. Other workers have conjectured an additional term, the Hopf term,[5] in the field theory Lagrangian which is responsible for fractional statistics. The Hopf term is non-local in the n-fields but can be formally written in terms of a conserved topological current $J^\mu = \varepsilon^{\mu\nu\lambda} e^{abc} n_a \partial_\nu n_b \partial_\lambda n_c$,

$$S_{Hopf} \sim \frac{\theta}{2\pi} \int d^3x d^3y \varepsilon^{\mu\nu\lambda} J_\mu(x) \frac{(x-y)_\nu}{|x-y|^3} J_\lambda(y) \ . \tag{2}$$

Formally, the solitons realize the mapping $\Pi_2(S^2) = Z$ and the instantons the Hopf map $\Pi_3(S^2) = Z$.[6] Note that the conserved charge $Q = \int d^2x J^0$ is the explicit expression for $\Pi_2(S^2)$ modulo factors of π.

The model as formulated with the n-fields has two unattractive features, the constraint field $\lambda(x)$ and the non-locality of the Hopf term. In the CP^1 formulation, one makes the map[7] $\vec{n} = Z^\dagger \vec{\sigma} Z$ with Z a two–spinor (α, β) and the resulting Lagrangian becomes

$$\begin{aligned} \mathcal{L} = {} & 2(\partial_\mu Z)^\dagger(\partial^\mu Z) \\ & + 2(Z^\dagger \partial_\mu Z)(Z^\dagger \partial^\mu Z) + \lambda(Z^\dagger Z - 1) \quad . \end{aligned} \tag{3}$$

A Hopf term can similarly be written starting from a conserved topological current[8] $J^\mu = i\varepsilon^{\mu\nu\lambda}\partial_\nu Z^\dagger \partial_\lambda Z$ which can also be written generally as $J^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu B_\lambda$. The Hopf term in the CP^1 language thus is written

$$S_{Hopf} = \frac{\theta}{4\pi^2}\int d^3x J^\mu B_\mu \quad , \tag{4}$$

which gives for the Lagrangian density

$$\mathcal{L}_{Hopf} = -\frac{\theta}{4\pi^2}\varepsilon^{\mu\nu\lambda}(Z^\dagger\partial_\mu Z)(\partial_\nu Z^\dagger \partial_\lambda Z) \quad . \tag{5}$$

Even in the CP^1 Lagrangian, a constraint field $\lambda(x)$ appears and even though the Hopf term is local, its instanton solutions are not obvious. One can rewrite the Lagrangian introducing gauge fields that make explicit the U(1) gauge invariance of the CP^1 model:

$$\mathcal{L}_{CP^1} = 2(D_\mu Z)^\dagger (D^\mu Z) + \frac{\theta}{4\pi^2}A_\mu\partial_\nu A_\lambda \tag{6}$$

where $D_\mu Z = (\partial_\mu + iA_\mu)Z$. The above Chern-Simons term is equal to the θ-term in (9) only to lowest order in θ. The CP^1 model has SU(2)$_{Global}\times$ U(1)$_{Local}$ given by, $Z \to GZ$ with $G\,\varepsilon SU(2)$, and

$$Z \to e^{i\delta(x)}Z \quad , \quad A_\mu \to A_\mu - \partial_\mu\delta \quad . \tag{7}$$

We will find maps of the nonlinear sigma model with the same symmetry but without the $\lambda(x)$ fields and a Hopf term that gives manifest integer solutions to lowest order in θ. We will find indeed that the Hopf fibrations are expressed uniquely as $\Pi_3(SU(2))$.

II. Principle Chiral Maps From The CP^1 Model

The complex field $Z = (\alpha, \beta)$ with the constraint $Z^\dagger Z = 1$, describes S^3 which is isomorphic to SU(2). The most general definition of an SU(2) field relative to the Z-field is

$$U = \begin{pmatrix} \alpha' & \beta' \\ -\beta'^* & \alpha'^* \end{pmatrix}$$

where

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = G \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{with } G \,\varepsilon\, SU(2)\ . \tag{8}$$

The constraint $Z^\dagger Z = 1$ thus implies $U^\dagger U = 1$ which is implemented simply via the expansion $U = e^{i\theta^a(x)\tau^a}$ where the τ^a matrices are the 2×2 Pauli matrices. ***All we have done is to rewrite the nonlinear sigma model which is a SO(3)/SO(2) model into one which is SU(2)/U(1).*** Whereas the CP^1 map uses a fundamental representation to do so, we have used an adjoint representation. The kinetic piece in the Lagrangian becomes in terms of the principle chiral fields,

$$\mathcal{L} = Tr(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{2} Tr(U^\dagger \sigma_z \partial_\mu U) Tr(U^\dagger \sigma_z \partial^\mu U) \tag{9}$$

which has the $SU(2)_{Global} \times U(1)_{Local}$ symmetry with $U \rightarrow UG$ for $G\ \varepsilon SU(2)$, and

$$U \rightarrow e^{i\theta(x)\sigma_z} U\ . \tag{10}$$

Introducing gauge fields to make this explicit, our Lagrangian becomes

$$\mathcal{L} = Tr\big[(D_\mu U)^\dagger (D^\mu U)\big] \quad , \tag{11}$$

where $D_\mu U \equiv (\partial_\mu + iA_\mu \sigma_z)U$. The above two Lagrangians are classically equivalent. To see this, simply solve for the gauge field equations of motion and substitute. A topological term starting with the gauge field similar to the CP^1 model uses the manifestly conserved current, $J^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$. This term changes the equations of motion for the gauge fields and to lowest order in theta gives the same topological term as starting from the topological current in the U(x) language,

$$J^{\mu}_{top} = i\varepsilon^{\mu\nu\lambda} Tr(\partial_\nu U^\dagger \sigma_z \partial_\lambda U) \ . \tag{12}$$

The Chern-Simons term is thus given by

$$\begin{aligned} \mathcal{L}_{CS} &= \frac{\theta}{8\pi^2}\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \\ &= -\frac{\theta}{16\pi^2}\varepsilon^{\mu\nu\lambda} Tr(U^\dagger \sigma_z \partial_\mu U) Tr(\partial_\nu U^\dagger \sigma_z \partial_\lambda U) + \mathcal{O}(\theta^2) \quad . \end{aligned} \tag{13}$$

Noting the following identity,

$$\begin{aligned} &\int d^3x \, \varepsilon^{\mu\nu\lambda} Tr(U^\dagger \sigma^a \partial_\mu U) Tr(\partial_\nu U^\dagger \sigma^b \partial_\lambda U) \\ &= \frac{\delta^{ab} 2}{3} \int d^3x \ \varepsilon^{\mu\nu\lambda} Tr(U^\dagger \partial_\mu U \ U^\dagger \partial_\nu U \ U^\dagger \partial_\lambda U) \ , \end{aligned} \tag{14}$$

the Chern-Simons term to lowest order in theta can be rewritten into the familiar form for the winding number of the vacuum in QCD,[9] or equivalently the Baryon number for skyrmions,[10]

$$\mathcal{L}_{Hopf} = -\frac{\theta}{24\pi^2}\varepsilon^{\mu\nu\lambda} Tr(U^\dagger \partial_\mu U \ U^\dagger \partial_\nu U \ U^\dagger \partial_\lambda U) + \mathcal{O}(\theta^2) \ . \tag{15}$$

Of course, the full Chern-Simons term is not an integer, but to lowest order in theta, we have shown that it gives the topological term which is explicitly an integer. Higher order corrections can be derived via the equations of motion for A_μ,

$$A_\mu = -\frac{J_\mu}{4} - \frac{\theta}{16\pi^2}\varepsilon_{\mu\nu\lambda}\partial^\nu A^\lambda \quad ,$$

which implies

$$\begin{aligned} A_\mu = &-\frac{J_\mu}{4} + \frac{\theta}{64\pi^2}\varepsilon_{\mu\nu\lambda}\partial^\nu J^\lambda \\ &+ \frac{\theta^2}{4\cdot 256\pi^4}\partial_\mu(\partial_\nu J^\nu) - \frac{\theta^2}{4\cdot 256\pi^4}\Box J_\mu + \mathcal{O}(\theta^3) \quad , \end{aligned} \tag{16}$$

where

$$J_\mu = -i[Tr(U^\dagger \sigma_z \partial_\mu U) - Tr(\partial_\mu U^\dagger \sigma_z U)] \quad . \tag{17}$$

III. Principle Chiral Maps from the n-Fields

The map from the previous section picks out an arbitrary direction along the the z-axis. Motivated partly to eliminate this specific choice, we start with the following definition,

$$\vec{n} = \frac{1}{\sqrt{4k^2}} Tr\left(U^\dagger (\sigma \cdot k) U \vec{\sigma}\right) \ , \tag{18}$$

where the three-vector k is some constant vector. One easily confirms $n^2 = 1$. The nonlinear sigma model Lagrangian now becomes

$$\mathcal{L} = \frac{1}{2} Tr[\partial_\mu U^\dagger \partial^\mu U] + \frac{1}{2k^2} Tr[\partial U^\dagger (\sigma \cdot k) U \partial_\mu U^\dagger (\sigma \cdot k) U] \ . \tag{19}$$

Expanding $iU\partial_\mu U^\dagger = \alpha^i_\mu \sigma_i$ for some alphas, we can show

$$\begin{aligned} &\frac{1}{2k^2} Tr\left(\partial_\mu U^\dagger (\sigma \cdot k) U \partial_\mu U^\dagger (\sigma \cdot k) U\right) \\ &= \frac{1}{2} Tr(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{2k^2}\left[Tr\left(U^\dagger (\sigma \cdot k) \partial_\mu U)\right]\left[Tr(U^\dagger (\sigma \cdot k) \partial^\mu U\right)\right] \ . \end{aligned} \tag{20}$$

Thus the Lagrangian becomes

$$\mathcal{L} = Tr[\partial_\mu U^\dagger \partial^\mu U] + \frac{1}{2k^2}\left[Tr\left(U^\dagger (\sigma \cdot k) \partial_\mu U\right)\right]\left[Tr\left(U^\dagger (\sigma \cdot k) \partial^\mu U\right)\right] \ . \tag{21}$$

Finally, introducing gauge fields, this becomes

$$\mathcal{L} = Tr[(D_\mu U)^\dagger (D^\mu U)] \ , \tag{22}$$

where $D_\mu U = (\partial_\mu + iA_\mu (\sigma \cdot k))\, U$. Hence, we see that the general construction reproduces the map derived from the CP^1 model for the special case $k_z = 1$,

$k_x = k_y = 0$. This Lagrangian has SU(2)$_{Global}\times$ U(1)$_{Local}$ symmetry realized as $U \to UG$ for $G\epsilon\ SU(2)$, and

$$U \to e^{i\theta(x)(\sigma\cdot k)}\ U \quad \text{with} \quad A_\mu \ \to A_\mu \ - \partial_\mu \theta\ . \tag{23}$$

Again, we add a Chern-Simons term,

$$\mathcal{L}_{CS} = \frac{\theta k^2}{8\pi^2}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\ . \tag{24}$$

Now expanding the A_μ fields in terms of the principle chiral fields, we get

$$\mathcal{L}_{CS} = -\frac{\theta}{16\pi^2k^2}\varepsilon^{\mu\nu\lambda}Tr\left(U^\dagger(\sigma\cdot k)\partial_\mu U\right)Tr\left(\partial_\nu U^\dagger(\sigma\cdot k)\partial_\lambda U\right) + \mathcal{O}(\theta^2)\ . \tag{25}$$

This implies that the real, gauge-invariant and SU(2) invariant topological current in the U fields is

$$J^\mu = \frac{i\varepsilon^{\mu\nu\lambda}}{\sqrt{k^2}}Tr\left(\partial_\nu U^\dagger(\sigma\cdot k)\partial_\lambda U\right)\ . \tag{26}$$

We can simplify the Hopf term by using identity (14). Therefore, we get the winding number formula for QCD vacua again,

$$\mathcal{L}_{Hopf} = -\frac{\theta}{24\pi^2}\epsilon^{\mu\nu\lambda}Tr(U^\dagger\partial_\mu UU^\dagger\partial_\nu UU^\dagger\partial_\lambda U)\ . \tag{27}$$

To summarize, when we expand the nonlinear sigma model with Hopf term using principle chiral fields, our most general Lagrangian is

$$\mathcal{L} = Tr[(D_\mu U)^\dagger(D^\mu U)] + \frac{\theta k^2}{8\pi^2}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \tag{28}$$

with $D_\mu U = (\partial_\mu + iA_\mu(\sigma\cdot k))U$ and the recursion relation

$$A_\mu = -\frac{J_\mu}{4k^2} - \frac{\theta k^2}{16\pi^2}\varepsilon_{\mu\nu\lambda}\partial^\nu A^\lambda\ ,$$

which when expanded to lowest order in theta gives $\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{Hopf}$ where $\mathcal{L}_{kin}$ is given in (21) and $\mathcal{L}_{Hopf}$ in (27).

IV. Solitons and Instantons

Let's first look at the soliton solutions in the U(x) fields. The Hamiltonian becomes, neglecting the toplogical term,

$$\mathcal{H} = Tr(\partial_0 U^\dagger \partial_0 U) - 2k^2 A_0 A_0 + Tr\left((D_i U)^\dagger (D_i U)\right) \ , \tag{29}$$

where $i = 1, 2$. For time independent solutions in the $A_o = 0$ gauge, the energy is

$$E = \int d^2x \; Tr[(D_i U)^\dagger (D_i U)] \ . \tag{30}$$

Using the inequality,

$$\int d^2x \; Tr \big| [D_i U \pm \frac{i\epsilon_{ij}}{\sqrt{k^2}} (\sigma \cdot k) D_j U] \big|^2 \geq 0 \ . \tag{31}$$

We find the energy is given by

$$E = \int d^2x \; Tr[(D_i U)^\dagger (D_i U)] \geq \frac{i}{\sqrt{k^2}} \int d^2x \; Tr\left(\partial_i U^\dagger (\sigma \cdot k) \partial_j U\right) \epsilon^{ij} \ . \tag{32}$$

The right-hand side is a total derivative (in fact, the quantized topological charge) and for U's at the boundary given by

$$U = e^{i\delta(\theta) \frac{\sigma \cdot k}{\sqrt{k^2}}} \ , \tag{33}$$

the right-hand side of (32) becomes

$$2 \int d\theta \frac{\partial \delta(\theta)}{\partial \theta} = 4\pi N \ ,$$

which implies $E \geq 4\pi N$. Thus, the Bogomol'nyi condition[11] is

$$D_i U = \pm i \frac{\epsilon_{ij}}{\sqrt{k^2}} (\sigma \cdot k) D_j U \ . \tag{34}$$

Solutions to the Bogomol'nyi condition have energy $4\pi N$ as required. Consider the special case $k_z = 1$, $k_x = k_y = 0$. If we expand U(x) as in (8), then we get the two equations

$$(\partial_i + iA_i)\alpha = \pm i\varepsilon_{ij}(\partial_y + iA_j)\alpha \ ,$$

$$(\partial_i + iA_i)\beta = \pm i\varepsilon_{ij}(\partial_j + iA_j)\beta \ , \tag{35}$$

where $A_i = -\frac{i}{2}Tr(\partial_i U^\dagger \sigma_z U) = -i(\partial_i Z^\dagger Z)$ with Z defined as a complex spinor $(\alpha,\ \beta)$. Writing in the CP^1 form, the two equations become $D_i Z = \pm i\varepsilon_{ij} D_j Z$, whose solutions are the well-known instantons in the euclidean 1+1 dimensional O(3) nonlinear sigma model.[4] Solutions to (34) are invariant under rescalings of $\vec{k}$. For general unit vector $\vec{k}$, the soliton configuration $U_{\vec{k}}$ that satisfies the Bogomol'nyi condition is

$$U_{\vec{k}} = G^{-1}(k)U_0$$

with U_0 being the solution gotten in the above special case and $G(k)$ being the SU(2) element satisfying

$$\vec{\sigma} \cdot \vec{k} = G^\dagger(k)\sigma_z G(k) \ . \tag{36}$$

The Hopf instantons are given by the winding numbers of QCD vacua which are also the hedgehog solutions in skyrmion physics. These solutions are well-known, for the general case of a Hopf N-instanton, the ansatz[12] is

$$U = \cos\theta(r) + i\ \sigma \cdot \hat{r} \sin\theta(r) \tag{37}$$

where $r^2 = (x_0^2 + x_1^2 + x_2^2)$ and $\theta(r)$ is any function with the property $\theta(0) = N\pi$ and $\theta(\infty) = 0$. With these general properties for $\theta(r)$, it can shown that expression (27) integrated over euclidean three space is an integer N. Thus, as promised, skyrmions of 3+1 chiral Lagrangians are like euclidean space Hopf instantons.

Now, using the defining map for the n-fields, we can derive the general expression for the Hopf instantons in the language of the nonlinear sigma model, in euclidean space,

$$\vec{n}(\vec{r}) = \frac{2\vec{k}}{\sqrt{4k^2}}\cos 2\theta(r) + \frac{2(\hat{r}\times\vec{k})}{\sqrt{4k^2}}\sin 2\theta(r) + \frac{4\sin^2\theta(r)}{\sqrt{4k^2}}(\vec{k}\cdot\hat{r})\hat{r} \, . \qquad (38)$$

The n-fields are directly proportional to the k-vectors at infinity and at the origin,

$$\vec{n}(\infty) = \frac{2\vec{k}}{\sqrt{4k^2}} = \vec{n}(0) \, . \qquad (39)$$

Similar expressions for the CP^1 fields could also be derived for the special case using the map (8). It would be interesting to map the solutions connected with the linking number for curves on S^3 to the solutions demonstrated here. The essential comparisons between the nonlinear sigma model, the CP^1 model and the SU(2) model are summarized in Table 1.

V. Conclusions

We have rewritten the nonlinear sigma model with Hopf term using principle chiral fields. This can be done either starting from the n-fields or the CP^1 fields. The Chern-Simons term to lowest order is manifestly an integer representing the Hopf instantons. The solitons in this language also satisfy a Bogomol'nyi identity. It is expected that higher order terms beyond the winding number expression are the relevant terms that stabilize the soliton against quantum corrections. This is the subject of our current work.

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Table 1

	Nonlinear Sigma Model	CP^1 Model	SU(2) Model
Gauge-invariant Lagrangian	$\frac{1}{2}(\partial_\mu n)(\partial^\mu n)$ $+\lambda(n^2-1)$	$2(D_\mu Z)^\dagger(D^\mu Z)+\lambda(Z^\dagger Z-1)$ $+\frac{\theta}{4\pi^2}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$	$Tr[(D_\mu U)^\dagger(D^\mu U)]$ $+\frac{\theta k}{8\pi^2}^2\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$
Symmetry	$0(3)_{Global}$	$SU(2)_{Global}\times U(1)_{Local}$	$SU(2)_{Global}\times U(1)_{Local}$
Conserved Topological Current	$J^\mu=\varepsilon^{abc}\varepsilon^{\mu\nu\lambda}$ $n_a\partial_\nu n_b\partial_\lambda n_c$	$J^\mu=i\varepsilon^{\mu\nu\lambda}(\partial_\nu Z^\dagger\partial_\lambda Z)$	$J^\mu=\frac{i}{\sqrt{k^2}}\varepsilon^{\mu\nu\lambda}$ $Tr(\partial_\nu U^\dagger(\sigma\cdot k)\partial_\lambda U)$
Topological Term	$\frac{1}{2\pi}\int d^3xd^3y\varepsilon^{\mu\nu\lambda}$ $J_\mu(x)\frac{(x-y)_\nu}{\|x-y\|^3}J_\lambda(y)$	$\frac{\theta}{4\pi^2}\int d^3x\varepsilon^{\mu\nu\lambda}$ $(Z^\dagger\partial_\mu Z)(\partial_\nu Z^\dagger\partial_\lambda Z)$	$\frac{\theta}{24\pi^2}\int d^3x\varepsilon^{\mu\nu\lambda}$ $Tr(U^\dagger\partial_\lambda UU^\dagger\partial_\nu UU^\dagger\partial_\lambda U)$
Bogomol'nyi Identity	$\partial_i n^a=\pm\varepsilon^{abc}$ $\varepsilon_{ij}n_b\partial_j n_i$	$D_iZ=\pm i\varepsilon_{ij}D_jZ$	$D_iU=\pm i\varepsilon_{ij}\frac{(\sigma\cdot k)}{\sqrt{k^2}}D_jU$