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Gauge Fixing and Renormalization in Topological Quantum Field Theory*

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ABSTRACT

In this letter we show how Witten's topological Yang-Mills and gravitational quantum field theories may be obtained by a straightforward BRST gauge fixing procedure. We investigate some aspects of the renormalization of the topological Yang-Mills theory. It is found that the beta function for the Yang-Mills coupling constant is not zero.

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1. Introduction

Recently, there has been an effort to study the topology of four dimensional manifolds using the methods of quantum field theory.[1-6] This was motivated by Donaldson's use of self-dual Yang-Mills equations to investigate the topology of low dimensional manifolds.[1] The work of Floer on three manifolds, as interpreted by Atiyah,[3] can be understood as a modified non-relativistic field theory. Atiyah then conjectured that a relativistic quantum field theory could be used to study Donaldson's invariants on four manifolds. This led Witten to work out a series of preprints in which just such topological quantum field theories (TQFT) were described.[4-6] In the first paper a TQFT was proposed whose correlation functions were purely topological and reproduced Donaldson's invariants.[4] A later paper extended these ideas to a more complicated Lagrangian which could be used to study similar invariants in topological gravity.[5] These TQFT's all possess a fermionic symmetry analogous to a BRST symmetry.

In this letter we will show how both of these theories may be obtained through a very simple gauge fixing procedure. A prescription for how quantum field theory calculations may be done for the TQFT will be presented. These calculations will allow us to study the perturbative renormalizability of the theory. It will be shown that the beta function for the coupling constant is not zero. This suggests that there is a quantum contribution to the stress-energy tensor which may indicate a metric dependence of the partition function.

Throughout this work we will use the notations of refs. [4] and [5]. However, in our discussion of the Yang-Mills TQFT, actions will be defined in terms of Lagrangians via $I = \int d^4x \, \sqrt{g} Tr \mathcal{L}$, where g is the determinant of the metric.

2. WITTEN'S TOPOLOGICAL YANG-MILLS THEORY

We begin by deriving Witten's Lagrangian for the non-abelian Yang-Mills (YM) theory [4] by gauge fixing the local transformation

$$\delta_I A_\alpha{}^a = \theta_\alpha{}^a \ . \tag{2.1}$$

Here, $A_\alpha{}^a$ is a gauge field in the adjoint representation (index a). The gauge fixing will be done using the following principles as guidelines: First, the BRST gauge fixing procedure will be employed in order that we manifestly maintain the fermionic symmetries of ref. [4]. Second, because of their importance in the latter work, we will seek to preserve the scaling and U symmetries. Finally, we would like a YM invariant action at the end of the process. Explicit coupling constant (e_0) dependence will be given in our expressions. For this we define the YM covariant derivative as $D_\alpha \equiv \partial_\alpha + e_0[A_\alpha, \ ]$.

We have found that the gauge fixing ansatz which leads to Witten's theory is $F_{\alpha\beta} + \tilde{F}_{\alpha\beta} \equiv 0$, where $\tilde{F}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$ is the dual of the YM field strength, $F_{\alpha\beta}$. To see this, let us assume that there is some Lagrangian, $\mathcal{L}_0$, which is both YM and δ_I invariant. Two examples are $\mathcal{L}_0 = 0$ and $\mathcal{L}_0 \propto [F_{\alpha\beta}\tilde{F}^{\alpha\beta}]$. Both of these are topological with the Euclidean integral of the second yielding the Pontryagin index (winding number). The former is trivially invariant while the latter is invariant under $\hat{\delta}_I$ transformations if: (i) the Bianchi Identity for the YM field-strength, $D_\alpha \tilde{F}^{\alpha\beta} = 0$, is used, and (ii) the parameter θ_α asymptotically drops off as one power faster than the gauge field. This is the requirement that θ_α does not change the Pontryagin number. Given, $\mathcal{L}_0$, we BRST gauge fix by writing down the gauge fixing and Faddeev-Popov Lagrangians:

$$\begin{aligned} \mathcal{L}^{(I)}_{GF+FP} &= \hat{\delta}_I[\frac{i}{4}\chi^{\alpha\beta}(F_{\alpha\beta} + \tilde{F}_{\alpha\beta} + \frac{1}{2}\alpha_0 \mathcal{B}_{\alpha\beta})] \\ &= [\frac{i}{4}\mathcal{B}^{\alpha\beta}(F_{\alpha\beta} + \tilde{F}_{\alpha\beta}) - i\chi^{\alpha\beta}D_\alpha\psi_\beta + \frac{i}{8}\alpha_0\mathcal{B}^{\alpha\beta}\mathcal{B}_{\alpha\beta}] \ , \end{aligned} \tag{2.2}$$

where we have introduced the BRST operator, $\hat{\delta}_I$, through the definition $\delta_I \equiv$

$i\epsilon\hat{\delta}_I$, for some constant anti-commuting parameter, ϵ. Traces in the adjoint representation are implied in all of our Lagrangians. This particular Lagrangian maintains the BRST symmetry:

$$\begin{aligned} \hat{\delta}_I A_\alpha{}^a &= \psi_\alpha{}^a \ , \\ \hat{\delta}_I \psi_\alpha{}^a &= 0 \ , \\ \hat{\delta}_I \chi^{\alpha\beta a} &\equiv \mathcal{B}^{\alpha\beta a} \ , \\ \hat{\delta}_I \mathcal{B}^{\alpha\beta a} &= 0 \ . \end{aligned} \tag{2.3}$$

The anti-ghost field, $\chi^{\alpha\beta}$, and the BRST auxiliary field, $\mathcal{B}^{\alpha\beta}$ are anti-symmetric and self-dual. The field, ψ_α, is the ghost conjugate to $\chi^{\alpha\beta}$.

A few remarks about eqns. (2.2) and (2.3) are in order. To obtain Witten's theory, we must pick the gauge $\alpha_0 = 0$ for the gauge parameter. The operator $\hat{\delta}_I$ yields the usual off-shell, nilpotent, BRST algebra. The equation of motion of this auxiliary field yields the transformation law for χ, as given in ref. [4]. After integrating out $\mathcal{B}$, eqn. (2.3) is only an on-shell symmetry, as to be expected. The equation of motion which must be used to maintain this symmetry is obtained from the anti-ghost variation: $D_{[\alpha}\psi_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}D^\gamma\psi^\delta = 0$. Now, without this restriction on ψ, there are four gauge parameters. However, χ only contains three "degrees of freedom". This suggests that there is further gauge fixing to be done which will reduce the freedom of the ghost field. There must be an additional symmetry in the theory. As we will soon see, this symmetry forces the introduction of two additional commuting fields, ϕ and λ, and another anti-commuting field, η. As a result, we will have the full multiplet of fields as given in ref. [4]. We now turn our attention to finding this symmetry while keeping our off-shell BRST invariance.

We do not have to look far, as the ghost Lagrangian above has a ghostly symmetry. It is invariant under the following transformations:

$$\begin{aligned} \hat{\delta}_G \psi_\alpha{}^a &= i(D_\alpha\phi)^a \ , \\ \hat{\delta}_G \mathcal{B}^{\alpha\beta a} &= -i e_0[\phi, \chi^{\alpha\beta}]^a \ , \end{aligned} \tag{2.4}$$

where ϕ^a is a secondary ghost (or G-ghost) field and $\hat{\delta}_G$ is the BRST operator of the ghost symmetry. To gauge fix this G-symmetry, we begin by writing

$$\mathcal{L}^{(G)}_{GF+FP} = \hat{\delta}_W[ic_0\lambda(D_\alpha\psi^\alpha + sb) + c_1\chi^{\alpha\beta}\mathcal{B}_{\alpha\beta}] \ , \tag{2.5}$$

where $\hat{\delta}_W \equiv \hat{\delta}_I + \hat{\delta}_G$, the c_i are arbitrary real constants, λ is the G-anti-ghost, s is the G-gauge parameter and b is the G-auxiliary field. However, there are two more symmetries we must worry about. They are the scaling and U symmetries[4]. The global scaling and U symmetry weights of the fields $(A, \phi, \lambda, \psi, \chi)$ are $(1, 0, 2, 1, 2)$ and $(0, 2, -2, 1, -1)$, respectively. In order to maintain these two invariances (in particular, the latter one), we must take $b \equiv e_0[\phi, \eta]$, for some anti-commuting field, η, which is the transform of λ; *i.e.*, $\hat{\delta}_G\lambda \equiv 2\eta$. This latter field has scaling and U weights 2 and -1, respectively. Observe that $\hat{\delta}_G\eta \neq 0$, contrary to what one would expect from traditional BRST gauge fixing. Short of introducing new fields, we find that the ansatz used above is unique. As the algebra of the symmetry, given in eqn. (2.4), closes only up to an ordinary YM gauge transformation[4], we have $\hat{\delta}_G\eta = -e_0\frac{i}{2}[\phi, \lambda]$. The natural choices for the constants c_i are $c_0 = -\frac{1}{2}$ and $c_1 = \frac{1}{8}$. This is the case since on manifolds of the form $M = Y^3 \times R^1$, these choices maintain the time-reversal symmetry of the Lagrangian. Putting all of the above together, we find

$$\begin{aligned}
\mathcal{L} - \mathcal{L}_0 &\equiv (\mathcal{L}^{(I)} + \mathcal{L}^{(G)})_{GF+FP} \\
&= -i\chi^{\alpha\beta}D_\alpha\psi_\beta - i\eta D^\alpha\psi_\alpha \\
&\quad + \frac{1}{2}\lambda D^\alpha D_\alpha\phi - i\frac{1}{2}e_0\lambda[\psi^\alpha, \psi_\alpha] - \frac{i}{8}e_0\phi[\chi^{\alpha\beta}, \chi_{\alpha\beta}] \\
&\quad + se_0(i\phi[\eta, \eta] + \frac{1}{4}e_0[\phi, \lambda]^2) \\
&\quad + \frac{1}{8}\mathcal{B}^{\alpha\beta}\mathcal{B}_{\alpha\beta} + \frac{i}{4}\mathcal{B}^{\alpha\beta}(F_{\alpha\beta} + \tilde{F}_{\alpha\beta}) \\
&= \frac{1}{8}(F + \tilde{F})^2 - i\chi^{\alpha\beta}D_\alpha\psi_\beta - i\eta D^\alpha\psi_\alpha \\
&\quad + \frac{1}{2}\lambda D^\alpha D_\alpha\phi - \frac{i}{2}e_0\lambda[\psi^\alpha, \psi_\alpha] - \frac{i}{8}e_0\phi[\chi^{\alpha\beta}, \chi_{\alpha\beta}] \\
&\quad + se_0(i\phi[\eta, \eta] + \frac{1}{4}e_0[\phi, \lambda]^2) \ ,
\end{aligned} \tag{2.6}$$

where the second equation follows from the first after use of the auxiliary field's equation of motion. The Lagrangian (2.6) can be brought to the form $\mathcal{L} - \mathcal{L}_0 = \frac{1}{e_0{}^2}\mathcal{L}'$, where $\mathcal{L}'$ is independent of the coupling constant, by a rescaling of the fields. It should be observed that for reasons of renormalizability we cannot choose $s = 0$. Once again naturalness arguments favor the choice $s = -\frac{1}{2}$, which maintains the time-reversal symmetry of the Lagrangian. We have thus obtained Witten's full YM Lagrangian (see eqns. (2.13) and (2.41) in ref. [4]) by straightforward gauge fixing.

3. RENORMALIZATION IN WITTEN'S YM TQFT*

Practical quantum field theory computations using the Lagrangian given in eqn. (2.6) become manageable if we make use of the following identification which is suggested by the time-reversal symmetry of the Lagrangian:

$$\chi^0 \equiv \eta, \quad \chi^i \equiv \chi^{0i} = \frac{1}{2}\epsilon^{ijk}\chi_{jk}. \tag{3.1}$$

Under time-reversal we have :

$$\begin{aligned} \phi &\to \lambda, \\ \lambda &\to \phi, \\ \psi^\alpha &\to \chi^\alpha, \\ \chi^\alpha &\to -\psi^\alpha. \end{aligned} \tag{3.2}$$

This symmetry suggests a "particle-antiparticle" relationship between ϕ and λ, and ψ^α and χ^α, respectively. This becomes even more suggestive when we rewrite the Lagrangian using the above identification:

$$\begin{aligned} \mathcal{L} - \mathcal{L}_0 =& \frac{1}{8}(F + \tilde{F})^2 - i\chi \not{D}_F \psi + \frac{1}{2}\lambda D^\alpha D_\alpha \phi \\ &- \frac{ie_0}{2}\lambda[\psi^\alpha, \psi_\alpha] - \frac{ie_0}{2}\phi[\chi^\alpha, \chi_\alpha] - \frac{e_0{}^2}{8}[\phi, \lambda]^2 \quad , \end{aligned} \tag{3.3}$$

* In this section, we will only consider a flat Euclidean background.

where $\not{D}_F = \Gamma^\mu D_\mu$, with

$$\Gamma^0 = \sigma^0 \otimes \sigma^0,\ \Gamma^1 = \sigma^3 \otimes i\sigma^2,\ \Gamma^2 = i\sigma^2 \otimes \sigma^0,\ \Gamma^3 = \sigma^1 \otimes i\sigma^2 \quad . \tag{3.4}$$

Here the σ^μ are the standard Pauli matrices. We then have that the Γ^μ matrices obey the following algebra:

$$\begin{aligned} \{\Gamma^i, \Gamma^j\} &= -\,2\delta^{ij}, \\ [\Gamma^i, \Gamma^j] &= 2\epsilon^{ijk}\Gamma^k, \\ [\Gamma^0, \Gamma^i] &= 0. \end{aligned} \tag{3.5}$$

Hence, the Γ^μ satisfy a $U(2)$ algebra. It should also be noted that it is not possible to factor out a matrix (e.g. γ^0) from the above in such a way that the $\tilde{\Gamma}^\mu = \gamma^0\Gamma^\mu$ obey the Clifford algebra. Instead, we have $\Gamma^\mu\bar{\Gamma}^\nu + \bar{\Gamma}^\mu\Gamma^\nu = 2g^{\mu\nu}$, where $\bar{\Gamma}^\mu = (\Gamma^0,\ -\Gamma^i)$.

Some simple calculations show that this theory is so constructed that it is renormalizable with only one coupling constant. Indeed, it is manifest from the derivation of the previous section that the time-reversal and BRST symmetries enforce the single coupling constant renormalization. It was suggested in ref [4] that the Lagrangian of eqn. (3.2) has an unconventional $N = 2$ supersymmetry. The latter theory also has a single coupling constant. We see that the $[\phi, \lambda]^2$ term is required as mentioned earlier, since counter-terms of such a form will be necessary to renormalize the box diagrams arising from the $\phi\chi^2$ and $\lambda\psi^2$ couplings.

We will now compute the coupling constant renormalization of this theory. Before proceeding we will need to gauge fix the Yang-Mills symmetry. In the absence of the δ_I symmetry, this would be done in the standard way by introducing the anti-commuting ghosts c and $\bar{c}$. However, we must be careful. For our calculations we will choose the covariant gauge, $\partial_\mu A^\mu = 0$, and the Feynman-'t Hooft gauge for the gauge boson propagator. Now, because of the gauge field's transformation law, given in eqn. (2.3), we will generate an additional ghost term of the form: $\bar{c}\partial_\alpha\psi^\alpha$. Notice that this is purely kinetic, so it will not contribute to our β-function. The Feynman rules for this theory are then given in Table 1. Through straightforward calculations we can now verify that the coupling constant is renormalized as follows:

$$\beta(e) \;=\; -\frac{2e^3C_2(G)}{(4\pi)^2} + O(e^5). \tag{3.6}$$

where $C_2(G)$ is the second Casimir in the adjoint representation of the group G.

For any computation of $\beta(e)$ we will need to know the field strength renormalization. The only new diagrams arising in this computation are given in Table 2. One immediately observes that the $\psi - \chi$ contribute like fermions in the adjoint representation of the gauge group; while the $\phi - \lambda$ give the standard scalar contribution. Adding the pure Yang-Mills diagrams we get that there is no field strength renormalization in the Feynman-'t Hooft gauge. Now to finish this calculation we just borrow the standard results from, for example, the boson-ghost-ghost vertex. It should be noted that the value for $\beta(e)$ given above agrees with those in $N = 2$ super-Yang-Mills theory.[7]

The nonvanishing of $\beta(e)$ may have some implications. The most significant will be that the energy-momentum tensor may not be a BRST commutator. Thus, we would expect that the theory may possess a dependence on the local properties of the background metric after all. In this case, the trace of the energy-momentum tensor will be proportional to the β-function. We did not find a way to generate a $F\tilde{F}$ counter-term in perturbation theory. Should it not be possible to generate such a counter-term, we must begin with $\mathcal{L}_0$ having a separate coupling constant for $F\tilde{F}$, ie.

$$\mathcal{L}_0 \;=\; \theta F\tilde{F} \;\; . \tag{3.7}$$

4. Witten's Topological Gravity

We proceed now to derive Witten's topological gravity action [5] following the same YM fixing procedure as we used above for the topological gauge field theory. Here the fundamental transformation law on the tetrad is

$$\delta_I e_\alpha^{A\dot{A}} = e_\alpha^{B\dot{B}} \theta_{AB,\dot{A}\dot{B}} \ . \tag{4.1}$$

We introduce the BRST gauge fixing Lagrangian and its associated action.

$$\mathcal{L}^{(I)} = \mathcal{L}^{(I)}_{GF+FG} = i\hat{\delta}_I[\chi^{ABCD} W_{ABCD}], \qquad I^{(I)} = \int d^4x \ det\ e \cdot \mathcal{L}^{(I)}, \tag{4.2}$$

where *det e* is the determinant of the tetrad. W_{ABCD} is the self-dual part of the Weyl tensor and transforms as spin (0,2) of $SU_L(2) \times SU_R(2)$. χ_{ABCD} is an anti-ghost with the same spin. The transformations of χ^{ABCD} and W_{ABCD} are given by

$$\begin{aligned}\hat{\delta}_I \chi_{ABCD} =& \mathcal{B}_{ABCD} \\ \hat{\delta}_I W_{ABCD} =& \frac{1}{6}[(\psi_{AB,}{}^{\dot{A}\dot{B}} R_{CD\dot{A}\dot{B}} - e^\alpha_{C\dot{A}} e^\beta_{D\dot{B}} D_\alpha D_\beta \psi_{AB,}{}^{\dot{A}\dot{B}}) \\ &+ (5 \ permutations\ of\ A,B,C,D)] \ ,\end{aligned} \tag{4.3}$$

where $\mathcal{B}_{ABCD}$ is an auxiliary commuting ghost field and the variation of W_{ABCD}, which is given in ref. [5], follows from the variation: $\hat{\delta}_I e_\alpha^{A\dot{A}} = i e_\alpha^{B\dot{B}} \psi_{AB,\dot{A}\dot{B}}$. As in the topological gauge theory, the ghost Lagrangian

$$\mathcal{L}^{(I)} = i\mathcal{B}^{ABCD} W_{ABCD} - i\chi^{ABCD} \hat{\delta}_I W_{ABCD} \ , \tag{4.4}$$

is invariant under the following local symmetry:

$$\begin{aligned}\hat{\delta}_G \psi_{AB,\dot{A}\dot{B}} =& -\frac{i}{4}(e^\alpha_{A\dot{A}} D_\alpha C_{B\dot{B}} + e^\alpha_{B\dot{A}} D_\alpha C_{A\dot{B}} + e^\alpha_{A\dot{B}} D_\alpha C_{B\dot{A}} + e^\alpha_{B\dot{B}} D_\alpha C_{A\dot{A}}) \ , \\ \hat{\delta}_G \mathcal{B}^{ABCD} =& \frac{1}{4}\chi^{EBCD}[(e^\alpha_{E\dot{X}} D_\alpha C^{A\dot{X}} + e^{\alpha A\dot{X}} D_\alpha C_{E\dot{X}} + (cyclic\ perm.)] \\ & -\frac{i}{2}\chi^{ABCD} e^\alpha_{X\dot{X}} D_\alpha C^{X\dot{X}} + iD_\alpha(e^\alpha_{X\dot{X}} C^{X\dot{X}} \chi^{ABCD})] \ .\end{aligned} \tag{4.5}$$

These transformation laws are covariant with respect to $SU_L(2) \times SU_R(2)$ transformations, global U symmetry and scale invariance. Whereas $\hat{\delta}_G\psi_{AB,\dot{A}\dot{B}}$ has the same stucture as $\hat{\delta}_G\psi_\alpha$ (in the topological YM theory with $C^\alpha = e^\alpha_{X\dot{X}}C^{X\dot{X}}$ being the analog of ϕ), $\hat{\delta}_G\mathcal{B}_{ABCD}$ is proportional to the covariant derivative of $C_{X\dot{X}}$, unlike $\hat{\delta}_G\mathcal{B}_{\alpha\beta}$. The source of this difference is that $\hat{\delta}_G F_{\mu\nu}$ has only one covariant derivative of ψ whereas eqn. (4.3) includes $D_\alpha D_\beta\psi$ terms. In any event the proof that $\mathcal{L}^{(I)}$ is invariant under (4.5) proceeds as follows. First we write

$$\hat{\delta}_G\mathcal{L}^{(I)} = i\hat{\delta}_G\mathcal{B}^{ABCD}W_{ABCD} - i\chi^{ABCD}\hat{\delta}_G\hat{\delta}_I W_{ABCD} \quad , \tag{4.6}$$

since $\hat{\delta}_G\chi^{ABCD} = \hat{\delta}_G W_{ABCD} = 0$. Hence

$$\begin{aligned}\hat{\delta}_G\mathcal{L}^{(I)} = iW_{ABCD}[\frac{1}{4}\chi^{EBCD}[(e^\alpha_{E\dot{X}}D_\alpha C^{A\dot{X}} + e^{\alpha A\dot{X}}D_\alpha C_{E\dot{X}}) + (\textit{cyclic perm.})] \\ + \frac{1}{2}\chi^{ABCD}e^\alpha_{X\dot{X}}D_\alpha C^{X\dot{X}} + (D_\alpha W_{ABCD})(e^\alpha_{X\dot{X}}C^{X\dot{X}}\chi^{ABCD})] \\ -i\chi^{ABCD}\hat{\delta}_G\hat{\delta}_I W_{ABCD}.\end{aligned} \tag{4.7}$$

Now by substituting

$$\begin{aligned}\hat{\delta}_G\hat{\delta}_I W_{ABCD} =& \frac{1}{6}[(\hat{\delta}_G\psi_{AB,}{}^{\dot{A}\dot{B}}R_{CD\dot{A}\dot{B}} - e^\alpha_{C\dot{A}}e^\beta_{D\dot{B}}D_\alpha D_\beta\hat{\delta}_G\psi_{AB,}{}^{\dot{A}\dot{B}}) \\ &+ (5 \textit{ permutations of } A,B,C,D)] \\ =& + \frac{i}{4}W^E{}_{BCD}[(e^\alpha_{E\dot{X}}D_\alpha C^{A\dot{X}} + e^{\alpha A\dot{X}}D_\alpha C_{E\dot{X}}) + (\textit{cyclic perm.})] \\ &- \frac{i}{2}W_{ABCD}e^\alpha_{X\dot{X}}D_\alpha C^{X\dot{X}} - iC^\alpha D_\alpha W_{ABCD} \quad ,\end{aligned} \tag{4.8}$$

which was given in ref. [5], it is clear that $\hat{\delta}_G\mathcal{L}^{(I)} = 0$. Continuing to follow the procedure of section 2 we now add to $\mathcal{L}^{(I)}$ a BRST gauge fixing term of the following form:

$$\mathcal{L}^{(G)} = \mathcal{L}^{(G)}_{GF+FG} = \hat{\delta}_W[\Phi e^{\alpha A\dot{A}}(D_\alpha B^{B\dot{B}})\psi_{AB,\dot{A}\dot{B}} + \frac{1}{2}\chi^{ABCD}\mathcal{B}_{ABCD}] \tag{4.9}$$

The Φ field has spin (0,0) and conformal weight 2. It is introduced here following the same reasoning as in ref. [5], namely so that $B_{A\dot{A}}$ has conformal weight 1.

This allows the first term in eqn. (4.9) to transform with a definite weight under local conformal transformations. Note that this term, which in our procedure follows naturally from the fixing of the ghost symmetry, is exactly the term that allowed Witten to introduce kinetic terms (denoted as Z in ref. [5]) for C and some of the components of ψ. Now using the equation of motion for $\mathcal{B}_{ABCD}$ we finally get

$$\begin{aligned}\mathcal{L} \; =& \; \frac{1}{2}W_{ABCD}W^{ABCD} - i\chi^{ABCD}(\psi_{AB,}{}^{\dot{A}\dot{B}}R_{CD\dot{A}\dot{B}} - e^{\alpha}_{C\dot{A}}e^{\beta}_{D\dot{B}}D_\alpha D_\beta\psi_{AB,}{}^{\dot{A}\dot{B}}) \\ &+ \frac{i}{2}\chi_{ABCD}\chi^{EBCD}(e^{\alpha}_{E\dot{X}}D_\alpha C^{A\dot{X}} + e^{\alpha A\dot{X}}D_\alpha C_{E\dot{X}}) \\ &+ \frac{i}{2}\chi^{ABCD}(e^{\alpha}_{X\dot{X}}C^{X\dot{X}}D_\alpha\chi_{ABCD}) \\ &+ \mathcal{L}_3 \;\; ,\end{aligned} \tag{4.10}$$

where $\mathcal{L}_3$ (eqn.(28) in [5]) is given by

$$\begin{aligned}\mathcal{L}_3 \;=\; \Phi[&\lambda^{B\dot{B}}e^{\alpha A\dot{A}}D_\alpha\psi_{AB,\dot{A}\dot{B}} \\ &- \frac{1}{4}(e^{\beta A\dot{A}}D_\beta B^{B\dot{B}})(e^{\alpha}_{A\dot{A}}D_\alpha C_{B\dot{B}} + e^{\alpha}_{B\dot{A}}D_\alpha C_{A\dot{B}} \\ &\qquad\qquad + e^{\alpha}_{A\dot{B}}D_\alpha C_{B\dot{A}} + e^{\alpha}_{B\dot{B}}D_\alpha C_{A\dot{A}}) \\ &- iB^{B\dot{B}}e^{\alpha}_{C\dot{C}}\psi^{AC,\dot{A}\dot{C}}D_\alpha\psi_{AB,\dot{A}\dot{B}} + iB^{C\dot{C}}e^{\alpha}_{C\dot{C}}\psi^{AB,\dot{A}\dot{B}}D_\alpha\psi_{AB,\dot{A}\dot{B}} \\ &+ \frac{i}{2}D_\alpha B^{B\dot{B}}(e^{\alpha}{}_{B}{}^{\dot{E}}\psi_{AC\dot{A}\dot{B}}\psi^{AC\dot{A}}{}_{\dot{E}} + e^{\alpha E}{}_{\dot{B}}\psi_{AB\dot{A}\dot{C}}\psi^{A}{}_{E}{}^{\dot{A}\dot{C}}) + \ldots] \;\; .\end{aligned} \tag{4.11}$$

where '...' denotes other Φ dependent terms. In analogy with the term $c_0 s\hat{\delta}_W[\lambda b]$ in eqn. (2.5) we can add to eqn. (4.9) a term of the form $\hat{\delta}_W[\Phi c^\alpha D_\alpha B^{A\dot{A}}\lambda_{A\dot{A}}]$ which will generate the analog of the last line in eqn. (2.6).

5. Conclusions

We have shown how to derive Witten's topological YM and gravitational theories by standard gauge fixing. This was done in a two-step process. First, the YM field-strength is gauge fixed to be anti-self-dual (an anti-instanton, for example). The resulting Lagrangian was then found to be invariant under a symmetry on the BRST ghost and auxiliary fields. Choosing a gauge slice for this last symmetry while maintaining the global scaling and ghost number (U) symmetries, led us to the full YM-TQFT. Following this procedure, we were also able to derive the gravitational TQFT.

One-loop calculations in the YM theory indicate that the coupling constant β-function does not vanish. Indeed, we find that it is the same as that of the N = 2 pure super-YM theory. This non-zero result means that the bare coupling constant in front of the F^2 term in the Lagrangian, is divergent. As $F\tilde{F}$ also appears with the same coupling constant, we must find a counter-term to remove this infinity. However, although a counter-term of the form F^2 is generated at the one-loop level, we were unable to find a $F\tilde{F}$ term. In this case we would have to start off with $\mathcal{L}_0 = \theta F\tilde{F}$ in order to maintain the renormalizability of the theory. Considering these results, it would be interesting to investigate the metric dependence of the theory.

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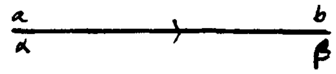
Note added: Upon completion of this work we received two preprints related to Witten's TQFT.[8]

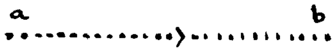
TABLE 1: Feynman Rules

$\chi\psi$ Propagator:



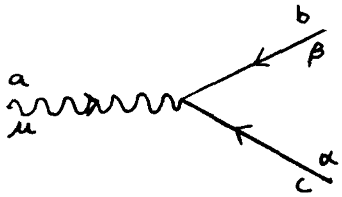
$$i\Delta^{ab}_{\alpha\beta}(p) \;=\; i(\bar{\Gamma}^{\mu})_{\alpha\beta}\frac{p_\mu}{p^2}\delta^{ab}$$

$\phi\lambda$ Propagator:



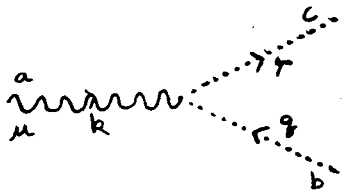
$$i\Delta^{ab}(p) \;=\; \frac{i}{p^2}\delta^{ab}$$

$A\chi\psi$ Vertex:



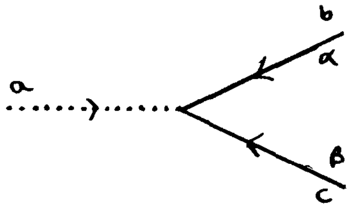
$$-ie(\Gamma^{\mu})_{\alpha\beta}f^{abc}$$

$A\phi\lambda$ Vertex:



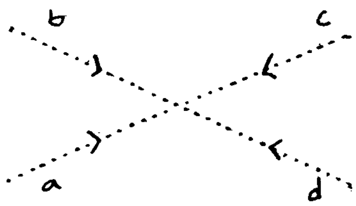
$$-ief^{abc}(r+q)^{\mu}$$

$\phi\psi\psi$ Vertex:



$$\frac{i}{2}ef^{abc}g^{\alpha\beta}$$

$\phi\lambda\phi\lambda$ Vertex:

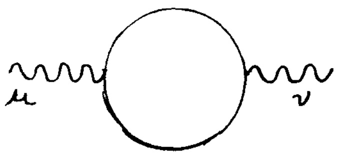
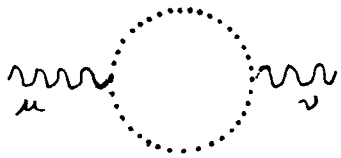


$$-\frac{1}{8}e^2 f^{abe}f^{ecd}$$

Γ-Traces:

$$
\begin{aligned}
Tr(\Gamma^{\mu}\bar{\Gamma}^{\nu}) &= 4g^{\mu\nu}\\
Tr(\Gamma^{\mu}\bar{\Gamma}^{\alpha}\Gamma^{\nu}\bar{\Gamma}^{\beta}) &= -4(\epsilon^{\mu\alpha\nu\beta} + g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\mu\beta} - g^{\alpha\mu}g^{\beta\nu})
\end{aligned}
$$

TABLE 2: One-loop contributions to $\beta(e)$

$\chi\psi$ Loop		$-\frac{4}{3}\frac{C_2(G)e^2}{16\pi^2}\frac{1}{\epsilon}(k^\mu k^\nu \; - \; k^2 g^{\mu\nu})$
$\phi\lambda$ Loop:		$\frac{1}{3}\frac{C_2(G)e^2}{16\pi^2}\frac{1}{\epsilon}(k^\mu k^\nu \; - \; k^2 g^{\mu\nu})$

REFERENCES

1. S. Donaldson, J. Diff. Geom. **18** (1983) 279; J. Diff. Geom. **26** (1987) 397; 'Polynomial Invariants for Smooth 4-Manifolds,' Oxford preprint.
2. A. Floer, Bull. AMS **16** (1987) 279; 'An Instanton Invariant for Three Manifolds,' Courant Institute preprint (1987).
3. M. F. Atiyah, 'New Invariants of Three and Four Dimensional Manifolds,' to appear in the Symposium on the Mathematical Heritage of Herman Weyl (Univ. of North Carolina, May 1987), ed. R. Wells, et al.
4. E. Witten, 'Topological Quantum Field Theory,' IAS preprint, IASSNS-HEP-87/72, February 1988.
5. E. Witten, 'Topological Gravity,' IAS preprint, IASSNS-HEP-87/2, February 1988.
6. E. Witten, 'Topological Sigma Models,' IAS preprint, IASSNS-HEP-87/7, February 1988.
7. P. S. Howe, K. S. Stelle, and P. C. West, Phys. Lett. **124B** (1983) 55; S. Ferrara and B. Zumino, Nucl. Phys. **B79** (1974) 413.
8. J. P. Yamron, 'Topological Actions from Twisted Supersymmetric Theories,' IAS preprint, IASSNS-HEP-88/12, April 1988; J. M. F. Labastida and M. Pernici, 'A Gauge Invariant Action in Topological Quantum Field Theory,' IAS preprint, IASSNS-HEP-88/13, April 1988.