SLAC–PUB–4664
June 1988
(T)

Topological Strings*

DAVID MONTANO AND JACOB SONNENSCHEIN†

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

ABSTRACT

Following some recent works of Witten, we explore topological non-linear sigma models (TSM). We construct 2-D topological gravity which is then coupled to the TSM resulting in a topological bosonic string theory. This theory has a nontachyonic vacuum and can be formulated on any even dimensional target manifold which admits some non-trivial topology. It is then shown that the metric independence of the target manifold can not be maintained at the quantum level. Quantum consistency at the one loop level of the sigma model requires that the Ricci scalar vanishes.

Submitted to *Nucl. Phys. B*

* Work supported by the Department of Energy, contract DE–AC03–76SF00515.
† Work supported by Dr. Chaim Weizmann Postdoctoral Fellowship.

1. Introduction

In the beginning of this year, Witten wrote a series of papers describing the construction of topological quantum field theories (TQFT).[1] [2] These are quantum field theories whose classical observables are topological invariants. Thus, in some sense, these TQFT's represent a phase of quantum field theories where general covariance is unbroken.[2] Witten then refers to the massless graviton as the Goldstone boson of general covariance. We have that the graviton is the fluctuation about a particular background metric (usually, Minkowski space). In the standard electroweak model considerations of renormalizability also require that the theory first be constructed in a gauge invariant symmetric phase. Perhaps, then, it is fruitful to formulate a theory in a phase of unbroken general covariance and then find a mechanism for the spontaneous breaking of this symmetry with the consequent generation of a graviton and thus local gravity. This may provide a nice way to maintain renormalizability and unitarity in a quantum theory of gravity and was a major motivation for our work.

In this paper we investigate a two dimensional nonlinear sigma model in a phase of unbroken general covariance. There has been recent speculation that such a nonlinear sigma model may describe string theory above the Hagedorn temperature.[3] This would be a topological string theory; i.e. a string theory where only world sheet instantons are physical states. Following the suggestions of Witten in ref. [1], we have suceeded in constructing such a sigma model without a critical dimension and with a non-tachyonic vacuum. Unfortunately, as in ref. [1] we must still require that the target manifold be symplectic. We also find, somewhat surprisingly, that consistent BRST quantization requires that the target manifold have a vanishing Ricci scalar. It thus appears impossible to quantum mechanically maintain the theory without a metric dependence. This constraint arises, as in string theory, by requiring that the world sheet scale invariance not be violated by quantum fluctuations.

Our construction follows the one we used in ref. [4] where it was shown that

Witten's TQFT's of ref. [2] could be derived using a simple two stage gauge fixing procedure. The key ingredient in formulating a topological nonlinear sigma model was the construction of two-dimensional topological gravity. This was attempted without complete success in ref. [1]. The incorporation of topological 2-D gravity was essential, for this system cancels the central charge due to the reparametrization ghosts of the two dimensional metric. It is then straightforward to add a topological matter Lagrangian (as given in ref. [1]) with a vanishing central charge. This is possible because the ghosts one introduces to project onto topological states (i.e. world sheet instantons) have a central charge which just cancels that of the matter fields.

In section 2, we will describe the details of the construction of the 2-D topological gravity. In section 3, we will review Witten's topological nonlinear sigma model and its coupling to topological gravity. We will then, in section 4, discuss the one loop conformal invariance of this theory. Section 5 will be devoted to the possible observables.

2. Topological Gravity in 2-D

Following the procedure of ref. [4] we begin with a topological invariant as the action. In 2-D gravity the obvious choice is the Euler number:

$$I_0 = \int d^2\sigma \sqrt{-g} R^{(2)}, \tag{2.1}$$

where g is the determinant of the metric and $R^{(2)}$ is the 2-D Ricci scalar. This action is clearly invariant under general coordinate transformations, $\delta g_{\alpha\beta} = D_\alpha \xi_\beta + D_\beta \xi_\alpha$, and scale transformations, $\delta g_{\alpha\beta} = 2 g_{\alpha\beta} \delta\sigma$. But it is further invariant under

$$\delta g_{\alpha\beta} = 2\Lambda_{\alpha\beta}(\sigma), \tag{2.2}$$

where $\Lambda_{\alpha\beta}$ is an arbitrary, differentiable, world sheet symmetric tensor. Under the above variation, equation (2.1) changes by a total derivative. It is convenient for some calculations to present (2.2) as a transformation of the zweibein:

$$\delta e^a_\alpha = e^b_\alpha \Lambda^a_b(\sigma), \quad \text{or} \quad \delta e^\alpha_a = -e^\alpha_b \Lambda^b_a(\sigma) \tag{2.3}$$

One should note that even though (2.3) has the structure of a local Lorentz transformation it is different since for the latter Λ_{ab} is antisymmetric.

We will now gauge fix this symmetry using the standard BRST procedure as done in [4]. In the first stage we fix part of the symmetry by gauge fixing the Ricci scalar, $R^{(2)}$. We introduce anticommuting ghosts $\psi_{\alpha\beta}$ and η, and a scalar commuting ghost, B, with the following BRST transformations:

$$\begin{aligned} \delta g_{\alpha\beta} &= 2i\epsilon\psi_{\alpha\beta}, \\ \delta\eta &= i\epsilon B, \end{aligned} \tag{2.4}$$

where ϵ is the anticommuting, BRST parameter.

We then write the gauge fixing and Faddeev-Popov Lagrangians as follows:

$$\mathcal{L}_I = \mathcal{L}_{(GF+FP)} = \hat{\delta}[\sqrt{-g}\eta R^{(2)}], \tag{2.5}$$

where $\delta = i\epsilon\hat{\delta}$. Using the transformation of $g_{\alpha\beta}$, it is straightforward to derive the variation of the Ricci scalar:

$$\hat{\delta} R^{(2)} = -(R^{(2)} + 2\Delta)\theta + 2D_\alpha D_\beta \psi^{\alpha\beta}, \tag{2.6}$$

where $\theta \equiv \psi^\alpha_\alpha$, D_α is a world sheet covariant derivative and Δ is the corresponding Laplacian. In ref. [1] ψ_{ab} was taken to be traceless, but for our derivation it was necessary that $\theta \neq 0$. Thus, eqn. (2.4) represents a larger symmetry. The Lagrangian (2.5) then takes the form:

$$\mathcal{L}_I = \sqrt{-g}\{BR^{(2)} - 2\eta D_\alpha D_\beta \psi^{\alpha\beta} + 2\eta\Delta\theta\}. \tag{2.7}$$

Just as in ref. [4] we now observe that the $\mathcal{L}_I$ has a remaining ghostly symmetry.

The corresponding transformations are:

$$\begin{aligned}\delta_G \psi_{\alpha\beta} &= \frac{\epsilon}{2}(D_\alpha C_\beta + D_\beta C_\alpha),\\ \delta_G B &= i\epsilon D_\alpha(\eta C^\alpha),\end{aligned} \tag{2.8}$$

where C_α is a commuting vector ghost. The ghostly symmetry of $\psi_{\alpha\beta}$ looks very much like reparametrization invariance but with the opposite statistics.

The next stage is then, obviously, to fix this symmetry. The reparametrization invariance allows us to fix $\psi_{\alpha\beta}$ to conformal gauge (just as for the 2-D metric). We are thus led to the following gauge fixing Lagrangian:

$$\mathcal{L}_{II} = \hat{\delta}_W\{\sqrt{-g}(\gamma^{\alpha\beta}\psi_{\alpha\beta} + \Omega\Delta B)\}, \tag{2.9}$$

where $\hat{\delta}_W = \hat{\delta}_I + \hat{\delta}_G$, and $\gamma^{\alpha\beta}$ and Ω are traceless symmetric tensor and scalar antighosts, respectively. They transform as follows:

$$\begin{aligned}\delta_W \gamma^{\alpha\beta} &= i\epsilon\lambda^{\alpha\beta},\\ \delta_W \Omega &= i\epsilon\phi\end{aligned} \tag{2.10}$$

It should be kept in mind that $\lambda^{\alpha\beta}$ and Ω are anticommuting ghosts while $\gamma^{\alpha\beta}$ and ϕ are commuting. Then using eqns. (2.9) and (2.10) we get:

$$\begin{aligned}\mathcal{L}_{II} =& \sqrt{-g}\{\gamma^{\alpha\beta}D_\alpha C_\beta + \phi\Delta B + \Omega\Delta D_\alpha(\eta C^\alpha)\\ &+ \lambda^{\alpha\beta}\psi_{\alpha\beta} - \theta(\gamma^{\alpha\beta}\psi_{\alpha\beta} + \Omega\Delta B)\}\end{aligned} \tag{2.11}$$

Using the equation of motion of the auxiliary field λ we can eliminate all of $\psi_{\alpha\beta}$, except for its trace θ. Altogether, we get for the 2-D topological gravity the following Lagrangian:

$$\begin{aligned}\mathcal{L} =& \mathcal{L}_0 + \mathcal{L}_I + \mathcal{L}_{II}\\ =& \sqrt{-g}\{R^{(2)} + B(R^{(2)} - \Delta\phi) + 2\eta\Delta\theta + \gamma^{\alpha\beta}D_\alpha C_\beta\,.\\ &+ \Omega[\Delta D_\alpha(\eta C^\alpha) + \theta\Delta B]\}\end{aligned} \tag{2.12}$$

We note that (2.12) is still reparametrization invariant. Fixing this last symmetry is standard and will not be repeated here. In addition, it is easy to verify that (2.12) is classically invariant under a global $U(1)$ "ghost number" symmetry and under global scale transformation. The U(1) charges of the fields $(e_{\alpha a}, \psi_{ab}, C_a, \eta, B, \Omega, \gamma^{ab}, \lambda^{ab}, \phi)$ are (0, 1, 2, −1, 0, −1, −2, −1, 0) respectively, and their conformal dimensions are (−1, 0, −1, 0, 0, 2, 0, 0, 2).

It was shown in ref. [1] that the algebra $(\delta_\eta \delta_\epsilon - \delta_\epsilon \delta_\eta)$ is closed only upto a diffeomorphism generated by $-2i\eta\epsilon C^\alpha$, a local Lorentz transformation with a parameter $\eta\epsilon\Lambda = -\eta\epsilon(\epsilon^{ab}\psi_{ac}\psi^c_b + i\epsilon^{ab}D_a C_b)$ and a Weyl rescaling by $D_\alpha C^\alpha$. (In our derivation the scale transformation would not appear.) It is, therefore, necessary for the consistency of the theory that there is no anomaly in the conformal symmetry. Obviously, this requirement also follows from the desire that the theory be topological. Hence, it is essential that the central charge of the Virasoro algebra vanish. Let us then list the contribution to the central charge from each of the kinetic terms in (2.12) :

$B\Delta\phi$	:	$c = 1$	commuting scalars
$\eta\Delta\theta$	:	$c = -1$	anticommuting scalars
$c^\alpha D^\beta \gamma_{\alpha\beta}$	:	$c = 26$	commuting vector and traceless tensor

If we add to the above the central charge from the metric reparametrization ghosts we get a complete cancellation! There is no magic here. Since $\psi_{\alpha\beta}$ and $g_{\alpha\beta}$ have the same gauge symmetry but opposite statistics, we expected that the contribution to the central charge from all of the ghosts would cancel.

3. Topological Sigma Models

3.1. On Flat World Sheets

In this section we review Witten's topological sigma model (TSM) on flat two dimensional world sheets.[1] The distinguishing property of these models is their BRST-like fermionic symmetry. This invariance indicates that the corresponding action is a BRST gauge fixed version of an action with a local symmetry. A sigma model is a theory of maps from a Riemann surface Σ to a Riemann manifold M expressed in terms of $u^i(\sigma)$, the coordinates of M where i=1,...D and σ is a point on Σ. We, therefore, want to construct a TSM which is invariant under arbitrary local deformations of u^i : $\delta u^i = \theta^i(\sigma)$.

In analogy with the construction of the Lagrangian for Witten's gauge fixed TQFT action,[4] we look for a topologically invariant action. Having in mind a world sheet without boundaries, we have to disregard actions which are world sheet total derivatives. We are then led to consider the following topological action:

$$I_0 = \int d^2\sigma \epsilon^{\alpha\beta} J_{ij} \partial_\alpha u^i \partial_\beta u^j = \int_T J \qquad (3.1)$$

where J_{ij} is the almost complex structure of the manifold M obeying $J_i^j J_j^k = -\delta_i^k$; J is the associated two form $J = \frac{1}{2} J_{ij} du^i \wedge du^j$ which we take to be closed $dJ = 0$, and T is the image of Σ in M. The condition, $dJ = 0$, which guarantees the topological nature of (3.1) means that only symplectic manifolds will be considered. In ref.[1] the more general case of almost complex manifolds is presented, but even there eventually only the symplectic case is related to physical systems. Moreover, as will be clarified in section 4, the condition $dJ = 0$ is required to maintain conformal symmetry at the quantum level.

We proceed now to the BRST-gauge fixing of the action (3.1), following the procedure developed in ref. [4]. There, we fixed the gauge configurations to be (anti) self-dual, namely (anti) instantons $F_{\alpha\beta} + \tilde{F}_{\alpha\beta} = 0$. In complete analogy we want to fix the sigma model configurations to be self dual (world sheet instantons), namely

$$\partial_\alpha u^i + \epsilon_\alpha^\beta J_j^i \partial_\beta u^j = 0. \tag{3.2}$$

This is exactly Witten's construction of the TSM action. The gauge fixing plus Faddeev Popov Lagrangian is

$$\mathcal{L} = \mathcal{L}_{(GF+FP)} = \frac{i}{2}\hat{\delta}[\rho_i^\alpha(\partial_\alpha u^i + \epsilon_\alpha^\beta J_j^i \partial_\beta u^j - \frac{1}{2}H_\alpha^i)], \tag{3.3}$$

where ρ_i^α is a self-dual anticommuting ghost ($\rho_i^\alpha = \epsilon_\beta^\alpha J_i^j \rho_j^\beta$) and H_α^i is a self-dual, commuting, auxiliary ghost. The BRST transformations of the various fields were constructed in ref. [1] in such a way that the closure of the algebra ($Q^2 = 0$), self-duality, and covariance under reparametrization of the u^i were assured. The resulting transformations are:

$$\begin{aligned}
&\delta_0 u^i = i\epsilon\chi^i,\\
&\delta_0 \chi^i = 0,\\
&\delta_0 \rho_{i\alpha} = \epsilon(H_\alpha^i - i\Gamma_{jk}^i \chi^j \rho_\alpha^k),\\
&\delta_0 H_\alpha^i = -\epsilon[\frac{1}{4}\chi^k\chi^l(R_{kl}{}^i{}_m + R_{kli'm'}J^{i'i}J_m^{m'})\rho_\alpha^m - i\Gamma_{jk}^i\chi^j H_\alpha^k],
\end{aligned} \tag{3.4}$$

where δ_0 denotes the variation on a flat world-sheet and Γ_{jk}^i, $R^i{}_{jkl}$ are the target manifold Christoffel connection and Riemann tensor, respectively. Indeed, as was shown in ref. [1], for a Kahler manifold (i.e $dJ = 0$) there are two fermionic symmetries generated by Q_R and Q_L. In this work we discuss only the case where the two anticommuting symmetry parameters are taken to be the same. Inserting the above transformation (3.4) into the Lagrangian (3.3) and eliminating the auxiliary field, H_α^i, one gets:

$$I = \int d^2\sigma[\frac{1}{2}(g_{ij}\partial_\alpha u^i \partial^\alpha u^j + \epsilon^{\alpha\beta}J_{ij}\partial_\alpha u^i \partial_\beta u^j) - i\rho_i^\alpha \partial_\alpha \chi^i - \frac{1}{8}\chi^k\chi^l\rho_i^\alpha\rho_{\alpha m}R_{kl}{}^{im}]. \tag{3.5}$$

It should be noted that unlike the case of the Yang-Mills TQFT [2], the gauge fixed action (3.5) does not possess a further ghost symmetry. The reason being that we have an equal number of symmetry parameters and degrees of freedom which were all fixed in (3.3). As was pointed out above, the physical configurations of the TSM are the world-sheet instantons. The BRST charge which annihilates the physical states is derived from the current:

$$J_\alpha = g_{ik}(\partial_\alpha u^i + \epsilon_\alpha^\beta J^i_j \partial_\beta u^j)\chi^k. \tag{3.6}$$

Therefore, only the mappings from Σ to M which admit instantons (3.2) are relevant. This question of classifying the world sheet instantons was addressed in ref. [5]. Here we want to only review some of the results: (i) The Manifold M must have, obviously, some non-trivial topology and more precisely $\pi_1(M) \neq 0$ or $\pi_2(M) \neq 0$. (ii) Denoting by N_h the homotopy classes of h-genus Riemann surfaces, then $N_0 = \pi_2(M)$; and if M is simply connected, namely $\pi_1(M) = 0$, then $N_h = \pi_2(M)$. (iii) For a simply connected manifold with non-trivial $\pi_2(M)$, it follows from Hurewicz' theorem that $H_2(M) \neq 0$, indeed $\pi_2(M) \simeq H_2(M)$. Thus, there exists on M some closed forms which are not exact. Recall that one such form was our starting point in the construction of the TSM action. In section 4 a further constraints on the possible target manifolds will emerge from the requirement of quantum conformal symmetry.

3.2. Coupling the Topological Sigma Model to 2-D Gravity

The coupling of the TSM to the gauge-fixed 2-D gravity that was described in section 2 follows the derivation of ref. [1]. We couple the TSM to the minimal gravity multiplet $e_{\alpha a}, \psi_{ab}$ and C^a.

Since the BRST algebra in the pure gravity case has $Q^2 = 0$ only up to an infinitesimal diffeomorphism and local Lorentz transformation, we require the same condition also for the coupled system. The BRST transformations which fulfil this conditions are

$$\begin{aligned}
\delta u^i &= \delta_0 u^i, \\
\delta \chi^i &= \epsilon C^\alpha \partial_\alpha u^i, \\
\delta \rho_i^\alpha &= \delta_0 \rho_i^\alpha, \\
\delta H_a^i &= \delta_0 H_a^i + i\epsilon(C^b D_b \rho_a^i + \frac{i}{2}\Lambda \epsilon_{ab}\rho^{bi}),
\end{aligned} \tag{3.7}$$

where the transformations δ_0 are given in eqn. (3.4) and Λ in section 2. To construct the Lagrangian we start with

$$\mathcal{L} = \mathcal{L}_{(GF+FP)} = \frac{i}{2}\hat{\delta}[\rho_i^a(D_a u^i + \epsilon_a^b J_j^i D_b u^j - \frac{1}{2}H_a^i)], \tag{3.8}$$

which is just eqn (3.3) on a curved world sheet. Substituting the transformation laws (3.7) and eliminating the auxiliary fields we get:

$$\begin{aligned}
I = \int d^2\sigma \; det \; e[&\frac{1}{2}(g_{ij}\partial_\alpha u^i \partial^\alpha u^j + \epsilon^{\alpha\beta} J_{ij}\partial_\alpha u^i \partial_\beta u^j) \\
&- i\rho_i^a D_a \chi^i - \frac{1}{8}\chi^k \chi^l \rho^{ai} \rho_a{}^m R_{lkim} \\
&- \frac{i}{2}\theta \rho_i^a D_a u^i \\
&\frac{i}{8}\epsilon_{ab}\rho^{ai}\rho_i^b \epsilon^{cd} D_c C_d + \frac{i}{4}\rho_i^a C^b D_b \rho_a^i]
\end{aligned} \tag{3.9}$$

The first two lines are just eqn. (3.5) on a non-flat world sheet; the third line emerges from the transformation of $e_{\alpha a}$ in the covariant derivative and *det e*, and the last line is generated by the modification of the transformations (3.7). Note the difference between this Lagrangian and the one of ref. [1]; here only the trace of $\psi_\alpha^\alpha = \theta$ appears due to the equation of motion of $\lambda_{\alpha\beta}$ eqn.(2.11). The physical states for the coupled TSM are determined by the TSM part of the BRST charge whose associated current has now the form:

$$\begin{aligned}
J_a = g_{ik}[&(D_b u^i + \epsilon_b^c J_j^i D_c u^j)(\delta_a^b \chi^k + \rho_a^k C^b - \frac{1}{2}\rho^{kb} C_a) \\
&- \frac{i}{2}\theta(\rho_a^i \chi^k - \frac{1}{4}\epsilon_{bc}\rho^{bi}\rho^{kc}\epsilon^{ad} C_d) + \frac{1}{4}\rho_b^i C_a \Gamma_{lm}^k \chi^l \rho^{mb}].
\end{aligned} \tag{3.10}$$

Thus again the BRST charge is projecting on to instanton configurations.

4. One Loop Conformal Invariance

In ref. [1] it was shown that for a flat target manifold the central charge of the topological sigma model vanishes. It still remains to be seen what constraints, if any, have to be placed on the target manifold if the world sheet conformal invariance is to be maintained quantum mechanically. This question will be addressed in this section.

We recall that in a general conformal field theory we have for the expectation value of the operator product expansion of the stress tensor:

$$\langle T_{++}(z)T_{++}(w)\rangle = \frac{c/2}{(z-w)^4}. \tag{4.1}$$

Only the TSM contributions to T_{++} are considered since we already showed that 2-D gravity contributes $c = 0$ and has no spacetime dependence. We now compute the central charge to one loop order in the sigma model. Following the work of ref. [6], we expand the background metric in normal coordinates,ξ^i:

$$\begin{aligned} g_{ij} =& \delta_{ij} + \frac{1}{3}R_{iklj}\xi^k\xi^l + \ldots, \\ \partial_\alpha u^i =& \partial_\alpha u^i_B + D_\alpha \xi^i + \frac{1}{3}R^i_{klj}\xi^k\xi^l\partial_\alpha u^j_B + \ldots, \\ g_{ij}\rho^i_+ D_+\chi^j =& (\delta_{ij} + \frac{1}{3}R_{iklj}\xi^k\xi^l)\rho^i_+\partial_+\chi^j + \frac{1}{2}R_{ijkl}\partial_+ u^l\xi^k\rho^i_+\chi^j + \ldots \end{aligned} \tag{4.2}$$

where u_B is the background field which satisfy the equation of motion. We make use of the following operator products

$$\begin{aligned} &\partial_+u^i(z)\partial_+u^j(w) \sim \frac{-\delta^{ij}}{(z-w)^2}, \qquad \rho^i_+(z)\chi^j(w) \sim \frac{\delta^{ij}}{(z-w)} \\ &\xi^i(z)\xi^j(w) = i\delta^{ij}\Delta_F(z-w), \qquad i\Delta_F(0) = \lim_{\epsilon\to 0}\frac{-\alpha}{2\pi\epsilon} \end{aligned} \tag{4.3}$$

For the bosonic stress tensor of the sigma model, we then get straightforwardly:

$$\langle T^B_{++}(z)T^B_{++}(w)\rangle = \frac{1}{(z-w)^4}\{\frac{D}{2} + \frac{\alpha}{2\pi}R\}, \tag{4.4}$$

with α the renormalized coupling constant and R the Ricci scalar of the target manifold . For the $\rho - \chi$ ghosts we get the following contribution:

$$\langle T^{\rho\chi}_{++}(z) T^{\rho\chi}_{++}(w) \rangle = \frac{1}{(z-w)^4} \{ -\frac{D}{2} - \frac{\alpha}{6\pi} R \}. \tag{4.5}$$

We thus see that for conformal invariance at one loop level to be maintained quantum mechanically, we must have $R = 0$. It is important to point out again that the spacetime dimensionality drops out. The curvature constraint is a curious result because we know that T_{++} is a BRST commutator. This indicates that the BRST algebra does not close unless the background Ricci scalar vanishes. There will also be a possible constraint on the Ricci tensor from the metric beta function, but it will be multiplying a BRST commutator. We will thus have a divergent quantity multiplying a BRST commutator. This is somewhat ambiguous, but we believe that such a term may be neglected without loss of consistency.

What are we to make of all of this? We began with a theory which was independent of the metric on the target manifold. The introduction of the metric arose in the gauge fixing procedure, since we needed a way to define an inner product. We must now conclude that the consistent BRST quantization of the topological sigma model requires that the target manifold has a vanishing Ricci scalar.

It should be noted that we neglected the contribution of the almost complex structure, J_{ij}, to the central charge correction, since it vanishes when $dJ = 0$. This is, of course, as expected since J only appears as a topological term.

5. Physical States

We have constructed in the previous sections a nonlinear sigma model without a critical dimension but with the constraint that the target manifold has a vanishing Ricci scalar. This was a theory of world sheet instantons. We must now investigate the possible physical states.

Firstly, it is simple to show that the vacuum state is not tachyonic. Let us reconsider the central charge cancellation in the world sheet Virasoro algebra. We write the algebras for the bosons, $u^i(z)$, and the spacetime ghosts, $\rho - \chi$:

$$\begin{aligned}[L_m^{(B)}, L_n^{(B)}] =& (m-n)L_{m+n}^{(B)} + \frac{D}{12}(m^3 - m)\delta_{m+n}, \\ [L_m^{(\rho\chi)}, L_n^{(\rho\chi)}] =& (m-n)L_{m+n}^{(\rho\chi)} + \frac{D}{2}A^{(\rho\chi)}(m)\delta_{m+n}.\end{aligned} \tag{5.1}$$

Here, $A^{(\rho\chi)}(m)$ is the ghost anomaly term which is given by:

$$A^{(\rho\chi)}(m) = -\frac{1}{12}[(12J^2 - 12J + 2)m^3 - 2m] \tag{5.2}$$

where J is the conformal spin of ρ (and χ has spin $1 - J$). Thus, the cubic and linear terms in the anomaly cancel exactly. Then, L_0 will annihilate the vacuum. This is precisely the statement that the intercept (i.e. the normal ordering constant for L_0, usually denoted by a) is zero. Typically, in string theory we require a non-vanishing intercept, a, so that the linear term in m is cancelled. This is the origin of the usual tachyonic vacuum. It is easy to check that since the topological Q-BRST (3.6) is linear in the field u^i, it cannot annihilate any of the ordinary oscillation modes of the bosonic string; therefore, these modes are not physical.

Consequently, this theory has a degenerate world sheet instanton vacuum. What are the observables? Such a theory certainly probes quantities like the second cohmology group of the target manifold, as discussed earlier. Indeed, Witten has shown that the *global* observables are of the form:[1]

$$W_A(\gamma) = \int_\gamma O_A^{(2)}$$

where

$$O_A^{(2)} = A_{i_1 \dots i_n} du^{(i_1)} \wedge du^{(i_2)} \chi^{i_3} \dots \chi^{i_n} \tag{5.3}$$

and A is a closed form and γ is a homology 2-cycle on the world sheet. This can , of course, be rewritten as an integral over the target manifold.

In the previous section we showed that one loop conformal invariance requires the vanishing of the Ricci scalar on the target manifold. We then expect that upon computing the partition function we will find states which are *not* only of topological origin. This will be interesting to see and will be addressed in future work.

6. Conclusion

We have now completed a tour through the strange world of topological nonlinear sigma models. Clearly, much work remains to be done, but some interesting results have already appeared. We have seen how such theories naturally have nontachyonic vacua without the need for world sheet supersymmetry. They also require no critical dimension. These were expected consequences from the topological origin of the theory where only world-sheet instantons are allowed physical states. More interestingly, we have seen how the topological symmetry of the target manifold could not be maintained quantum mechanically. To one loop in the sigma model the target manifold had to have a vanishing Ricci scalar. We also needed manifolds that would have some nontrivial topology, in particular, $H_2(M) \neq 0$. The most general solution for the target manifold is a compact Ricci flat Kahler manifold.

This brings us back to the question of whether these theories may be describing a high temperature phase of string theory.[3] If so, a possible order parameter could be the three form, H, given by the exterior derivative of the almost complex structure, J. In the symmetric phase H would be zero so that our initial Lagrangian (3.1) would be a topological invariant. At some critical temperature H would get an expectation value and the topological symmetry would be broken. Finite temperature calculations of the sigma model beta functions might lead to a potential for H with a nonzero expectation value. It would be interesting to test the above ideas quantitatively.

Finally, we would like to mention that it seems possible to extend this work to an arbitrary topological theory of extended objects. There are some subtleties that must be overcome. The theory of 4-D topological gravity of ref.[2] would have to be completed and generalized. Some of the results in this paper and in ref. [4] may be of some help in this respect. These would be theories that probe the higher even homotopy groups of the chosen target manifolds.

Acknowledgements: We would like to thank M. Peskin for his encouragement and for fruitful discussions. We would especially like to express our gratitude to R. Brooks who took part in the first stage of the work. His insightful comments and critical reading of the manuscript were greatly valued.

REFERENCES

1. E. Witten, 'Topological Sigma Models,' IAS preprint, IASSNS-HEP-87/7, February 1988.
2. E. Witten, 'Topological Quantum Field Theory,' IAS preprint, IASSNS-HEP-87/72, February 1988; 'Topological Gravity,' IAS preprint, IASSNS-HEP-87/2, February 1988.
3. J. Atick and E. Witten, 'The Hagedorn Transition and the Number of Degrees of Freedom of String Theory,' IAS preprint, IASSNS-HEP-88/14, April 1988.
4. R. Brooks, D. Montano, and J. Sonnenschein, 'Gauge Fixing and Renormalization in Topological Quantum Field Theory,' SLAC-PUB-4630, May 1988.
5. Y. Gao and M. Li, Phys. Lett. **196B** (1987) 339.
6. L. Alvarez-Gaume, D. Z. Freedman and S. Mukhi, Ann. Phys. (N.Y.) **134** (1985) 85; T. Banks, D. Nemeschansky, and A. Sen, Nucl. Phys **277B** (1986) 67.