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Bosonization of Parafermions*

PAUL GRIFFIN†

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

and

DENNIS NEMESCHANSKY‡

Physics Department, University of Southern California
Los Angeles, CA 90089

ABSTRACT

We develop a general method of solving non-trivial conformal field theories by coset imbeddings into free boson models. In particular, we consider Z_N parafermion models represented by the coset model constructions $su(N)_1 \oplus su(N)_1/su(N)_2$, and bosonize the $su(N)_1$ models. Correlation functions of these parafermion theories are shown to be calculable without detailed information of the diagonal $su(N)_2$ theory. A differential equation for the four point function of order operators σ_1 is derived as an example. We also derive differential equations with respect to the modular parameter τ for the parafermion characters in terms of the $su(N)_1$ characters only. In particular, the equations for the Z_2 and Z_3 characters are first and second order respectively.

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† Address after Sept. 1, 1988: Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510

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1. Introduction

Chiral Bosonization of two-dimensional conformal field theories is a powerful method of extracting correlation and partition functions. In addition to the original work on fermions [1], the Wess-Zumino-Witten (WZW) models[2,3] of level one admit representations of their currents and vertex operators as simple functions of free bosons[4]. Also, the background charge method pioneered by Feigen and Fuchs[5], and subsequent generalizations [6], effectively bosonize a large class of models including the minimal conformal series[7]. The purpose of this paper is to describe how to extract correlators and characters by embedding a theory into a larger bosonizable one. In particular, we shall focus on embedding Z_N Parafermion (PF) algebras of Zamolodchikov and Fateev[8] into a direct sum of two affine level 1 Kac-Moody algebras $su(N)_1 \oplus su(N)_1$, where the subscript denotes the level 1 algebras.

The classification of two dimensional conformal field theories (CFT's) is relevant to the study of universality properties of statistical models, and for understanding the breath of possible string compactifications. One particular classification scheme which embodies a large subset of known rational CFT's is the coset construction of Goddard, Kent and Olive[9]. Consider a WZW model with holomorphic and antiholomorphic ('left' and 'right') Kac-Moody symmetry algebras $G \oplus G$. Let the stress tensor of the left sector be T_G and the central charge of G by c_G. If H is a subalgebra of G with corresponding T_H and c_H, then a coset model is labeled by G/H, and has a stress tensor $T_G - T_H$ with central charge $c_G - c_H$. That this construction yields an exactly solvable CFT is discussed by Douglas[10]. In particular, the exactly solvable models based on the background charge method fall under this classification, as do other general Z_N models[8, 11]. The Z_N PF models to be considered are classified as both $su(2)_N/U(1)$ and $su(N)_1 \oplus su(N)_1/su(N)_2$.

While correlators and characters of the G/H theory are fixed by the information contained in the G and H theories[10], for the case at hand we will argue that *the G/H theory can solved with information from the G theory alone.* In particular, differential equations for correlation functions, and for the characters of each highest weight state, can be obtained from the bosonized $SU(N)_1$ theories. We denote this technique as *Coset Bosonization* , because it eliminates the need for non-trivial information from the H theory.

An intriguing interplay between physics and mathematics occurs with WZW models and their corresponding Kac-Moody symmetry algebras. The characters [12] of the irreducible representations of the untwisted affine Kac-Moody (KM) algebras are building blocks of the partition function of the WZW model[13]. The characters decompose under the infinite dimensional Weyl group of the KM algebra. Each character is a finite sum of string functions[14] multiplied by theta functions which describe a sum over cosets of the Weyl Group. The string functions are building blocks for the partition functions of the generalized parafermions of Gepner[15]. So a direct calculation of the parafermion characters yields information about the KM representations. Moreover, the field space of generalized parafermions contains most of the non-trivial information of the KM representations. In particular, the characters for each highest weight state of the Zamolodchikov and Fateev PF models considered here are essentially the string functions which generate the WZW characters of $SU(2)_N$[16]. So one of the motivations for the calculation of PF characters from conformal field theory methods is to generate the structure of KM theory from a physics point of view.

The Z_N PF theories are building blocks for large classes of interesting theories. By combining them with a single free boson, one can obtain either the $su(2)_N$ models discussed above or models with $N = 2$ world sheet supersymmetry[17]. By combining them with Feigen and Fuchs bosonic models with background charge, most known rational conformal field theories are generated[18].

The contents of this paper are organized as follows: section 2 is a review of the operator content and algebra of Z_N PF theories with emphasis on results crucial to later results. In section 3 we describe the bosonization of the parafermion current algebra, relations between bosonic $su(N)_1$ vertex operators and operators in the PF field space, and a sample calculation of the differential equation for an order operator four point function. Section 4 contains relations between level one characters of $SU(N)$ and characters of the PF theory, and differential equations are derived for these characters via a calculation of the parafermion two-point function on the torus. Special integrable cases, including the Ising (Z_2) model, are discussed. We conclude with a discussion of our prescription and comment on future applications in section 5. Appendix A contains background material on bosonization of $SU(N)_1$ and Appendix B describes an alternate method of calculating the parafermion two-point function on the torus based on an N-fold cover argument.

2. Parafermion Current Algebra and Field Space

In this section, we review properties of parafermionic theories which were developed in [8]. The Z_N theories contain parafermion currents $\psi_k(z)$ and $\overline{\psi}_k(\overline{z})$ ($k = 1, 2, \cdots, N-1$) which satisfy

$$\partial_{\overline{z}}\psi_k(z) = 0 \quad \partial_z\overline{\psi}_k(\overline{z}) = 0. \tag{2.1}$$

The fields ψ_0 and $\overline{\psi}_0$ are defined as identity operators. For the Z_2 case, ψ_1 is the Ising fermion. The parafermions ψ_k and $\overline{\psi}_k$ have $Z_N \times \widetilde{Z}_N$ charge (k, k) and $(k, -k)$ respectively, where the charge is defined mod N. This labeling of charge under the Z_N symmetries is useful for discussing mutual semi-locality properties of fields, and manifests the self-dual behavior. It is also convenient to express the charges (k, l) in the form $[k+l, k-l]$, defined mod $2N$, in which case the parafermions ψ_k and $\overline{\psi}_k$ have charges $[2k, 0]$ and $[0, 2k]$ respectively. This basis is particularly useful when comparing PF fields with those of the corresponding $su(2)_N$ models, as we will discuss at the end of this section. Furthermore, holomorphic (antiholomorphic) fields have charges $[l, 0]$ ($[0, \overline{l}]$) in this basis; the parafermions generate decoupled chiral algebras so we will focus mainly on the fields ψ_k since similar results for the $\overline{\psi}_k$ fields can be easily transcribed. The conjugate is defined as $\psi_k^\dagger \equiv \psi_{N-k}$. Fateev and Zamolodchikov have shown that the following choice for the dimensions of the parafemions is consistant:

$$\Delta_k = k(N-k)/N. \tag{2.2}$$

The operator product expansion defining the parafermion algebra is

$$\begin{aligned} \psi_k(z)\psi_l(w) &= c_{k,l}(z-w)^{-2kl/N}[\psi_{k+l}(w)+\mathcal{O}(z-w)] \quad k+l<N\ , \\ \psi_k(z)\psi_l^\dagger(w) &= c_{k,N-l}(z-w)^{-2k(N-k)/N}[\psi_{k-l}(w)+\mathcal{O}(z-w)] \quad k>l\ , \\ \psi_k(z)\psi_k^\dagger(w) &= (z-w)^{-2\Delta_k}[I+(2\Delta_k/c)(z-w)^2T(w)+\mathcal{O}(z-w)^3]. \end{aligned} \tag{2.3}$$

Associativity of the OPE's determine the $c_{k,l}$. Note that there is no $\mathcal{O}(z-w)$ term in the $\psi(z)\psi^\dagger(w)$ OPE, because any dimension 1 current would violate the assumption that $Z_N \times Z_N$ is the maximal symmetry of the model. The stress tensor $T(z)$ obeys the operator products

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial_w T(w)}{(z-w)}+\mathcal{O}(1), \\ T(z)\psi_k(w) &= \frac{\Delta_k\psi_k(w)}{(z-w)^2}+\frac{\partial_w\psi_k(w)}{(z-w)}+\mathcal{O}(1). \end{aligned} \tag{2.4}$$

The central charge of the Virasoro algebra consistent with (2.4) is given by

$$c = 2(N-1)/(N+2). \tag{2.5}$$

Generalizing the Ising model, there exist order and disorder operators σ_k and μ_k $(k = 1, 2, \cdots, N-1)$, with charge $[k, k]$ and $[k, -k]$, and conjugates σ_{N-k} and μ_{N-k}. They are highest weight under the Virasoro algebra with conformal dimensions $(h, \overline{h})$ given as (d_k, d_k), where

$$2d_k = k(N-k)/N(N+2) \tag{2.6}$$

(The value $k = 0$ corresponds to the identity.) The operator product of order and disorder fields contains the parafermion currents :

$$\sigma_k(z,\bar z)\mu_k(w,\overline{w}) = Q_k(z-w)^{\Delta_k-2d_k}(\bar z-\overline{w})^{-2d_k}[\psi_k(w)+\cdots],$$
$$\sigma_k(z,\bar z)\mu_k^\dagger(w,\overline{w}) = Q_k(z-w)^{\Delta_k-2d_k}(\bar z-\overline{w})^{-2d_k}[\psi_k(w)+\cdots], \tag{2.7}$$

where $Q_k = k!(N-k)!/N!$ is determined by normalizing the two point functions of ψ_k, σ_k, and μ_k. The Ising model is a special case where order and disorder fields are self-conjugate, so that the operator product of σ_k and μ_k contains both ψ_k and $\overline{\psi}_k$.

The parafermion field space is generated by applying parafermion currents to the highest weight states under the PF algebra – i.e. the order and disorder operators. It decomposes into a direct sum of subspaces specified by Z_{2N} charges:

$$\mathcal{H} = \tfrac{1}{2} \oplus [\phi^{(k)}_{[l,\bar l]}]. \tag{2.8}$$

The superscript denotes the highest weight order operator, the subscripts denote Z_{2N} charges, where $(l,\bar l) = k \bmod 2Z$. The factor of $\frac{1}{2}$ denotes the double counting of fields:

$$\phi^{(k)}_{[l,\bar l]} = \phi^{(N-k)}_{[N+l,N+\bar l]}. \tag{2.9}$$

In this language, the order (disorder) operators σ_k (μ_k) are $\phi^{(k)}_{[k,k]}$ ($\phi^{(k)}_{[k,-k]}$), and the currents ψ_k ($\overline{\psi}_k$) are $\phi^{(0)}_{[2k,0]}$ ($\phi^{(0)}_{[0,2k]}$). The modes of the currents ψ_k and $\psi_k^\dagger$ are defined via the operator product expansions

$$\psi_1(z)\phi^{(k)}_{[l,\bar l]}(0,0) = \sum_{m=-\infty}^{\infty} z^{-l/N+m-1}A_{(1+l)/N-m}\phi^{(k)}_{[l,\bar l]}(0,0),$$
$$\psi_1^\dagger(z)\phi^{(k)}_{[l,\bar l]}(0,0) = \sum_{m=-\infty}^{\infty} z^{l/N+m-1}A^\dagger_{(1-l)/N-m}\phi^{(k)}_{[l,\bar l]}(0,0). \tag{2.10}$$

Order and disorder operators are annihilated by operators A_ν and $A^\dagger_\mu$ for all $\nu,\mu >$

0. Upon each order operator the sequence $\phi^{(k)}_{[l,k]}$ is generated as

$$\begin{aligned}\phi^{(k)}_{[k+2l,k]} &= A_{(k-1+2l)/N-1}A_{(k-1+2l-2)/N-1},\cdots A_{(1+k)/N-1}\sigma_k \qquad l=0,1,\cdots,N-k,\\ \phi^{(k)}_{[k-2l,k]} &= A^{\dagger}_{-(k+1-2l)/N}A^{\dagger}_{-(k+1-2l+2)/N},\cdots A^{\dagger}_{-(1-k)/N}\sigma_k \qquad l=0,1,\cdots,k.\end{aligned} \tag{2.11}$$

The operators $\overline{A}$ and $\overline{A}^{\dagger}$ are similarly defined and the resulting fields $\phi^{(k)}_{[l,\bar{l}]}$ have conformal dimension $(d^{(l)}_k, d^{(\bar{l})}_k)$ given by

$$\begin{aligned}d^{(l)}_k &= d_k + (l-q)(k+l)/4N \qquad -k \leq l \leq k,\\ d^{(l)}_k &= d_k + (l-k)(2N-k-l)/4N \qquad k \leq l \leq 2N-k.\end{aligned} \tag{2.12}$$

Note that the disorder operator μ_k is in the field space generated by the order operator σ_k. All fields above are primary under the Virasoro algebra, and the other fields in the spectrum differ in dimensions from these by positive integers.

As implied above, the Z_N theory has a simple relation to the $su(2)_N$ WZW model. Introduce a free massless boson $\Phi = \phi(z) + \overline{\phi}(\overline{z})$, with two point functions on the conformal plane

$$\begin{aligned}\langle\phi(z)\phi(w)\rangle &= -\ln(z-w),\\ \langle\overline{\phi}(\overline{z})\overline{\phi}(\overline{w})\rangle &= -\ln(\overline{z}-\overline{w}),\\ \langle\phi(z)\overline{\phi}(\overline{w})\rangle &= 0.\end{aligned} \tag{2.13}$$

The holomorphic stress tensor is $T(z) = -\frac{1}{2}{:}\,\partial_z\phi\partial_z\phi{:}$; it generates a Virasoro algebra with central charge $c = 1$. Holomorphic currents are defined by

$$\begin{aligned}J^{+}(z) &= \sqrt{N}\psi_1(z){:}\exp\left[+i\sqrt{\frac{2}{N}}\phi(z)\right]{:},\\ J^{-}(z) &= \sqrt{N}\psi_1(z){:}\exp\left[-i\sqrt{\frac{2}{N}}\phi(z)\right]{:},\\ J^{0} &= \sqrt{2N}{:}\,\partial_z\phi(z){:}\ .\end{aligned} \tag{2.14}$$

All three currents are of dimension $(1,0)$, and generate the correct operator products for the $su(2)_N$ current algebra by virtue of (2.3) and (2.13) . The primary fields under this algebra are given by the simple relation

$$\Phi^{(j)}_{m,\overline{m}} = \phi^{(2j)}_{[2m,2\overline{m}]} : \exp\left\{ im\sqrt{\frac{2}{N}}\phi(z) + i\overline{m}\sqrt{\frac{2}{N}}\overline{\phi}(\overline{z})\right\} : \qquad (2.15)$$

Correlators of the PF and WZW theories are therefore equivalent up to free boson correlators. The double counting (2.9) of the PF Hilbert space is related to the Z_2 outer automorphism of $su(2)_N$. The quantum numbers of the PF theory are one-half the spin of the associated $su(2)$ fields, and a Weyl reflection, $\overline{m} \to -\overline{m}$, applied to the antiholomorphic part of the representation converts 'order' to 'disorder' in the PF theory.

It is useful to decompose the fields as a product of holomorphic and antiholomorphic parts

$$\phi^{(k)}_{[l,\overline{l}]} = \phi^{(k)}_{l}(z)\overline{\phi}^{(k)}_{\overline{l}}(\overline{z}).$$

Note that both the holomorphic and antiholomorphic fields have the same value of k. More general constraints between the holomorphic and antiholomorphic parts generate other models, as discussed by Gepner and Qiu [16].

3. Coset Bosonization of the PF vertex operators

In this section we will describe how correlation functions of the PF field theory can be extracted from the correlators of $su(N)_1$. There are three basic principles we shall combine. First, the highest weights of the PF theory are imbedded in the $SU(N) \otimes SU(N)$ highests weights. Secondly, the relations between coset model correlation functions developed by Douglas[10] can be applied. Finally, the stress tensor method of Friedan[19] generates simple differential equations for the PF correlators because the stress tensor and parafermions are completely bosonizable.

3.1. Complete Bosonization of PF Current Algebra

Since the central charges of $su(N)_1 \oplus su(N)_1/su(N)_2$ are equivalent to the central charges (2.5) of the PF theories, it is natural to search for a combination of bosonic vertex operators which reproduces the PF algebra (2.3). In fact, this combination has been found by previous authors* [20,21], and also applied by Bernard and Thierry-Mieg [23]. Introduce $(N-1)+(N-1)$ free bosons which satisfy periodic boundary conditions on a $2(N-1)$ dimensional torus, generated by the sum of root lattices of $SU(N)$ and $\widetilde{SU}(N)$. The bosons $\Phi^\mu = \phi^\mu(z) + \overline{\phi}^\mu(\overline{z})$ and $\widetilde{\Phi}^\mu = \widetilde{\phi}^\mu + \widetilde{\overline{\phi}}^\mu(\overline{z})$, $(i = 1, \cdots, N-1)$, have the following two point correlators on the sphere :

$$\begin{aligned} \langle \phi^\mu(z)\phi^\nu(w)\rangle &= -\delta^{\mu\nu}\ln(z-w), \\ \langle \widetilde{\phi}^\mu(z)\widetilde{\phi}^\nu(w)\rangle &= -\delta^{\mu\nu}\ln(z-w), \\ \langle \phi^\mu(z)\widetilde{\phi}^\mu(w)\rangle &= 0. \end{aligned} \tag{3.1}$$

Mixed holomorphic and antiholomorphic two point correlators vanish. We refer the reader to Appendix A for a brief discussion of the bosonization of $SU(N)_1$. The bosonic operator below is found to satisfy the parafermion operator products (normal ordering is implied):

$$\psi_i(z) \propto \sum_{\omega \in [i]} \exp[i\omega \cdot (\phi(z) - \widetilde{\phi}(z))], \tag{3.2}$$

where ω is a weight of the i^{th} basic representation of $su(N)$, which is in the i^{th} conjugacy class of $SU(N)$. In Young Tableaux notation, the representation is denoted by i vertical boxes; the conjugacy class is given by the number of Young Tableaux boxes mod N. The conjugate of the operator is given by interchange of representations $[i] \rightarrow [N-i]$, which changes the signs in the exponential. Normalization of the two point correlator determines the overall coefficient.

⋆ We are grateful to L. Dixon for pointing out reference [21].

A straightforward method of obtaining the stress tensor is to take the ψ_1 and $\psi_1^\dagger$ operator product and to extract the second leading term. (The linear term always vanishes since the elements of each basic representation form symmetric representations centered about the origin.) By comparision with (2.3) we find the holomorphic stress tensor

$$T_{PF}(z) = \frac{1}{N+2} \left\{ -\partial_z \chi \cdot \partial_z \chi + \sum_{\alpha} \exp i\sqrt{2}\alpha \cdot \chi \right\}. \tag{3.3}$$

where $\chi \equiv (\phi - \widetilde{\phi})/\sqrt{2}$, and the sum on α is over roots in the adjoint representation of $su(N)$ with non-vanishing weight. The stress tensor can also be obtained in a different fashion. The diagonal $su(N)_2$ current is given by the sum of the level one currents. The diagonal stress tensor is found via the Sugawara construction; it is proportional to bilinears of the normal ordered diagonal currents. The PF stress tensor is the difference between this and the total. This yields (3.3) up to a factor proportional to cocycles which accompany the currents. We take (3.3) to be the stress tensor consistent at the operator product level with our definition of the parafermions (3.1), and with central charge (2.5).

From these explicitly bosonized representations, it is clear that any correlation function of the operators (3.2) and (3.3) can be calculated on the sphere. On the other hand, these correlators are also simple to calculate without the bosonic representation[8]. This is because the parafermion fusion rules are simple — this type of correlator has only one current block. We will apply these bosonized operators to find correlators for the rest of the operators of the PF theory in section (3.3). We will also use the bosonized version of the parafermion to calculate parafermion correlators on the torus in section (4.1).

3.2. Order Operators and the PF Field Space

We now isolate products of PF fields and $su(N)_2$ fields which combine into bosonized vertex operators. The order and disorder operators of the PF theory are of particular interest. The bosonic operators initially available are the highest weights of the $su(N)_1$ theories, $G^{(1)}_{[k]}$ and $\widetilde{G}^{(1)}_{[k]}$. The index k denotes the k^{th} basic representation and the labeling of the elements of each representation is suppressed. It is useful to define the holomorphic decompositions

$$\begin{aligned} G^{(1)}_{[k]} &= g^{(1)}_{[k]}(z)\overline{g}^{(1)}_{[k]}(\overline{z}), \\ \widetilde{G}^{(1)}_{[k]} &= \widetilde{g}^{(1)}_{[k]}(z)\widetilde{\overline{g}}^{(1)}_{[k]}(\overline{z}). \end{aligned} \tag{3.4}$$

These holomorphic and antiholomorphic operators are exponentials of free bosons

$$\begin{aligned} g^{(1)}_{[k]} &= \exp[i\omega\cdot\phi(z)], \\ \overline{g}^{(1)}_{[k]} &= \exp[i\overline{\omega}\cdot\overline{\phi}(\overline{z})], \end{aligned} \qquad (\omega,\overline{\omega})\in[k]. \tag{3.5}$$

The operators $g^{(1)}_{[k]}$ and $\widetilde{g}^{(1)}_{[k]}$ have conformal dimensions $(\Delta^{(1)}_{[k]},0)$, where

$$\Delta^{(1)}_{[k]} = k(N-k)/2N. \tag{3.6}$$

The operators $\widetilde{g}^{(1)}_{[k]}$ and $\widetilde{\overline{g}}^{(1)}_{[k]}$ are defined as above with replacement of ϕ by $\widetilde{\phi}$, etc. In the diagonal $su(N)_2$ theory, the highest weights , labeled as $G^{(2)}_{[i,j]}$ have the holomorphic decomposition

$$G^{(2)}_{[i,j]} = g^{(2)}_{[i,j]}(z)\overline{g}^{(2)}_{[i,j]}(\overline{z}). \tag{3.7}$$

There is no explicit representation for these holomorphic fields in the bosonic language. The operator $g^{(2)}_{[i,j]}$ has conformal dimension $(\Delta^{(2)}_{[i,j]}, 0)$, where

$$\Delta^{(2)}_{[i,j]} = \frac{N+1}{2N(N+2)}[(i(N-i)+j(N-j)] + \frac{j(N-i)}{N(N+2)}, \quad \begin{matrix} N > i,j \geq 0 \\ i \geq j \end{matrix} \tag{3.8}$$

This is the explicit evaluation of the dimension formula $\Delta(R) = C_2(R)/(N+2)$, where $C_2(R)$ denotes the quadratic Casimir operator of the representation R.

Since the parafermion currents and the stress tensor are bosonized as functions of χ, there exists the symmetry

$$(\phi, \widetilde{\phi}) \rightarrow (-\widetilde{\phi}, -\phi) \tag{3.9}$$

under which (3.2) and (3.3) are invariant. Any two bosonic vertex operators equivalent under this symmetry will have the same properties with respect to the PF current and Virasoro operators. Secondly, Z_N neutrality of non-vanishing correlators is equivalent to Coulomb charge balance in the exponentials of the bosonized operators. Hence Z_N charges of bosonized operators are essentially specified by their coulomb charge. Finally, the parafermion currents are not well defined operators in the $su(N)_1$ theories that the bosons describe, because the differences between left and right momenta do not lie on the root lattice (this constraint is discussed in Appendix A). Therefore, since operators in the PF theory are constructed by operating modes of these currents on highest weights, operators in the PF theory will not in general be well defined operators in the $su(N)_1 \oplus su(N)_1$ model.

The operator products with parafermions provide a way of identifying Z_N charges. Consider the operator product

$$\psi_1(z) G^{(1)}_{[k]}(0,0) \sim z^{-k/N}[G'_{[k]}(0,0) + \cdots], \quad (\omega, \overline{\omega}) \in [k], \tag{3.10}$$

where $G'_{[k]}$ is the excited field onto which the OPE factors. This and the OPE with $\overline{\psi}_1$ imply via (2.10) that $G^{(1)}_{[k]}$ behaves as the order operator σ_k , with charge

$[k,k]$, under the PF current algebra. The OPE of $G^{(1)}_{[k]}$ with T_{PF} shows that it has dimension (d_k, d_k) under the coset Virasoro algebra. Similarly, $G^{(1)}_{[k]}$ behaves as $G^{(2)}_{[k]}$ under the diagonal Virasoro and current algebras. (The sum of the dimensions $\Delta^{(2)}_{[k]}$ and d_k equals $\Delta^{(1)}_{[k]}$.) However, because of the symmetry (3.9) , $\widetilde{G}^{(1)}_{[N-k]}$ has the same properties under the PF algebras. We identify this symmetry as the double counting (2.15) of the PF field space. This leads to the following consistent identifications

$$\begin{aligned} G^{(1)}_{[k]} &= G^{(2)}_{[k]}\phi^{(k)}_{[k,k]} \\ \widetilde{G}^{(1)}_{[k]} &= G^{(2)}_{[k]}\phi^{(k)}_{[-k,-k]}. \end{aligned} \tag{3.11}$$

The rest of the PF field space is generated by applying parafermion currents to these operators. For example, from (2.10) and (2.11), the disorder operator $\mu_1 = \overline{A}^{\dagger}_0\sigma_1$, and has the contour integral representation

$$\mu_1(0,0) = \frac{1}{2\pi i}\oint d\overline{z}\ \overline{z}^{-1/N}\ \overline{\psi}^{\dagger}_1(\overline{z})\sigma_1(0,0) \tag{3.12}$$

Insert $\widetilde{G}^{(2)}_{[1]}(0,0)$ into both sides of (3.12) , and use (3.2) and (3.11) to evaluate the residue of the simple pole. The result is

$$G^{(2)}_{[1]}\mu_1 = \exp\left[i\omega\cdot\phi + i\overline{\omega}\cdot\widetilde{\overline{\phi}}\right], \qquad (\omega,\overline{\omega}) \in [1]. \tag{3.13}$$

The bosonic vertex operators in this case are not $su(N)_1$ fields of the original theory. General formulas for the relation of holomorphic operators in the three theories are

$$\begin{aligned} g^{(1)}_{[k]} &= g^{(2)}_{[k]}\phi^{(k)}_{k}, \\ \widetilde{g}^{(1)}_{[k]} &= g^{(2)}_{[k]}\phi^{(k)}_{-k}. \end{aligned} \tag{3.14}$$

The exponentials $g^{(1)}_{[k]}$ and $\widetilde{g}^{(1)}_{[k]}$ defined by (3.5) have charges $[k,0]$ and $[-k,0]$ respectively.

The self-consistency of these identifications is illustrated by a calculation of order-disorder operator products. We first check the normalization of order operators by interpreting the OPE of (3.11) with its conjugate (summing over elements of the representation $[k]$) as

$$\sum_{(\omega,\overline{\omega})\in[k]} G^{(2)}_{[k]}(z,\overline{z})G^{\dagger(2)}_{[k]}(0,0) = \binom{N}{k}^2 \; z^{-2\Delta^{(2)}_{[k]}} \; \overline{z}^{-2\Delta^{(2)}_{[k]}}[1+\cdots],$$
$$\sigma_k(z,\overline{z})\sigma^{\dagger}_k(0,0) = z^{-2d_k} \; \overline{z}^{-2d_k}[1+\cdots]. \tag{3.15}$$

The order-disorder OPE is embedded in the following OPE of the bosonic theory:

$$\sum_{(\omega,\overline{\omega})\in[k]} \sigma_k(z,\overline{z})G^{(2)}_{[k]}(z,\overline{z})\mu_k(0,0)G^{\dagger(2)}_{[k]}(0,0) = \binom{N}{k} \; \overline{z}^{-2\Delta^{(1)}_{[k]}}[\psi_k(z)\cdots]. \tag{3.16}$$

Application of (3.15) to identify the $G^{(2)}_{[k]}\; G^{\dagger(2)}_{[k]}$ contribution to (3.16) and use of the identity $2\Delta^{(1)}_{[k]} = \Delta_k - d_k$ shows that the correct leading term in the order-disorder OPE (2.7) is obtained. This also generates the normalization factor $Q_k = \binom{N}{k}^{-1}$.

3.3. Correlation Functions in the PF Field Space

We will now argue that correlation functions in PF field space can be calculated in terms of the bosonic correlation functions of $su(N)_1$. We will apply the G/H conformal field theory ideas of Douglas [10] and the stress tensor method of Friedan [19]. We have found above linear relations between the operators of the $G = su(N)_1 \oplus su(N)_1$ and $G/H = PF$ field theories. The G/H conformal field theory relation between correlators of the three theories is for the holomorphic (or antiholomorphic) decomposition. The full correlation function is reconstructed by demanding crossing symmetry, and we will not consider this here. Let $\langle G \rangle$ be a holomorphic correlator in the G theory, $\langle H \rangle$ and $\langle F \rangle$ be those of the H and G/H theories with the holomorphic operators related by the highest weight relations (3.14) , or excitations generated by modes of the PF currents (3.2) acting on

(3.14). Then the correlators are related by

$$\langle G\rangle_A = \langle H\rangle_A^p \langle F\rangle_p. \tag{3.17}$$

The A indices denote the number of group invariants, i.e. channels, that can be constructed from the group indices of the G and H operators. The group invariants are the same for the G and H correlators in our case because of the identification of the highest weights given by (3.11). This is a simplification of the general case discussed by Douglas[10]. The p index, which is summed over, labels the number of primary fields that the H correlator factors into for each channel. The sub-correlator $\langle H\rangle_A^p$ is known as a current block [2]. More general cases have a primary field label for the G correlator but for the bosonized level 1 theory, there is only one primary on which the highest weights factor. The G/H correlator is subsequently labeled by the number of G and H primaries, which in our case is the single index p. If we know $\langle G\rangle_A$ and $\langle H\rangle_A^p$, then $\langle F\rangle_p$ can be determined [10]. But our philosophy will be to attempt to determine $\langle F\rangle_p$ without knowledge of correlators in the H theory.

At our disposal are the explicitly bosonized PF currents (3.2) and stress tensor (3.3). Let $A_{pf}^{(i)}$ be an operator consisting of PF creation operators defined via the OPE's (2.10). They commute with operators of the H theory

$$\langle G\rangle_A^{(i)} = \langle H\rangle_A^p \langle F\rangle_p^{(i)} \tag{3.18}$$

The label (i) denotes correlator obtained via the action of $A_{pf}^{(i)}$ on an initial set of order operators. Different PF correlators are obtained by applying different raising operators onto a set of order operators. If this results in enough different non-vanishing correlators, then (3.18) can be used to determine the matrix $\langle H\rangle_A^p$. We conjecture that this can be achieved; that the number of primaries p in the H theory is equal to the number of independent correlators in the PF theory generated by the action of A_{pf} onto order operator correlators. If there are more group invariants than primaries, then we take a subset of the invariant channels equal in number to the primaries.

Next, if the PF stress tensor (3.3) is inserted into the correlators (3.18), then the derivative with respect to insertion points of the PF correlators is obtained by isolating the single poles in the OPE of the fields with the stress tensor [19]. This is written as

$$\langle DG\rangle_A^{(i)} = \langle H\rangle_A^p \langle \partial_x F\rangle_p^{(i)} \tag{3.19}$$

where x is an insertion point, and $\langle DG\rangle_A^{(i)}$ is the bosonic correlator obtained by isolating the single pole. Then by applying (3.18) to eliminate $\langle H\rangle_A^p$, the following matrix equation for the correlators $\langle F\rangle_A^{(i)}$ in terms of the calculable functions $\langle G\rangle_A^{(i)}$ and $\langle DG\rangle_A^{(i)}$ is found

$$\langle \partial_x F\rangle_p^{(i)} = \langle F\rangle_p^{(j)} {\langle G\rangle_A^{(j)}}^{-1} \langle DG\rangle_A^{(i)} \tag{3.20}$$

This is a useful equation because it generates differential equations generically of order equal to the number of primaries p for each of the correlators $\langle F\rangle_p^{(i)}$, in terms of calculable bosonic correlators of the G theory.

As an example, consider the four-pt. function

$$\langle \phi_1^{(1)}(z_1)\phi_1^{\dagger(1)}(z_2)\phi_1^{(1)}(z_3)\phi_1^{\dagger(1)}(z_4)\rangle \tag{3.21}$$

This is the holomorphic part of the correlator $\langle \sigma_1\sigma_1^\dagger\sigma_1\sigma_1^\dagger\rangle$. The fusion rule for the field $\phi_1^{(1)}$ and its conjugate is

$$[\phi_1^{(1)}][\phi_1^{\dagger(1)}] = [I] + [\phi_0^{(2)}], \tag{3.22}$$

where $\phi_0^{(2)}$ is the holomorphic part of the energy operator $\epsilon^{(1)}$ [8]. These two primary field channels are matched with the identity and adjoint of the level 2 theory.

Hence two different sets of correlator relations are required, one of which must contain (3.21) as the G/H factor. Two relevant equations are

$$\begin{aligned}\langle G\rangle^{(1)}_A &= \langle g^{(1)}_{[1]} g^{\dagger(1)}_{[1]} g^{(1)}_{[1]} g^{\dagger(1)}_{[1]}\rangle_A = \langle g^{(2)}_{[1]} g^{\dagger(2)}_{[1]} g^{(2)}_{[1]} g^{\dagger(2)}_{[1]}\rangle^p_A \langle \phi^{(1)}_1 \phi^{\dagger(1)}_1 \phi^{(1)}_1 \phi^{\dagger(1)}_1\rangle_p,\\ \langle G\rangle^{(2)}_A &= \langle g^{(1)}_{[1]} g^{\dagger(1)}_{[1]} \tilde{g}^{(1)}_{[1]} \tilde{g}^{\dagger(1)}_{[1]}\rangle^p_A = \langle g^{(2)}_{[1]} g^{\dagger(2)}_{[1]} g^{(2)}_{[1]} g^{\dagger(2)}_{[1]}\rangle^p_A \langle \phi^{(1)}_1 \phi^{\dagger(1)}_1 \phi^{(1)}_{-1} \phi^{\dagger(1)}_{-1}\rangle_p.\end{aligned} \tag{3.23}$$

The label A runs over the two 's' and 't' channel group invariants:

$$\begin{aligned} I_1 &= \delta_{\omega_1-\omega_2}\delta_{\omega_3-\omega_4},\\ I_2 &= \delta_{\omega_1-\omega_4}\delta_{\omega_2-\omega_3},\end{aligned} \tag{3.24}$$

where ω_i is the weight of the field at z_i. Define the anharmonic quotient:

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}. \tag{3.25}$$

The bosonic four point correlation functions can be written as

$$\langle G\rangle^{(i)}_A = [(z_1-z_3)(z_2-z_4)]^{-2\Delta^{(1)}_{[1]}} \langle G(x)\rangle^{(i)}_A, \tag{3.26}$$

where $\langle G(x)\rangle^{(i)}_A$ is a function of the anharmonic quotient only. Similarly, we define the matrix

$$\langle DG\rangle^{(i)}_A = [(z_1-z_3)(z_2-z_4)]^{-2\Delta^{(1)}_{[1]}} \langle DG(x)\rangle^{(i)}_A. \tag{3.27}$$

We find the following by straightforward calculation:

$$\langle G(x)\rangle^{(i)}_A = (x(1-x))^{1/N} \begin{pmatrix} \frac{1}{x} & \frac{1}{1-x} \\ \frac{1}{x(1-x)^{1/N}} & 0 \end{pmatrix}, \tag{3.28}$$

and

$$\langle DG(x)\rangle^{(i)}_A = \frac{-(x(1-x))^{1/N}}{N(N+2)} \begin{pmatrix} \frac{1}{x(1-x)} + \frac{N-1}{x^2} & -\frac{1}{x(1-x)} - \frac{N-1}{(1-x)^2} \\ y^{-1/N}(\frac{N-1}{x^2} - \frac{1}{x(1-x)}) & y^{-1/N}\frac{N}{x(1-x)} \end{pmatrix}. \tag{3.29}$$

The indices (i) and A label row and column respectively. Insert these matrices into

(3.20) to find the desired differential equation for $\langle F\rangle_p^{(1)}$:

$$\left\{\frac{N(N+2)}{2(N+1)}\partial_x{}^2+\frac{N(N+2)-2}{2(N+1)}\left(\frac{1}{x}-\frac{1}{1-x}\right)\partial_x-d_1\frac{1}{x^2(1-x)^2}\right\}\langle F\rangle_p^{(1)}=0, \tag{3.30}$$

where d_1 is the dimension (2.6) of the order operator σ_1, and the two solutions correspond to the two primaries labeled by p. The $N=2$ case of (3.30) was originally found in [7] ; solutions to (3.30) are given in [8].

Note that the G/H Ising model case is special because the $SU(2)_2$ H theory is an Ising model plus a free boson. Then the G correlators yield squares of Ising correlators, up to trivial $U(1)$ factors. Hence it is possible to generate the square of Ising model correlators directly. This is equivalent to the approach taken by Boyanovsky[24].

4. Differential Equations for PF Characters

4.1. How to Generate a Differential Equation

The partition functions for the Z_N parafermion theories can be calculated by relating them to characters $\chi_l^{(N)}(\tau)$ of the spin $l/2$ representations of $su(2)_N$, which expresses the number of states generated by holomorphic current creation operators acting on a highest weight state with spin $l/2$. String functions c_m^l , $(l-m\in 2Z)$ are defined by the relation[14]

$$\chi_l^{(N)}(\tau)=\sum_{m=-N+1}^{N}c_m^l(\tau)\Theta_{m,N}(\tau). \tag{4.1}$$

This expression is a decomposition of the character under the infinite dimensional Weyl group of $su(2)_N$; the theta function contains all of the translational subgroup and corresponds to the $U(1)$ current subalgebra of $su(2)_N$. String functions are the holomorphic building blocks of the PF partition functions. More precisely, the combination

$$Z_m^{(l)} = \eta(\tau) c_m^l(\tau) \tag{4.2}$$

are the PF characters; how to generate the PF partition functions from (4.2) is discussed in [16]. (The functions $\Theta_{m,N}$ and $\eta(\tau)$ are discussed in Appendix A.) The PF characters (4.2) describe the number of states in each sector, originating from highest weight σ_l and with Z_{2N} holomorphic charge m , by the operation of holomorphic creation operators in the theory. They admit two symmetries

$$\begin{aligned} Z_m^{(l)} &= Z_{-m}^{(l)} \\ Z_m^{(l)} &= Z_{k+m}^{(k-l)} \end{aligned} \tag{4.3}$$

Self-duality of the PF theory generates the first and the overcounting (3.9) corresponds to the second. Equivalently, by relating the PF theory to $su(2)_N$, the first is $SU(2)$ Weyl symmetry of the spectrum and the second is due to the Z_2 outer automorphism of $su(2)_N$. In addition, the charges are defined only mod $2N$ –

$$Z_m^{(l)} = Z_{m+2N}^{(l)}. \tag{4.4}$$

It is also useful to define $Z_m^{(l)} \equiv 0$ if $l - m$ is odd.

One method to determine WZW characters is to apply the Kac-Weyl character formula [12]. The characters also satisfy differential equations in the modular parameter τ, as discussed in [25]. Via the GKO construction,they are related to the conformal characters of the minimal models [26], which also satisfy differential equations [27]. In this section we will find differential equations for the PF characters directly.

The essential tool for the analysis is the two-point function of parafermions on the torus as determined by conformal field theory techniques. Because the parafermions are explicitly bosonized, this correlator can be calculated exactly in the $SU(N)_1 \otimes \widetilde{SU}(N)_1$ vacua, which are labeled by the highest weights of each of the $SU(N)$. The bosonic parafermion operators are not well defined vertex operators in the level 1 theories; we calculate the holomorphic square of the correlator, because the holomorphic square of the parafermion is a well defined operator in the level 1 theories. After the classical contributions (windings about the lattice) are properly included and a Poisson resummation on one set of the windings is performed, the resulting function is the sum over highest weights of holomorphic squares. We interpret the holomorphic square root in each sector as the parafermion correlator in a given level 1 vacuum sector:

$$\langle \psi_1(z)\psi_1^\dagger(0)\rangle_{i,j} = F(\tau)\vartheta^{-2\Delta_1}(z)\sum_{\omega\in[1]}\Theta_{[i];1}(-\omega z,\tau)\Theta_{[j];1}(+\omega z,\tau). \qquad (4.5)$$

The notation $[i]$ denotes the i^{th} basic representation of $su(N)_1$ ($[j]$ denotes the representation of $\widetilde{SU}(N)_1$), and the theta functions ϑ, and $\Theta_{[i];1}$ are defined in Appendix A. The correlator (4.5) factors onto the product of characters $\chi^{(1)}_{[i]}\tilde{\chi}^{(1)}_{[j]}$ with singularity $z^{-2\Delta_1}$ as $z \to 0$.

How is this used to generate differential equations? Because of the coset construction, the level 1 vacua are a sum of products of $su(N)_2$ and PF vacua :

$$\chi^{(1)}_{[i]}\tilde{\chi}^{(1)}_{[j]} = A^{k,l}_{i,j}\chi^{(2)}_{[k]}Z_l. \qquad (4.6)$$

The integer valued matrix A will be determined below. As $z \to 0$, the leading term of (4.5) factors onto (4.6). The second leading term of the $\psi_1\psi_1^\dagger$ OPE (2.3) factors onto the PF stress tensor, which on the torus is the modular derivative of the PF partition function [19, 22]. Therefore, the second leading term is of the form

$$\langle \psi_1(z)\psi_1^\dagger(0)\rangle''_{i,j}|_{z=0} \simeq A^{k,l}_{i,j}\chi^{(2)}_{[k]}\partial_\tau Z_l. \qquad (4.7)$$

We will show that in general (4.6) eliminates $\chi^{(2)}_{[k]}$ from (4.7). This generates a differential equation for the PF functions Z_l. First we illustrate this with the Z_2 (Ising) and Z_3 cases.

4.2. Z_2 and Z_3 Character Relations

The relationships between characters of $SU(2)_1$, $SU(2)_2$ and Z_2 theories are given in [9], since the Ising model is the first in the minimal conformal series. To simplify the analysis, it is useful to define

$$Q^l_m(\tau) = Z^l_m(\tau)\eta^c(\tau) \tag{4.8}$$

where c is the central charge (2.5) of the Z_N models. The differential equations for Q^l_m will simplify and involve only level one $SU(N)$ theta functions we denote as $\Theta_{[i]}$, suppressing the level 1 subscript. The $su(N)_1$ characters are given by $\chi^{(1)}_{[i]} = \Theta_{[i]}/\eta^{N-1}$, up to an offset in the vacuum energy which is of no concern. The factors of $\eta(\tau)$ in these characters and in the definition (4.8) can be absorbed in a redefinition of the level 2 characters. For the Ising model, the relation between characters, reexpressed in terms of the Q^l_m, is given as

$$\begin{aligned}
\Theta_{[0]}\Theta_{[0]} &= \hat{\chi}^{(2)}_{[0]}Q^{(0)}_0 + \hat{\chi}^{(2)}_{[1,1]}Q^{(2)}_0,\\
\Theta_{[1]}\Theta_{[0]} &= \hat{\chi}^{(2)}_{[1]}Q^{(1)}_1,\\
\Theta_{[1]}\Theta_{[1]} &= \hat{\chi}^{(2)}_{[1,1]}Q^{(0)}_0 + \hat{\chi}^{(2)}_{[0]}Q^{(2)}_0.
\end{aligned} \tag{4.9}$$

The characters $Z^{(0)}_0$, $Z^{(2)}_0$, and $Z^{(1)}_1$ have highest weights I, ψ, and σ respectively. The second term of the $\psi_1\psi_1^\dagger$ OPE (2.3) factors onto the stress tensor. For the flat, doubly periodic torus defined by the periods 1 and τ in the complex plane, the unnormalized stress tensor one point function is the modular derivative of the partition function[19, 22]:

$$\langle T(z)\rangle = 2\pi i\partial_\tau Z(\tau,\overline{\tau}). \tag{4.10}$$

When the partition function decomposes into a sum of a holomorphic square of characters, then the expectation value of the stress tensor in a given sector (corresponding to a character) is the modular derivative of the character of that sector.

By applying this to the second term of (4.5) for the Z_2 case, the following equations are generated

$$\begin{aligned}
\tfrac{1}{4}(\Theta_{[0]}\Theta_{[0]})' &= \hat{\chi}^{(2)}_{[0]} Q'^{(0)}_0 + \hat{\chi}^{(2)}_{[1,1]} Q'^{(2)}_0 , \\
\tfrac{1}{4}(\Theta_{[1]}\Theta_{[0]})' &= \hat{\chi}^{(2)}_{[1]} Q'^{(1)}_1 , \\
\tfrac{1}{4}(\Theta_{[1]}\Theta_{[1]})' &= \hat{\chi}^{(2)}_{[1,1]} Q'^{(0)}_0 + \hat{\chi}^{(2)}_{[0]} Q'^{(2)}_0 ,
\end{aligned} \tag{4.11}$$

where $'$ denotes ∂_τ. Note that the $\Theta_{[0]}\Theta_{[1]}$ equations are redundant because $Z_1^{(1)}{=}Z_{-1}^{(1)}$. The equations (4.9) and (4.11) generate first order differential equations for the Q's which are solved by inspection. The results are –

$$\begin{aligned}
Q_0^{(0)} &\propto \tfrac{1}{2}\left[({\Theta_{[0]}}^2 + {\Theta_{[1]}}^2)^{1/4} + ({\Theta_{[0]}}^2 - {\Theta_{[1]}}^2)^{1/4}\right], \\
Q_1^{(1)} &\propto (\Theta_{[0]}\Theta_{[1]})^{1/4}, \\
Q_0^{(2)} &\propto \tfrac{1}{2}\left[({\Theta_{[0]}}^2 + {\Theta_{[1]}}^2)^{1/4} - ({\Theta_{[0]}}^2 - {\Theta_{[1]}}^2)^{1/4}\right],
\end{aligned} \tag{4.12}$$

Arbitrary integration constants are determined by demanding that the $Z_0^{(0)}$ highest weight is a singlet, and that the $Z_1^{(1)}$ and $Z_0^{(2)}$ highest weights are doubly degenerate; they correspond to the states (σ, μ) and $(\psi, \psi^\dagger)$ respectively.

Products of $SU(2)_1$ theta functions can be reexpressed as products of Riemann theta functions by redefining variables in the definitions of the functions. The relations relevant to the functions (4.12) are –

$$\begin{aligned}
\Theta_{[0]}\Theta_{[1]} &= \left(\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}\right)^2 , \\
{\Theta_{[0]}}^2 + {\Theta_{[1]}}^2 &= 2\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right)^2 , \\
{\Theta_{[0]}}^2 - {\Theta_{[1]}}^2 &= 2\left(\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}\right)^2 ,
\end{aligned} \tag{4.13}$$

Applying these to (4.12) leads to the standard results.

The rules which govern the combination of characters become clear when the Z_3 case is displayed. Again, some factors of $\eta(\tau)$ are absorbed into $\chi^{(2)}_{[i]}$ –

$$
\begin{aligned}
\Theta_{[0]}\Theta_{[0]} &= \hat{\chi}^{(2)}_{[0]}Q^{(0)}_0 + \hat{\chi}^{(2)}_{[2,1]}Q^{(2)}_0, \\
\Theta_{[2]}\Theta_{[1]} &= \hat{\chi}^{(2)}_{[0]}Q^{(3)}_1 + \hat{\chi}^{(2)}_{[2,1]}Q^{(1)}_1, \\
\Theta_{[1]}\Theta_{[0]} &= \hat{\chi}^{(2)}_{[1,0]}Q^{(1)}_1 + \hat{\chi}^{(2)}_{[2,2]}Q^{(3)}_1, \\
\Theta_{[2]}\Theta_{[2]} &= \hat{\chi}^{(2)}_{[1,0]}Q^{(2)}_0 + \hat{\chi}^{(2)}_{[2,2]}Q^{(0)}_0,
\end{aligned}
\tag{4.14}
$$

The characters $Z^{(0)}_0$, $Z^{(2)}_0$, $Z^{(3)}_1$, and $Z^{(1)}_1$ correspond to the highest weights I, ϵ_1, ψ, and σ_1 respectively. (The field $\epsilon^(1)$ is a Z_N neutral field of dimension 2/5 discussed in [8]). Therefore, (4.14) is two decoupled systems consisting of two equations with two unknowns. This will lead to second order differential equations for the Q's. But since the information for the general case is at hand, further discussion of Z_3 will be differed until the general structure is obtained.

First note that many combinations of level one characters yield redundant information; label each combination of $\Theta_{[i]}\Theta_{[j]}$ by $n = i + j \bmod N$ $(0 \leq n < N)$. Then the products in the $(N-n)^{th}$ class yield the same information as the n^{th}, by the identity $\Theta_{[i]} = \Theta_{[N-i]}$, and the relations (4.3) and (4.4) . Second, in the Z_2 and Z_3 examples above, the level 2 highest weights (and associated characters) which appear are in the product of the level 1 highest weights or their descendants. Therefore they belong to the same conjagacy class n as the product of level 1 weights. All terms in the products appear which generate unitary representations of $su(N)_2$. (This restricts the highest weights to one or two columns of Young Tableaux.) They are multiplied by characters of the PF theory, such that the sum of conformal dimensions of the level-2 and PF highest weights equals the sum of level 1 highest weights, mod Z. Third, the Z_{2N} charge of the PF characters equals the sum of the charges of the level 1 highest weights. (This charge is the difference $i - j$ mod N.) Finally, integer-valued coefficients which in principle multiply the products of level-2 and PF characters are all unity. This is surprising because the highest weight of level-2 characters appears more than once in the product of level 1 states. In these cases however, the PF characters have degenerate highest weights which account for this.

4.3. Character Relations and Differential Equations

By the above discussion, the general case is deduced by including all level-2 characters whose highest weights appear in the product of level 1 states, and matching these with PF characters with correct charges and dimensions. The general relation is

$$\begin{aligned}\Theta_{[i]}\Theta_{[n+\kappa N-i]} = &\sum_{2n\geq 2\alpha\geq n} \hat{\chi}^{(2)}_{[\alpha,n-\alpha]} Q^{(2\alpha-n)}_{2i-n} \\ &+ \sum_{2\alpha\geq n+N} \hat{\chi}^{(2)}_{[\alpha,n+N-\alpha]} Q^{(2(N-\alpha)+n)}_{2i-n},\end{aligned} \tag{4.15}$$

where $\kappa = 0$ if $2n \geq 2i \geq n$ and $\kappa = 1$ if $2i \geq n + N$. For the second case, the Z_{2N} charge of Q is converted to $2i - n - N$ by application of (4.3). The sum over α is over all level 2 highest weights in the conjagacy class n. That they all appear is discussed in [28], and we have verified this for simple cases. Note that $i \geq j$ in this construction since the rest of the equations are redundant.

Let β be the Z_{2N} charge of the PF characters, and n be the conjugacy class of the product $\Theta_{[i]}\Theta_{[j]}$. Then

$$\begin{aligned} i &= \frac{\beta+n}{2} + \ell^n_\beta(N-\beta), \\ j &= \frac{n-\beta}{2} + \ell^n_\beta \beta, \end{aligned} \tag{4.16}$$

for all $N \geq \beta \geq 0$ such that $\beta - n \in 2Z$. The integer ℓ^n_β is defined as

$$\begin{aligned} \ell^n_\beta &= 0 \quad n \geq \beta \geq 0, \\ \ell^n_\beta &= 1 \quad N \geq \beta > n. \end{aligned} \tag{4.17}$$

Since the products of theta functions and the level-2 characters are specified by n and β , introduce the notation

$$\begin{aligned} \Lambda^n_\beta &\equiv \Theta_{[i]}\Theta_{[j]}, \\ \chi^n_\beta &\equiv \hat{\chi}^{(2)}_{[i,j]}, \end{aligned} \tag{4.18}$$

with (i,j) given by (4.16) . The character relations (4.15)are equivalent to

$$\Lambda^n_\beta = \sum_{N \geq \gamma \geq 0} \chi^n_\gamma Q^{(\gamma)}_\beta. \tag{4.19}$$

These are defined only for $\beta - n \in 2Z$ and $\gamma - n \in 2Z$. It is useful to define Λ^n_β, χ^n_β to be zero if $n - \beta$ is odd. Because of the symmetry $n \rightarrow N - n$, we restrict the conjugacy class n such that $N \geq 2n \geq 0$ with no loss of information.

It is not clear that we have enough data to solve for χ^n_β via (4.19) and generate a differential equation for Q in terms of the theta functions, because the domain of β is greater than that of n. The solution to this puzzle is to apply the relations (4.3) to the Q's , thus reducing the system to matrix equations. The analysis leads naturally to the following quantities –

$$\begin{aligned} \Lambda^n_{\pm\beta} &\equiv \Lambda^n_\beta \pm \Lambda^n_{N-\beta}, \\ \chi^n_{\pm\beta} &\equiv \chi^n_\beta \pm \chi^n_{N-\beta}. \end{aligned} \tag{4.20}$$

The point of these definitions is $\Lambda_\pm$ and $\chi_\pm$ for $N \geq 2\beta \geq 0$, contain all the information of Λ and χ for $N \geq \beta \geq 0$. Note though , that (4.20) may in certain cases vanish identically ; this is true for N even and $n-\beta$ odd. This leads to different cases that will be discussed below. Similarly, we define

$$\begin{aligned} Q^{(\gamma)}_{\pm\beta} &\equiv Q^{(\gamma)}_{\beta} \pm Q^{(\gamma)}_{N-\beta} \quad (N > 2\gamma \geq 0), \\ Q^{(\gamma)}_{+\beta} &\equiv \tfrac{1}{2}(Q^{(\gamma)}_{\beta} + Q^{(\gamma)}_{N-\beta}) \quad (2\gamma = N). \end{aligned} \tag{4.21}$$

These definitions satisfy the matrix equations

$$\begin{aligned} \Lambda^n_{+\beta} &= \chi^n_{+\gamma} Q^{(\gamma)}_{+\beta} \quad N \geq 2n, 2\beta, 2\gamma \geq 0, \\ \Lambda^n_{-\beta} &= \chi^n_{-\gamma} Q^{(\gamma)}_{-\beta} \quad N > 2n, 2\beta, 2\gamma \geq 0. \end{aligned} \tag{4.22}$$

A sum over γ is implied. The second term in the parafermion two-point function generates the equations

$$\begin{aligned} \tfrac{1}{N+2}\Lambda'^n_{+\beta} &= \chi^n_{+\gamma} Q'^{(\gamma)}_{+\beta} \quad N \geq 2n, 2\beta, 2\gamma \geq 0, \\ \tfrac{1}{N+2}\Lambda'^n_{-\beta} &= \chi^n_{-\gamma} Q'^{(\gamma)}_{-\beta} \quad N > 2n, 2\beta, 2\gamma \geq 0. \end{aligned} \tag{4.23}$$

We have applied all the symmetries of the level 1 and PF characters ; (4.22) is used to solve for $\chi_{\pm}$ in (4.23) . Hence there is enough information in the two-point correlator of parafermions on the torus to find differential equations for the PF characters in the general case. The final equation clearly takes the form

$$Q_{\pm}\Lambda'_{\pm} = (N+2)\Lambda_{\pm}Q'_{\pm}. \tag{4.24}$$

For manifestly integrable cases when the rank of $\Lambda_{\pm}$ is one, the solution is clearly $Q \propto \Lambda^{1/(N+2)}$. Note that (4.24) generates a power series solution in the variable $q = e^{2\pi i\tau}$ for all characters , given the dimensions of the highest weights as initial condition information. We now consider the different cases and determine the order of the differential equation equation for each element, given by the rank of the matrix.

N Odd

By observation of (4.20) the matrices $\Lambda_\pm$, $\chi_\pm$ and $Q_\pm$ never generically vanish when N is odd. Each matrix is square and of the form A^α_β, with $\alpha = 0, 1, \cdots, (N-1)/2$. Hence the order of the differential equation for each element, given by the rank of the matrix, is $(N+1)/2$.

The lowest order example is the Z_3 case, with the Q matrices given by –

$$Q_\pm = \begin{pmatrix} Q^{(0)}_0 & \pm Q^{(0)}_2 \\ \pm Q^{(1)}_3 & Q^{(1)}_1 \end{pmatrix} \tag{4.25}$$

The differential equation (4.24) is specified by the Λ matrices

$$\Lambda_\pm = \begin{pmatrix} {\Theta_{[0]}}^2 & \pm\Theta_{[1]}\Theta_{[2]} \\ \pm{\Theta_{[2]}}^2 & \Theta_{[0]}\Theta_{[1]} \end{pmatrix} \tag{4.26}$$

This system can be converted into ordinary second order differential equations in a single variable. First note that $\Theta_{[2]} = \Theta_{[1]}$ for $SU(3)$. Then rewrite $\Lambda_\pm$ as

$$\Lambda_\pm = {\Theta_{[1]}}^2 \begin{pmatrix} s^2 & \pm 1 \\ \pm 1 & s \end{pmatrix} \equiv {\Theta_{[1]}}^2 M_\pm \tag{4.27}$$

where $s = \Theta_{[0]}/\Theta_{[1]}$. The overall factor ${\Theta_{[1]}}^2$ leads to the redefinition – $Q_\pm = {\Theta_{[1]}}^{2/5} R_\pm$. Then let $R_\pm = R_\pm(s)$. The factor of $ds/d\tau$ cancels on both sides of (4.24)and an ordinary differential equation in the variable s results –

$$R_\pm \frac{d}{ds} M_\pm = 5 M_\pm \frac{d}{ds} R_\pm . \tag{4.28}$$

This clearly generates two second order differential equations which after futher studies yield series solutions.

N Even and n Even

The matrix $\Lambda^n_{\pm\beta}$ vanishes when $n-\beta$ is odd Hence the matrices in (4.24) are of the form $A^{2\alpha}_{2\beta}$. If $N/2$ is odd, the dimension is given by $(N+2)/4$ for both Λ_+ and Λ_-. The first example in this series is the Z_2 sector

$$\begin{aligned} Q_\pm &= Q_0^{(0)} \pm Q_2^{(0)}, \\ \Lambda_\pm &= {\Theta_{[0]}}^2 \pm {\Theta_{[1]}}^2 \end{aligned} \tag{4.29}$$

The other half of the Z_2 theory is in the n odd case. If $N/2$ is even, then the dimension of the matrix $\Lambda_\pm$ is given by $(N+2)/4 \pm 1/2$. The first in the series is the Z_4 case –

$$\begin{aligned} Q_+ &= \begin{pmatrix} Q_0^{(0)} + Q_4^{(0)} & 2Q_2^{(0)} \\ \frac{1}{2}(Q_0^{(2)} + Q_4^{(2)}) & Q_2^{(2)} \end{pmatrix} \\ Q_- &= Q_0^{(0)} - Q_4^{(0)} \end{aligned} \tag{4.30}$$

with Λ matrices –

$$\begin{aligned} \Lambda_+ &= \begin{pmatrix} {\Theta_{[0]}}^2 + {\Theta_{[2]}}^2 & 2\Theta_{[1]}\Theta_{[3]} \\ 2{\Theta_{[1]}}^2 & 2\Theta_{[0]}\Theta_{[2]} \end{pmatrix} \\ \Lambda_- &= {\Theta_{[0]}}^2 - {\Theta_{[2]}}^2 \end{aligned} \tag{4.31}$$

N Even and n Odd

The matrix $\Lambda^n_{\pm\beta}$ vanishes when $n-\beta$ is odd so the matrices in (4.24) are of the form $A^{2\alpha+1}_{2\beta+1}$. If $N/2$ is odd, the dimensions are $(N\pm 2)/4$. The first example in this series is the Z_2 case –

$$\begin{aligned} Q_+ &= Q_1^{(1)}, \\ \Lambda_+ &= 2\Theta_{[0]}\Theta_{[1]}. \end{aligned} \tag{4.32}$$

The second member of this series, Z_6 , has an integrable equation for Λ_-

$$Q_- = Q_1^{(1)} - Q_5^{(1)},$$
$$\Lambda_- = \Theta_{[0]}\Theta_{[1]} - \Theta_{[3]}\Theta_{[4]}. \tag{4.33}$$

The rest of the Z_6 system (Λ_+ for n odd and $\Lambda_\pm$ for n even) has rank two matrix structure.

Finally, if $N/2$ is even, the dimension of the matrices is $N/4$ and the lowest representative of this series is the Z_4 odd sector

$$Q_\pm = Q_1^{(1)} \pm Q_3^{(1)},$$
$$\Lambda_\pm = \Theta_{[0]}\Theta_{[1]} \pm \Theta_{[2]}\Theta_{[3]}. \tag{4.34}$$

This completes the analysis of the different cases. Only the Z_2 system is completely integrable (first order). The Z_4 and Z_6 systems are the only partially integrable systems.

5. Discussion

The process of extracting information about a CFT via coset bosonization is clearly not unique to the case of Z_N parafermions we have discussed. To clarify the general structure of this method, we offer the following prescription:

Let G be a bosonizable theory with current algebra we also label G, and H the theory of the closed subalgebra H.

<u>A. Find the chiral algebra fields of G/H.</u> For the PF case, these are given by ψ_i and T_{PF}. We expect that such fields can in general be completely bosonized because their simple fusion rules can be reproduced by bosonic vertex operators.

B. Identify relations between primaries of the chiral algebras. Begin with the primaries under G and deduce their coset decomposition by considering their charges under the coset chiral algebra fields.

C. Understand the fusion rules of H and G/H fields. This fundamental information of the H and G/H theories, is required to formulate the coset relations for correlation functions.

D. Calculate correlation functions $\langle F \rangle_p^{(i)}$, via the differential equation (3.20).

E. Determine the relations between characters of the theories. Use symmetries of G and H and detailed analysis of highest weight relations (i.e. charge conservation and level matching).

F. Factor chiral field correlators on the torus. This generates differential equations for the characters. With no current algebra in the G/H theory, the second leading term of neutral non-vanishing chiral field two-pt functions factor onto the stress-tensor, which is the modular derivative of the character of the particular sector.

This prescription immediately applies to the cases $G_k \subset (G_1)^k$, where G_k is the level k WZW model of the semi-simple group G[30]. The currents of G_k are the diagonal sum of the level 1 currents and the stress tensor is obtained via the Sugawara construction. In particular, the roles of H and G/H can be interchanged in our construction to discuss aspects of $SU(N)_2$. We also know of a bosonic representation for the $SU(3)_2/U(1)$ parafermion, for which this formalism can be applied. It is clear that elements of this prescription can be combined to calculate G/H correlators on the torus, and that a formulation of these ideas should be possible for arbitrary genus surfaces. We hope that this construction can supplement other techniques and shed additional light on the structure of the space of CFT's.

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APPENDIX A

Bosonization of $su(N)_1$

The diagonal $su(N)_1$ theory is represented by an even Lorentzian self-dual lattice [29] compactifcation. It is equivalently formulated on a conventional $N-1$ dimensional torriodal compactification [31] with constant background metric and antisymmetric tensor field, with action

$$S = \frac{1}{2\pi}\int d^2z[\partial_z\Phi^\mu\partial_{\bar{z}}\Phi^\mu + B_{\mu\nu}\partial_z\Phi^\mu\partial_{\bar{z}}\Phi^\nu]. \tag{A.1}$$

We now define the constant background metric g_{ij} and antisymmetric tensor field $B_{\mu\nu}$, following Ginsparg[32]. Consider the root lattice Λ_R of $SU(N)$ generated by the simple roots e_i^μ, where $i = 1,\cdots,N-1$ labels the roots and $\mu = 1,\cdots,N-1$ denotes the euclidean component of each root. The dual vectors e_μ^{*i} satisfy $e_\mu^{*i}e_j^\mu = \delta_j^i$ and generate the weight lattice Λ_W. The metric is given by $g_{ij} = e_i^\mu e_j^\mu/4 = A_{ij}/4$ where A_{ij} is the Cartan matrix of $SU(N)$. Spacetime vielbeins are $\frac{1}{2}$ the simple roots. Points Φ^μ and $\Phi^\mu + 2\pi n^i(\frac{1}{2}e_i^\mu)$ are identified to form the spacetime torus. The constant antisymmetric field is defined as $B_{\mu\nu} = b_{ij}(2e_\mu^{*i})(2e_\nu^{*j})$, where

$$b_{ij} = \begin{cases} g_{ij} & i<j, \\ 0 & i=j\ , \\ -g_{ij} & i>j\ . \end{cases} \tag{A.2}$$

One method of obtaining the spectrum of the model is by the path integral approach on the world sheet torus. Represent the torus by the parallelogram defined by the vectors 1 and $\tau = \tau_1 + i\tau_2$ in the complex plane, where $\tau_2 > 0$. Two canonical homology cycles A and B correspond to motions in the 1 and τ directions. The zero modes $\Phi^\mu = C^\mu z + \overline{C}^\mu\bar{z}$, where C_μ and $\overline{C}^\mu$ are constants, must undergo shifts by spacetime lattice vectors when carried about the cycles

$$\oint_A C^\mu + \overline{\oint_A C^\mu} = \pi n^i e_i^\mu,$$
$$\oint_B C^\mu + \overline{\oint_B C^\mu} = \pi m^i e_i^\mu. \tag{A.3}$$

The solutions to (A.3) are given by

$$C^\mu = i\pi[n_i\overline{\tau} - m_i]\tfrac{1}{2}e_i^\mu. \tag{A.4}$$

The classical contribution to the partition function is $Z_{cl} = \sum_{nm} exp[-S_{nm}]$ where S_{nm} is the classical action of the solution (A.4). By application of the Poisson resummation formula to the lattice vector $m^i e_i^\mu/2$, the classical partition function becomes

$$Z_{cl} = \sum_{\gamma_L,\gamma_R} \exp(i\pi\overline{\tau}\gamma_L^2 - i\pi\tau\gamma_R^2). \tag{A.5}$$

The sums on left and right momenta γ_L and γ_R are over the integers (n^i, p_j) where p_j is introduced via the Poisson resummation

$$\gamma_L^\mu = [p_i - \tfrac{1}{2}(b_{ij} - g_{ij})n^j]e_\mu^{*i},$$
$$\gamma_R^\mu = [p_i - \tfrac{1}{2}(b_{ij} + g_{ij})n^j]e_\mu^{*i}. \tag{A.6}$$

The lattice (γ_L, γ_R) is Lorentzian even and self dual, which follows from $\gamma_L - \gamma_R \in \Lambda_R$ and $\gamma_L + \gamma_R \in \Lambda_W$.

Theta functions of $su(N)_k$ are given by

$$\Theta_\lambda(z,\tau) = \sum_{\gamma\in\Lambda_R+\lambda/k} \exp[i\pi k\tau\gamma^2 - 2\pi i k\gamma\cdot z], \tag{A.7}$$

where λ is a weight vector. The notation $\Theta_{[i]}(z)$ is used to denote the $su(N)_1$ theta function with $\lambda \in [i]$, where $[i]$ denotes the i^{th} basic representation. The classical partition function (A.5) is given by

$$Z_{cl} = \sum_{i=0}^{N-1} \Theta_{[i]}\overline{\Theta_{[i]}}. \tag{A.8}$$

The quantum contribution to the partition function is easily found via the stress tensor method. The two point function for the free bosons on the torus is given by

$$\langle \Phi^\mu(z,\overline{z})\Phi^\nu(0,0)\rangle = -\delta^{\mu\nu}\big[\ln|\vartheta(z)/\eta|^2 - 2\pi(\mathrm{Im}z)^2/\tau_2\big]. \tag{A.9}$$

The theta function $\vartheta = \vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]$ is a Riemann theta function on the torus with zero at $z = 0$. The general definition of these functions on the torus is

$$\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z|\tau) = \sum_{n\in Z} \exp\big[i\pi\tau(n+a)^2 + i2\pi(n+a)(z+b)\big]. \tag{A.10}$$

The function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. The stress tensor for the bosonic theory is

$$T(z) = -\tfrac{1}{2}\!:\partial_z\Phi^\mu\partial_z\Phi^\mu\!: \tag{A.11}$$

By taking derivatives of (A.9) , the quantum expectation value of the stress tensor can be isolated; subsequent application of (4.10) generates the modular derivative of the quantum partition function Z_{qu}. The correlator (A.9) is normalized to factor onto Z_{qu} ; this normalization is fixed by the procedure. The result is

$$Z_{qu} = 1/\eta^{N-1}. \tag{A.12}$$

Up to a shift $-(N-1)/24$ in vacuum energy, the level 1 characters are $\chi^{(1)}_{[i]} = \Theta_{[i]}/\eta^{N-1}$ and the full partition function is a sum over i of the holomorphic square of these.

APPENDIX B

Cover-Space approach to the PF Two-Point Correlator on the Torus

It is natural to search for parafermion correlators on the torus by applying the relative locality properties of the parafermion currents. In this appendix, only the Z_3 theory will be considered. The 2-point function between the parafermion ψ_1 and its conjugate must be of the form

$$\langle\psi_1(z)\psi_1^\dagger(w)\rangle = \frac{F(z-w)}{\vartheta^{4/3}(z-w)}, \tag{B.1}$$

where $\vartheta = \vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]$ as defined by (A.10) . The theta function ϑ essentially describes the quantum contribution to the correlator, as discussed in section 4.1 and Appendix A, including the relative semi-locality between parafermion and anti-parafermion. The function F must describe the relative semi-locality between the parafermion and the highest weights of the vacuum sectors. Consider the torus for large τ_2 with τ_1 fixed. Then if $z \to z-\tau$ then the parafermion factors onto the highest weight. This generates a phase of the form ω^n, where $\omega \equiv e^{2\pi i/3}$, and factors the correlator onto another vacuum state, both as given by (2.10) . Under this action, the parafermion adds charge $[4,0]$ to the highest weight. For the case at hand there are two sequences –

$$\begin{array}{cccc} \left\langle\psi\psi^\dagger\right\rangle_I \to & \left\langle\psi\psi^\dagger\right\rangle_\psi \to & \left\langle\psi\psi^\dagger\right\rangle_{\psi^\dagger} \to & \left\langle\psi\psi^\dagger\right\rangle_0 \\ (\phi_0^{(0)} \to & \phi_4^{(0)} \to & \phi_2^{(0)} \to & \phi_0^{(0)}) \end{array} \tag{B.2}$$

and –

$$\begin{array}{cccc} \left\langle\psi\psi^\dagger\right\rangle_\sigma \to & \left\langle\psi\psi^\dagger\right\rangle_{\sigma^\dagger} \to & \left\langle\psi\psi^\dagger\right\rangle_{\epsilon_1} \to & \left\langle\psi\psi^\dagger\right\rangle_\sigma \\ (\phi_1^{(1)} \to & \phi_5^{(1)} \to & \phi_3^{(1)} \to & \phi_1^{(1)}) \end{array} \tag{B.3}$$

For the second sequence, the identity (2.9) has been used to identify $\epsilon_1 = \phi^{(1)}_{[3,3]}$. Under $z \to z-1$, the parafermion circles about the highest weight and the correlator also obtains a phase from the semi-locality with the highest weight. If the contribution to the phases from the anti-parafermion is factored out by normalizing the vacuum (identity) sector, then the following phases are obtained under this transformation

$$\begin{aligned}
&\left(\left\langle\psi\psi^\dagger\right\rangle_I, \left\langle\psi\psi^\dagger\right\rangle_{\epsilon_1}\right) \rightarrow \left(\left\langle\psi\psi^\dagger\right\rangle_I, \left\langle\psi\psi^\dagger\right\rangle_{\epsilon_1}\right), \\
&\left(\left\langle\psi\psi^\dagger\right\rangle_\psi, \left\langle\psi\psi^\dagger\right\rangle_{\sigma^\dagger}\right) \rightarrow \omega\left(\left\langle\psi\psi^\dagger\right\rangle_\psi, \left\langle\psi\psi^\dagger\right\rangle_{\sigma^\dagger}\right), \\
&\left(\left\langle\psi\psi^\dagger\right\rangle_{\psi^\dagger}, \left\langle\psi\psi^\dagger\right\rangle_\sigma\right) \rightarrow \omega^\dagger\left(\left\langle\psi\psi^\dagger\right\rangle_{\psi^\dagger}, \left\langle\psi\psi^\dagger\right\rangle_\sigma\right).
\end{aligned} \tag{B.4}$$

The correlator returns to its original sector with no phase under the motions $z \longrightarrow z - 3\tau$ and $z \longrightarrow z - 3$. This situation is pictured by drawing cube root branch cuts about the two cycles; the parafermion and anit-parafermion are connected also by a cut – as the parafermion moves about cycles, it deposits this branch cut. The cover of the torus with branch cuts is the three-fold torus, on which the correlator is doubly-periodic (meromorphic). The functions $F(z - w)$, which generate no branch cut between z and w, are meromorphic on the cover. One convenient representation of these functions is given by

$$F_a(z - w|\tau) = \vartheta\begin{bmatrix} a/12 \\ 0 \end{bmatrix}(4(z - w)|12\tau) \quad a = 0, 1, \cdots, 11. \tag{B.5}$$

These functions give $\left\langle\psi\psi^\dagger\right\rangle_a \equiv F_a/\vartheta^{4/3}$ the required properties; the denominator generates 9 poles of order 4/3 on the cover so the numerator has 12 zeroes as dictated by the meromorphic constraint on the cover. The functions $\left\langle\psi\psi^\dagger\right\rangle_a$ clearly form a basis on the cover. Transformation properties of the F_a are

$$\begin{aligned}
F_a(z + 1) &= \omega^a F_a(z), \\
F_a(z + \tau) &= F_{a+4}(z).
\end{aligned} \tag{B.6}$$

By comparing these transformation properties with (B.2), (B.3), and (B.4), the highest weights (I,ϵ_1) behaive as $\langle\psi\psi^\dagger\rangle_a$ for $a=0$ mod 4. The other sectors are generated by motions about the B cycle.

There are four linearly independent functions $\langle\psi\psi^\dagger\rangle_a$, with τ dependent coefficients, which describe the different correlators for each sector. This freedom is is reduced by two additional constraints. First, the correlator for the (I,ϵ_1) sector is invariant under the interchange of ψ and its conjugate since the highest weights have no charge. This reduces the $\langle\psi\psi^\dagger\rangle_{I,\epsilon_1}$ correlators to

$$\left\langle\psi\psi^\dagger\right\rangle_{I,\epsilon} = a_{I,\epsilon_1}(\tau)\left\langle\psi\psi^\dagger\right\rangle_0 + b_{I,\epsilon_1}(\tau)\left\langle\psi\psi^\dagger\right\rangle_6 + c_{I,\epsilon_1}(\tau)\left[\left\langle\psi\psi^\dagger\right\rangle_3 + \left\langle\psi\psi^\dagger\right\rangle_9\right] \tag{B.7}$$

Finally, there can be no $U(1)$ currents in the theory so the second leading term in the $(z-w)$ expansion must vanish. This reduces the problem to two decoupled systems with two unknown τ dependent coefficients. This is precisely the amount of information supplied by the coset bosonization procedure of section 4 (compare (B.7) to (4.5) and (4.14)).

REFERENCES

1. D.Friedan, E.Martinec, S. Shenker, *Nucl. Phys.* **B271** (1986), 93 ;
   E. Verlinde, H. Verlinde, *Nucl. Phys.* **B288** (1987), 357 ;
   L. Alvarez-Gaume, J. Bost, P. Nelson, C. Vafa, *Comm. Math. Phys.* **112** (1987), 503 .
2. V.G. Knizhnik and A.B. Zamolodchikov, *Nucl. Phys.* **B247** (1984), 83.
3. E. Witten, *Comm. Math Phys.* **92** (1984), 455 .
4. I.B. Frenkel and V.G. Kac, *Invent. Math* **62** (1980), 23 ;
   G. Segal, *Comm. Math Phys.* **80** (1999), 301 ;
   P. Goddard, W. Nahm, D. Olive, and A. Schwimmer, *Comm. Math Phys.* **107** (1986), 179 .
5. B.L Feigen and D.B. Fuchs, Moscow preprint (1983) ;
   Vl.S. Dotsenko and V.A. Fateev, *Nucl. Phys.* **B240** (1984), 312 .

6. V.A. Fateev and A.B. Zamolodchikov, *Nucl. Phys.* **B280** (1987), 644 ;
V.A. Fateev and A.B. Zamolodchikov, *Int. Journal of Mod. Phys.* **A3** (1988), 507 ;
F.A. Bais, P. Bouwknegt, M. Surridge, and K. Schoutens, Amsterdam preprint IFTA 87-18 (1987) .
7. A.A. Belavin A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys.* **B241** (1984), 333 .
8. A.B. Zamolodchikov and V.A. Fateev, *Sov. Phys. JETP* **62** (1985), 215.
9. P. Goddard, A. Kent, and D. Olive, *Phys. Lett.* **152B** (1985), 88;
and, *Comm. Math. Phys.* **103** (1986), 105
10. M. Douglas, Caltec Preprint CALT-68-1453 (1987).
11. J. Lykken, Santa Cruz Preprint SCIPP-88/10 (1988).
12. V. Kac, *Infinite dimensional Lie Algebras*, Cambridge Unviversity Press (1985).
13. D. Gepner and E. Witten, *Nucl. Phys.* **B278** (1986), 493 .
14. V.G. Kac and D. Peterson, *Adv. Math.* **53** (1984), 125 .
15. D. Gepner, *Nucl. Pyhs.* **B290** (1987), 10.
16. D. Gepner and Z. Qiu, *Nucl. Phys.* **B285** (1987), 423 .
17. A.B. Zamolodchikov and V.A. Fateev, *Sov. Phys. JEPT* **63** (1986), 913 ,
Z. Qiu, *Phys. Lett.* **B188** (1987), 207 .
18. D. Kastor, E. Martenec, Z. Qiu , Preprint EFI-87-58-Chicago Aug. 1987.
J. Bagger , D. Nemeschansky, S. Yankielowicz, *Phys. Rev. Lett.* **60** (1988), 389 .
19. D. Friedan, Les Houches Sum. School 1982, 839;
L. Dixon, D.Friedan, E. Martinec, S. Shenker, *Nucl. Phys.* **B999** (1987), 999 .

20. M. Douglas , private communication.

21. T. Eguchi and K. Higashijima, *Vertex Operators and Non-abelian Bosonization*, in Recent Dev. in Q.F.T., Ambjorn et. al. eds., Elsevier, 1985 ; P. Goddard, and A. Schwimmer, *Phys. Lett.* **206B** (1988), 62 .

22. T. Eguchi and H. Ooguri, *Phys. Lett* **B203** (1988), 44 .

23. D. Bernard and J. Thierry-Mieg, *Comm. Math. Phys.* **181** (1987), 111 .

24. D. Boyanovsky, Preprint PITT/01/88 .

25. S.D. Mathur, S. Mukhi and A. Sen, Preprint TIFR/TH/88-22 and TIFR/TH/88-32;
T. Eguchi and H Ooguri, preprint UT-531-Tokyo (1988).

26. A. Rocha-Caridi, in *Vertex Operators in Mathematics and Physics* , p. 451, ed. J. Lepowsky et. al. , Springer Verlag (1985).

27. T. Eguchi and H. Ooguri, *Nucl. Phys.* **B282** (1987), 308 .

28. V. Kac and M. Wakimoto, *Adv. Math.* **70** (1988), 156 .

29. K.S. Narain, *Phys. Lett.* **169B** (1986), 41 .

30. J. Bagger and D. Nemeshansky, USC Preprint USC-88/010 .

31. K.S. Narain, M.H. Sarmadi, and E. Witten, *Nucl. Phys.* **B279** (1987), 369 .

32. P. Ginsparg,, *Phys. Rev.* **D35** (1987), 648 .