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# HIGH-FREQUENCY LIMIT OF THE LONGITUDINAL IMPEDANCE OF AN ARRAY OF CAVITIES*

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## ABSTRACT

The longitudinal impedance of an array of cylindrically symmetric cavities connected by side pipes is estimated in the high-frequency limit. The expression for the impedance is obtained for an arbitrary number of the cavities. The transition from the case of a single cavity to a periodic structure is studied. The impedance per cell decreases with frequency $\omega$ as $\omega^{-1/2}$ for a small number of cells. For a large number of cells the impedance decreases as $\omega^{-1/2}$ or as $\omega^{-3/2}$ depending on a certain relation between the frequency and the number of cells. The parameter which governs the transition from one regime to the other is found. In particular, for the infinite periodic structure there is only the second regime and the impedance decreases as $\omega^{-3/2}$ for all frequencies.

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## 1. INTRODUCTION

There is a pronounced trend in the recent conceptual designs of the next generation of linear colliders, FEL drivers and synchrotron light sources: they all use many bunches with small bunch lengths. Evaluations of the stability of the particle motion and of the corresponding current limitations in all such devices require an accurate estimate of the impedances[1] for the high-frequency region.

There is a certain discrepancy at present time in the estimates of the longitudinal impedance, obtained for different models. The optical resonator model,[2] which is applied to an infinite periodic set of thin discs, predicts a decrease in the longitudinal impedance with frequency as $\omega^{-3/2}$ and some numerical calculations are consistent with this result.[3] On the other hand, the analytical evaluations of the longitudinal impedance for a single cavity[4,5] give a quite different behavior: the impedance goes down as $\omega^{-1/2}$. This dependence for a single cavity was also obtained in a simple diffraction model.[6–8] In this paper we find agreement with the results obtained in both models. The observed difference in behavior of the impedance obtained in the two models can be attributed to the fact that the region of applicability for each model is different.

Here we present an analytical evaluation of the longitudinal impedance for an idealized RF system, namely, for a linear array of $M$ cylindrically symmetric cavities connected by a pipe of a radius $a$. The frequencies $\omega$ under consideration are well above the cutoff frequency of the pipe but at the same time small in comparison to the particle Lorentz factor $\gamma$:

$$1 \ll \omega a/c \ll \gamma \quad . \tag{1}$$

Figure 1 gives the layout of the geometry considered and the coordinate system used. Each cell consists of a cavity with a side pipe. The length of a cavity is $g$, the radii of the cavity and of the pipe are $b$ and $a$, respectively. The total length of a cell is $L$. At the entrance and at the exit the system of cells is connected to semi-infinite pipes of the same radius $a$. The number of cavities $M$ is considered to be a variable parameter. In the limit of small $M \approx 1$ our calculation gives the same result as one obtained for a single cavity. When $M$ is very large there are two frequency regions with different behavior of the impedance. In the asymptotic region of extremely large frequencies the impedance decreases as $\omega^{-1/2}$. For moderate (but still very large) frequencies, which satisfy a criterion developed in the paper, the impedance falls as $\omega^{-3/2}$. We give the parameter which governs the transition from one regime to another. In particular, for the infinite periodic structure, there is only the second regime and the impedance decreases as $\omega^{-3/2}$ for all high frequencies. Such behavior of the impedance agrees perfectly with the results mentioned above.

The crucial point in calculations of the radiation fields for any periodic structure is an accurate description of the interference of waves produced in different cells. The interference pattern of a field in a periodic structure can be built up only if the structure is long enough. Let us consider the phase shift

$$\Delta\phi = k_{\|}(z_2 - z_1) - \omega(t_2 - t_1) \tag{2}$$

between two waves radiated by a relativistic particle with velocity $v$ at two obstacles with coordinates $z_1$ and $z_2 = z_1 + S$ along the structure at the moments $t_1$ and $t_2 = t_1 + S/v$, correspondingly. The longitudinal component of the wave vector $k_{\|}$ for a structure with a transverse dimension $a$ is related to the frequency in the following way:

$$k \equiv \omega/c = \sqrt{k_{\|}^2 + \nu^2/a^2} \quad ,$$

where $\nu$ is of the order of one. For a relativistic particle $v \approx c$ and in the high-frequency limit $ka \gg \nu$ the phase shift is of order of $S/2ka^2$. For the interference to be of any importance the phase shift in Eq. (2) has to be of the order of $\pi$ or larger. This means that for a periodic structure with the cell length $L$ the number of cells $M = S/L$ must satisfy the following inequality:

$$M > \pi(ka)(a/L) \quad . \tag{3}$$

Note that this requires larger $M$ if the frequencies under consideration become higher. If condition (3) is satisfied, behavior similar to that of an infinite periodic structure can be expected. Otherwise, the impedance per cell is close to the impedance of a single cavity. For an accelerator with a given length or a given $M$ the behavior of the impedance per cell in the limit $k \to \infty$ is always the same as that for a single cavity. The condition in Eq. (3) is necessary but not sufficient. The sufficient criterion is given below.

In the next section the solution of the Maxwell equations, with appropriate boundary and matching conditions, is represented by an expansion of the Fourier harmonics of the field in a series of eigenmodes with unknown coefficients. An exact infinite system of linear algebraic equations for these coefficients is derived. The impedance is expressed in terms of these coefficients. In Sec. 3 the solution of the system is found in the *zeroth* approximation. In the next section an improved *diagonal* approximation is described for a particular case of a single cavity. We demonstrate how the basic system of equations can be simplified to make it solvable. Here we follow the derivation of our previous paper[5] and reproduce the result for a single cavity. In Sec. 5 the developed method is used to derive an explicit expression for the longitudinal impedance of an array of cavities with an arbitrary number of cells. In the next section this expression is used to derive the longitudinal impedance averaged over a suitable frequency interval for a system with a small number of cavities. We obtain here the first two terms of a series in a parameter which depends on $M$ and the frequency. The main term reproduces the same dependence as that for a single cavity. In Sec. 7 we consider the system with a large number of cells. We show that for an infinite periodic structure the behavior of the longitudinal averaged impedance decreases as $\omega^{-3/2}$ in agreement with the optical resonator model. For a *finite number* of cavities there is always an asymptotic region of frequencies where the average longitudinal impedance per one cell decreases as $\omega^{-1/2}$, i.e., in the same way as for a single cavity. The criterion for the transition from the regime $\omega^{-3/2}$ to the regime $\omega^{-1/2}$ is given. In conclusion we discuss the approximations used and the implication of the results for accelerator design and for the evaluation of the total energy loss for a bunch of particles distributed over a finite length.

## 2. THE BASIC SYSTEM OF EQUATIONS

Due to the axial symmetry of the problem, the cylindrical coordinate system with radial coordinate $r$ and longitudinal coordinate $z$ is appropriate. We choose the plane $z = 0$ to coincide with the beginning of the first cavity. For cylindrically symmetric (monopole) modes, Fourier harmonics of the electric field generated by a particle with charge $e$ and velocity $v$ moving along the axis of the system can be written as a sum of the field of a particle in a pipe and the radiation field $E^{rad}$ produced due to the presence of the cavities. For the region inside the pipe $r \leq a$, the radial and longitudinal Fourier components of the electric field are

$$E_r = Q\gamma e^{ikz} G_1(r,a) + E_r^{rad} \quad , \tag{4}$$

$$E_z = -iQe^{ikz} G_0(r,a) + E_z^{rad} \quad , \tag{5}$$

where $k = \omega/c, \quad Q = ek/\pi c\gamma^2$ and

$$G_{0,1}(r,a) = K_{0,1}(kr/\gamma) \mp I_{0,1}(kr/\gamma)\frac{K_0(ka/\gamma)}{I_0(ka/\gamma)} \quad . \tag{6}$$

Here and throughout the rest of this paper the subscript $\omega$ is omitted. The radiation field components inside the $N$-th cavity $a < r < b, \quad NL \leq z < NL+g$ are

$$E_r^N = \sum_{n=0}^{\infty} \lambda_n g_n^{(1)}(r) \sin(\lambda_n \varsigma_N) D_n^N \quad , \tag{7}$$

$$E_z^N = \sum_{n=0} (\mu_n/b) g_n^{(0)}(r) \cos(\lambda_n \varsigma_N) D_n^N \quad , \tag{8}$$

where

$$g_n^{(0,1)}(r) = J_{0,1}(\mu_n r/b) N_0(\mu_n) - N_{0,1}(\mu_n r/b) J_0(\mu_n) \quad , \tag{9}$$

$$\mu_n = b\sqrt{k^2 - \lambda_n^2}, \qquad \lambda_n = n\pi/g, \qquad \varsigma_N = z - NL \quad . \tag{10}$$

$I_{0,1}, K_{0,1}, J_{0,1}, N_{0,1}$ are the modified and regular Bessel functions of the first or second kind of the *zeroth* or first order, respectively, and $D_n^N$ are unknown coefficients for the $N$-th cavity, $N = 0, 1, .., .M-1$. The field components in Eqs. (4) and (5) and Eqs. (7) and (8) are constructed in such a way that their tangential projections are equal to zero on all the metallic surfaces: at $r = a$ in the pipe and $r = b$ in the cavities for an appropriate values of $z$, and at $z = NL$ and $z = NL + g$ for an arbitrary values of $r$ in the interval $b > r > a$.

Matching the radial components of the field from Eqs. (4) and (7) in the $N$-th cavity on the surface $r = a$, $0 \leq \varsigma_N < g$ defines the coefficients $D_n^N$ in terms of the radial component of the radiation field $E_r^{rad}$. Matching the $z$-components of the field from Eq. (5) and (8) at $r = a$ gives a relation between the $r$ and $z$ components of the field $E^{rad}$, produced in the $N$-th cavity. In each region $0 \leq \varsigma_N \leq g$

$$E_z^{rad}(a,z) = \frac{2}{g} \sum_n \left(\frac{\mu_n}{b\lambda_n}\right) \frac{g_n^{(0)}(a)}{g_n^{(1)}(a)} \cos(\lambda_n \varsigma_N) e^{ikNL} \{\gamma Q G_1(a,a) U_n(k) +$$

$$e^{-ikNL} \int_0^g d\varsigma \sin(\lambda_n \varsigma) E_r^{rad}(a, NL + \varsigma)\} \quad . \tag{11}$$

Here

$$U_n(k) = \int_0^g d\varsigma e^{ik\varsigma} \sin(\lambda_n \varsigma) \quad . \tag{12}$$

In each region $g \leq \varsigma_N \leq L$

$$E_z^{rad}(a, z) = 0 \quad . \tag{13}$$

The radiation field $E^{rad}$ satisfies the homogeneous wave equation and has to be finite at $r = 0$. It can be represented as a superposition of cylindrical eigenfunctions with unknown coefficients $A(q)$:

$$E_z^{rad}(r, z) = \frac{e}{\pi v} \int_{-\infty}^{\infty} dq A(q) J_0(\chi_q r/a) e^{iqz} \quad , \tag{14}$$

$$E_r^{rad}(r, z) = -i \frac{ea}{\pi v} \int_{-\infty}^{\infty} dq A(q) \frac{q}{\chi_q} J_1(\chi_q r/a) e^{iqz} \quad , \tag{15}$$

where

$$\chi_q = a\sqrt{k^2 - q^2 + i\epsilon} \quad . \tag{16}$$

An infinitely small imaginary part $\epsilon$ is added in Eq. (16) to comply with the radiation condition. Notice, that the longitudinal impedance is given by the coefficient $A(k)$

$$Z(k) = -Z_0 A(k), \qquad Z_0 = 377 \quad \text{Ohm} \quad . \tag{17}$$

Substituting Eqs. (14) and (15) into Eqs. (11) and (13) one obtains the following integral equation for the function $A(q)$:

$$A(q) = \sum_{N=0}^{M-1} \sum_{n=0}^{\infty} \frac{C_n(k)}{\pi g \lambda_n} \frac{V_n(q)}{J_0(\chi_q)} e^{i(k-q)NL} \left[ \frac{k}{\gamma} U_n(k) G_1(a,a) \right.$$

$$\left. -ia \int_{-\infty}^{\infty} q' dq' U_n(q') \frac{J_1(\chi_{q'})}{\chi_{q'}} e^{i(q'-k)NL} A(q') \right] \quad , \tag{18}$$

where the following notations are introduced:

$$C_n(k) = \frac{\mu_n}{b} \frac{g_n^{(0)}(a)}{g_n^{(1)}(a)} \approx \sqrt{k^2 - \lambda_n^2} \tan\left( (b-a)\sqrt{k^2 - \lambda_n^2} \right) \quad , \tag{19}$$

$$V_n(q) = \int_0^g d\varsigma e^{-iq\varsigma} \cos(\lambda_n \varsigma) \quad . \tag{20}$$

Notice that

$$V_n^*(q) = -i \frac{q}{\lambda_n} U_n(q) \tag{21}$$

and

$$|V_n(q)|^2 = \frac{4q^2 \sin^2 \frac{g}{2}(q - \lambda_n)}{(q^2 - \lambda_n^2)^2} \quad . \tag{22}$$

In the rest of the paper we assume that $b > a$ since when $b = a$ all $C_n(k) = 0$ identically. Consequently, all $A(q) = 0$ and there is no radiation produced as it should be in a smooth pipe.

We look for the solution of Eq. (18) in the form

$$A(q) = \sum_{N=0}^{M-1} \sum_{n=0}^{\infty} \frac{V_n(q)}{J_0(\chi_q)} B_n^N(q) e^{i(k-q)NL} \quad . \tag{23}$$

Functions $V_n(q)e^{-qNL}$ are orthogonal

$$\int_{-\infty}^{\infty} dq V_n^*(q) V_m(q) e^{-iqL(N-N')} = \pi g \delta_{n,m} \delta_{N,N'} \quad . \tag{24}$$

That enables us to obtain the following system of equations for $B_n^N(q)$:

$$B_n^N(q) = \frac{C_n(k)}{\pi g} \left\{ \frac{k}{\lambda_n \gamma a} U_n(k) G_1(a,a) + \right.$$
$$\left. \sum_{N'} \sum_{m} \int_{-\infty}^{\infty} \frac{dq'}{\chi_{q'}} \frac{J_1(\chi_{q'})}{J_0(\chi_{q'})} V_n^*(q') V_m(q') e^{i(q'-k)L(N-N')} B_m^{N'}(q') \right\} \quad . \tag{25}$$

The right-hand side of this equation does not depend on $q$. That means that the coefficients $B_n^N$ do not depend on $q$ either but are functions of $k$ only. Thus the problem of solving the integral equation (18) for function $A(q)$ is reduced to the problem of finding a solution of the system of linear algebraic equations (25) for coefficients $B_n^N$. For a relativistic particle with $\gamma \gg ka$, an asymptotic expansion of modified Bessel functions gives $G_1(a,a) = \gamma/ka$ and Eq. (25) takes the form:

$$B_n^N = \frac{a}{\pi g} C_n(k) \left\{ \frac{i}{ka^2} V_n^*(k) + \sum_{N'=0}^{M-1} \sum_{m} \Gamma_{nm}^{N-N'} B_m^{N'} \right\} \quad , \tag{26}$$

where $N = 0, 1, ..., M-1$ and the following notation for matrix elements is introduced:

$$\Gamma_{nm}^{N-N'} = \int_{-\infty}^{\infty} \frac{dq'}{\chi_{q'}} \frac{J_1(\chi_{q'})}{J_0(\chi_{q'})} V_n^*(q') V_m(q') e^{i(q'-k)L(N-N')} \quad . \tag{27}$$

To satisfy the radiation condition, Eq. (16), the path of integration in the integral in Eq. (27) must be shifted above the negative real axis and below the positive real axis of the complex plane $q$. The integral can be replaced by the sum over residues in the zeros of the Bessel function $J_0(\nu_l) = 0$. Nondiagonal matrix elements are

$$\Gamma_{nm}^{N-N'} = \sum_l \frac{2\pi i}{u_l a^2} \begin{cases} V_n^*(u_l) V_m(u_l) e^{iL(u_l-k)(N-N')} & \text{for} \quad \mathrm{N} > \mathrm{N}' \\ V_m^*(u_l) V_n(u_l) e^{-iL(u_l+k)(N-N')} & \text{for} \quad \mathrm{N} < \mathrm{N}' \end{cases} , \tag{28}$$

where

$$u_l = \begin{cases} \sqrt{k^2 - (\nu_l/a)^2} & \text{for} \quad \nu_l < ka \\ \\ i\sqrt{(\nu_l/a)^2 - k^2} & \text{for} \quad \nu_l > ka \end{cases} . \tag{29}$$

In the sum in Eq. (28) all terms with $\nu_l > ka$ are exponentially small. Hence, the summation over $l$ may be truncated at $\nu_l = ka$. The imaginary part of the diagonal term is:

$$\mathrm{Im}\ \Gamma_{nn}^0 = \sum_l \frac{2\pi}{u_l a^2} |V_n(u_l)|^2 , \tag{30}$$

where the summation is performed up to $l$ which satisfy inequality $\nu_l \leq ka$.

The longitudinal impedance in terms of the coefficients $B_n^N$ is

$$Z(k) = -Z_0 \sum_{N=0}^{M-1} \sum_{n=0}^{\infty} V_n(k) B_n^N(k) . \tag{31}$$

So far the system, Eq. (26), is the exact set of equations defining the radiation of an ultrarelativistic particle.

## 3. THE ZEROTH ORDER APPROXIMATION

The system, Eq. (26), is too complicated to be solved exactly. In the high-frequency limit we can expect that it can be solved by the method of iterations. In the *zeroth* order approximation we neglect the second term in the brackets in Eq. (26):

$$B_n^N = \frac{iC_n(k)}{\pi gka} V_n^*(k) \quad . \tag{32}$$

Then the impedance per cell is

$$\frac{Z}{M} = -iZ_0 \sum_{n=0}^{\infty} |V_n(k)|^2 \frac{C_n(k)}{\pi gka} \quad . \tag{33}$$

Notice that in the *zeroth* order approximation the impedance per cell does not depend on the number of cells in the array.

For large wave numbers $k$ the impedance is a fast changing function of $k$ and goes to infinity at the resonance values

$$k_{nl} = \sqrt{\left(\frac{\pi n}{g}\right)^2 + \left(\frac{\pi(l+1/2)}{(b-a)}\right)^2} \quad , \tag{34}$$

which are defined by the equation $C_n^{-1}(k_{nl}) = 0$. The impedance can be presented as a sum of the Breit–Wigner resonances with infinitely small widths. Representing $C_n^{-1}(k)$ in the vicinity of a resonance as

$$C_n^{-1}(k) = R_{nl}^{-1}(k - k_{nl} + i\epsilon) \tag{35}$$

with

$$R_{nl} = -\frac{[\pi(l+1/2)]^2}{k_{nl}(b-a)^3} \quad , \tag{36}$$

the real part of the impedance is given by the sum of $\delta$-functional terms

$$\mathrm{Re}\frac{Z}{M} = Z_0 \sum_{n,l} |V_n(k)|^2 \frac{\pi^2}{gka} \frac{(l+1/2)^2}{k_{nl}(b-a)^3} \delta(k - k_{nl}) \quad . \tag{37}$$

Practically, we are interested in Re$Z$ averaged over some interval of wave numbers $\Delta k$. It is clear that $\Delta k$ has to be large in comparison with the difference between resonance frequencies $\delta k$

$$\delta k = k_{n(l+1)} - k_{nl} \approx \frac{\pi l}{k(b-a)^2} \quad , \tag{38}$$

The choice of an appropriate $\Delta k$ can be made in the following way. The factor $|V_n(k)|^2$ given by Eq. (22) has a maximum value of order of $(g/2)^2$ for $n = n_0$ and rapidly decreases as $(\pi/g|n-n_0|)^{-2}$ for $n \neq n_0$ where

$$n_0 = \left[\frac{kg}{\pi}\right] \quad . \tag{39}$$

Here the square brackets mean the integer part of the argument. The main contribution to the impedance is therefore given by mode $n_0$ which, of course, is different for different $k$. Hence, it is convenient to choose the averaging interval as

$$\Delta k \approx \pi/2g \quad , \tag{40}$$

which for large $k$ is large in comparison with the difference Eq. (38).

For a rough estimate of the real part of the impedance in Eq. (37) it is enough to take into account only the term $n = n_0$. The average impedance therefore is

$$\left\langle \mathrm{Re}\frac{Z}{M} \right\rangle = Z_0 \frac{\pi^2 g}{4ka} \frac{1}{\Delta k} \sum_{l=0}^{l_{max}} \frac{(l+1/2)^2}{k_{nl}(b-a)^3} \quad , \tag{41}$$

where

$$l_{max} = (b-a)\sqrt{k/\pi g} \quad . \tag{42}$$

The impedance, estimated in this way with $\Delta k$ from Eq. (40), differs from Lawson's estimate[6]

$$\left\langle \mathrm{Re}\frac{Z}{M} \right\rangle = \frac{Z_0}{2\pi} \sqrt{\frac{g}{\pi a}} \frac{1}{\sqrt{ka}} \tag{43}$$

only by a numerical factor of $\pi/3$. Numerical calculations confirm that this result is independent of the choice of the size of the interval $\Delta k$.

We conclude that with good accuracy the main contribution to the impedance comes from eigenmodes with eigennumbers

$$n = n_0 \qquad \text{and} \qquad 0 \leq l \leq l_{max} \quad . \tag{44}$$

This result has a very simple physical meaning. For a pillbox cavity the eigenmode with the eigennumbers $(n, l) \gg 1$ is characterized by the wave number

$$k_{nl} = \sqrt{(n\pi/g)^2 + (\nu_l/a)^2}, \qquad \nu_l \approx \pi l \quad . \tag{45}$$

The mode can be considered as a wave with wave vector components $k_\perp = \pi l/g$ and $k_\| = n\pi/g$. Interaction of a particle with the wave gives a contribution to the impedance for a frequency $\omega$ if $\omega/c \approx k_{nl}$ and the phase shift within the time of flight through the cavity $g/v$ is small:

$$(\omega - k_\| v)(g/v) < \pi/2 \quad . \tag{46}$$

For a relativistic particle and for $n = n_0$ from Eq. (39), this condition means that $l < \sqrt{ka^2/\pi g}$ which is essentially the same as given by Eq. (42).

The *zeroth* order approximation does not take into account either the interference of the radiation from different cavities or the finite widths of the resonances contributing to the impedance. In the next section we derive a method which allows us to improve the calculations.

## 4. DIAGONAL APPROXIMATION

We start with a somewhat simpler case of a single cavity. In this particular case Eq. (26) takes the form:

$$B_n = \frac{a}{\pi g} C_n(k) \left\{ \frac{i V_n^*(k)}{ka^2} + \sum_m \Gamma^0_{nm} B_m \right\} \quad . \tag{47}$$

In the *zeroth* order approximation the sum on the right hand side of this equation was neglected altogether. The approximation can be improved by taking into account the fact that the main contribution to the sum is given by the diagonal term $m = n$. All the other terms give only small corrections and can be taken into account by the method of iterations. In this *diagonal approximation*[5] we get for the impedance the following expression:

$$Z(k) = -iZ_0 \sum_n \frac{|V_n(k)|^2}{ka^2 y(k)} \quad , \tag{48}$$

in which the definition

$$y(k) \equiv \frac{\pi g}{a} C_n^{-1} - \Gamma_{nn}^0 \approx \frac{\pi g}{a} \frac{\cot\left((b-a)\sqrt{k^2 - \lambda_n^2}\right)}{\sqrt{k^2 - \lambda_n^2}} - \Gamma_{nn}^0 \tag{49}$$

is used. The sum in Eq. (48) is again determined mainly by terms $n \approx n_0$.

Similarly to what is done in Eq. (33) the impedance in Eq. (48) can be represented as a sum over the Breit–Wigner terms. The resonance frequencies are now given by the condition Re $y(k) = 0$ and finite resonance widths are defined by Im $\Gamma_{nn}^0$. Evaluation of $\Gamma_{nn}^0$ has been done in Ref. 5. For $k$ within the range $n\pi/g < k < (n+1)\pi/g$ a good estimate for $\Gamma_{nn}^0$ is:

$$\Gamma_{nn}^0 = (i-1)\frac{g}{a}\sqrt{\frac{\pi g}{2k}} \quad . \tag{50}$$

The resonance frequency shift, given by Re$\Gamma_{nn}^0$ is small and the expansion around a resonance frequency $k_{nl}$ takes the form:

$$y(k) = R_{nl}^{-1}(k - k_{nl} + i\gamma_{nl}) \quad , \tag{51}$$

where

$$R_{nl} = -\frac{al^2}{g^2(b-a)l_{max}^2} \quad , \tag{52}$$

$$\gamma_{nl} = \frac{1}{g\sqrt{2}}\left(\frac{l^2}{l_{max}^3}\right) \quad . \tag{53}$$

Hence, in the diagonal approximation Re$Z$ is not singular, as it was in the zero-th approximation in Eq. (37), although it may have rather sharp peaks if $\gamma_{nl}$ is small. That is the main qualitative feature of the diagonal approximation for a single cavity.

The ratio of the resonance width $\gamma_{nl}$ to the distance between adjacent resonances $\delta k$ is small for the resonances with $l < l_{max}$:

$$\frac{\gamma_{nl}}{\delta k} \approx \frac{l}{l_{max}} < 1 \quad . \tag{54}$$

Therefore, the averaging over $\Delta k$ for resonances with different $l$ may be performed independently. Since the integral over a Breit–Wigner resonance does not depend on its width, the result for the real part of the impedance is the same as in Eq. (43). The diagonal approximation allows us to estimate corrections, given by the next iterations, and to prove that in the high-frequency limit they are small.[5]

## 5. GENERAL EXPRESSION FOR THE LONGITUDINAL IMPEDANCE

Consider now a structure consisting of $M$ cells. The interference of waves, generated in different cells is crucial to the evaluation of the impedance for the multi-cell structure and has to be taken into account. We describe the interaction of a particle with each cell in the same way as it is done above for a single cavity. Therefore, we consider Eq. (26) in the diagonal approximation for the lower indices, retaining only terms $m = n = n_o$, but keeping the summation over the upper indices $N'$. It gives:

$$B_n^N y(k) = \frac{iV_n^*}{ka^2} + \sum_{N'=0, N' \neq N}^{M-1} \Gamma_{nn}^{N-N'} B_n^{N'} \quad , \tag{55}$$

where $N = 0, 1, ..., M-1$; $\Gamma_{nn}^{N-N'}$ is defined in Eq. (28) and $y(k)$ is defined in Eq. (49).

It should be noticed that the system Eq. (55) is difficult to solve numerically for the interesting case $M \sim ka \gg 1$. Indeed, the rank of the corresponding matrix is $M$. In addition, the coefficients in Eq. (55) oscillate rapidly with a typical period of $1/M$. Therefore the computational time for the calculation of the averaged impedance increases with $M$ as $M^3$.

To simplify Eq. (55) consider the behavior of its matrix elements as given in Eq. (28). All the elements with $N < N'$ contain factors which oscillate with the sum frequencies $u_l + k \sim 2k$. After averaging over the frequency interval they would give only a negligibly small contribution. On the other hand, all the elements with $N > N'$ contain factors which oscillate with the small difference frequencies $u_l - k$. These terms describe the interaction of a particle with the waves traveling in the same direction. Therefore, we may assume that

$$\Gamma_{nn}^{N-N'} = 0 \qquad \text{for} \qquad N < N' \quad , \tag{56}$$

and rewrite Eq. (55) in the form

$$B_n^N y(k) = \frac{iV_n^*}{ka^2} + \sum_{N'=0}^{N-1} \Gamma_{nn}^{N-N'} B_n^{N'} \quad . \tag{57}$$

By omitting the terms with $N' > N$ we neglect the interaction of a particle with the waves traveling in the opposite direction. In particular, we neglect the decay of the modes in the cavities into these waves. Since we do that in the non-diagonal terms, for consistency the same should be done in the diagonal terms as well. In other words, Im $\Gamma_{nn}^0$ in the definition of $y(k)$ Eq. (49) should be divided by 2.

Equations (57) are the recurrence relations between coefficients $B_n^N$. All the coefficients can be found sequentially starting with the *zeroth* one :

$$B_n^0 = \frac{iV_n^*}{ka^2 y(k)} \quad . \tag{58}$$

Notice that expression (58) gives the impedance of a single cavity.

It is also possible to solve the system of Eqs. (57) explicitly. To do that we notice that the $N$-th coefficient is expressed through coefficients with indices $N' < N$. Although we are interested in only the first $M$ coefficients, the procedure can be formally extended to any $N$. Since the kernel $\Gamma_{nn}^{N-N'}$ depends only on the differences $N - N'$, Eq. (57) can be solved by applying the discrete Laplace transformation. The discrete Laplace transforms of $B_n^N$ and $\Gamma_{nn}^N$, respectively are defined in the complex plane $s$ as follows:

$$B_n(s,k) = \sum_{N=0}^{\infty} e^{-Ns} B_n^N \quad , \tag{59}$$

$$\Gamma_n(s,k) = \sum_{N=1}^{\infty} e^{-Ns} \Gamma_{nn}^N \quad , \tag{60}$$

with $\sigma \equiv \mathrm{Re}\ s > 0$.

Hence, the Laplace transform of a solution of Eq. (57) is:

$$B_n(s,k) = \frac{iV_n^*}{ka^2} \frac{1}{(y(k) - \Gamma_n(s,k))(1 - e^{-s})} \quad . \tag{61}$$

The inverse transformation now gives the solution of Eq. (57):

$$B_n^N = \int_{-i\pi+\sigma}^{i\pi+\sigma} \frac{ds}{2\pi i} e^{Ns} B_n(s,k), \qquad \sigma > 0 \quad . \tag{62}$$

The impedance in Eq. (31) of an array with arbitrary number of cells $M$ is now given by the following expression:

$$Z(k) = -\frac{Z_0}{4\pi k a^2} \sum_{n=0}^{\infty} |V_n(k)|^2 \int_{-i\pi+\sigma}^{i\pi+\sigma} ds \frac{e^{Ms} - 1}{(y(k) - \Gamma_n(s,k))(\cosh s - 1)} \quad . \tag{63}$$

Here

$$\Gamma_n(s,k) = \sum_{l=0}^{\infty} \frac{2\pi i |V_n(u_l)|^2}{u_l a^2} \frac{1}{e^{iL(k-u_l)+s} - 1} \tag{64}$$

and

$$u_l = \sqrt{k^2 - (\nu_l/a)^2 + i\epsilon}, \qquad \nu_l \approx \pi l \quad . \tag{65}$$

The integrand in Eq. (63) is the same on two parallel lines $s = -i\pi + \sigma$ and $s = +i\pi + \sigma, -\infty < \sigma < 0$ . Therefore, we can add integrals over these two lines extending the contour of integration in the complex plane $s$ from $-\infty - i\pi$, then from $-i\pi + \sigma$ to $i\pi + \sigma$ and back to $-\infty + i\pi$. The integral is then equal to the sum of the residues at the root of the equation $\cosh s = 1$ and at the roots of the equation

$$y(k) = \Gamma_n(s,k) \quad . \tag{66}$$

Expression (63) gives the longitudinal impedance of an array with an arbitrary number of cavities $M$. The next two sections contain a more detailed analysis of the general formula of Eq. (63) for small and large $M$, respectively.

## 6. SMALL NUMBER OF CAVITIES

For an array with small $M$, it is convenient to rewrite Eq. (63) introducing a new variable $t = e^{-s}$. That inverts the infinite point into zero and transforms the contour of integration into a circle with radius $|t| < 1$:

$$Z = -\frac{Z_0}{2\pi k a^2} \sum_n |V_n(k)|^2 I(k) \quad , \tag{67}$$

where

$$I(k) = \oint \frac{dt(1 - t^M)}{t^M (1-t)^2 \big(y - T_n(t,k)\big)} \quad . \tag{68}$$

Here the function $T_n(t,k) \equiv \Gamma_n(s,k)$. The only singularity inside the contour is at the point $t = 0$. The function $T_n(t,k)$ may be expanded into series over $t$:

$$T_n(t,k) = t f(t,k) \quad , \tag{69}$$

where

$$f(t,k) = \sum_{m=1}^{\infty} \sigma_m t^{m-1} \quad , \tag{70}$$

with the coefficients

$$\sigma_m(k) = \sum_l \frac{2\pi i}{a^2 u_l} |V_n(u_l)|^2 e^{-imL(k-u_l)} \quad . \tag{71}$$

The integral in Eq. (68) is given by the finite double sum:

$$I(k) = \frac{2\pi i}{y} \sum_{j=0}^{M-1} \left(\frac{1}{y}\right)^j \sum_{m=0}^{M-j-1} \frac{M-j-m}{m!} \left[\frac{\partial^m f^j(t)}{\partial t^m}\right]_{t=0} \quad . \tag{72}$$

The same result can be obtained by solving the triangular matrix equation (57) directly.

In particular, for $M = 1$, $I(k) = 2\pi i/y$. That again gives the impedance of Eq. (48), and after averaging over a frequency interval, the result of Eq. (43) for a single cavity.

In general, for small $M$ the main contribution in Eq. (72) is given by terms with small $j, m$:

$$I(k) = \frac{2\pi i}{y} \left[M + \frac{1}{y} \sum_{m=0}^{M-2} (M-m-1)\sigma_{m+1} + ...\right] . \tag{73}$$

Suppose now that the following condition is fulfilled:

$$\chi_{1/2} \equiv \frac{ML}{2k} \left(\frac{\pi}{a}\right)^2 \ll 1 \quad . \tag{74}$$

An estimate for $\sigma_m$ gives in this case:

$$\sigma_m \simeq (1+i) \left(\frac{g}{2a}\right)^2 \sqrt{\frac{\pi a^2}{mkL}} \quad , \tag{75}$$

and from Eq. (73) we obtain

$$I(k) = \frac{2\pi i M}{y}\left[1 + \frac{4}{3}\frac{\sqrt{M}}{y}\sigma_1 + ...\right] \quad . \tag{76}$$

The average impedance per cell is now given as an expansion:

$$\left\langle \mathrm{Re}\left(\frac{Z}{M}\right)\right\rangle = \frac{Z_0}{2\pi}\sqrt{\frac{g}{\pi k a^2}}\left[1 + \frac{2}{5k(b-a)}\sqrt{\frac{\pi M}{kL}} + ...\right] \quad , \tag{77}$$

in a parameter $\tau \sim M^{1/2}/(ka)^{3/2}$ which is small, provided the condition in Eq. (74) is fulfilled. In this case, the real part of the impedance per cell is the same as that for a single cavity. Notice that the condition in Eq. (74) is the opposite of the condition in Eq. (3).

For large $M$ expansion (77) is not applicable. This case is considered in the next section.

## 7. LARGE NUMBER OF CAVITIES

All roots of Eq. (66) are pure imaginary. To prove that it is convenient to change the independent variable of the integrand in Eq. (63): $s = it$. Neglecting exponentially small terms with $l > ka/\pi$, the function $P_n(t,k) \equiv \Gamma_n(it,k)$ can be rewritten as

$$P_n(t,k) = \sum_{l=0}^{ka/\pi} \frac{\pi |V_n|^2}{u_l a^2}\left[\cot\frac{\psi_l}{2} - i\right] \quad , \tag{78}$$

with real $\psi_l = L(k - u_l) + t$. The imaginary part of $P_n(t,k)$ does not depend on $t$ and cancels the imaginary part of $y(k) = \Gamma^0_{nn}(k)$. Therefore, all the roots $t_m(k)$ of equation

$$S(t,k) \equiv y(k) - P_n(t,k) = 0 \tag{79}$$

are real. The pole which arises from the term $(\cosh s - 1)$ in Eq. (63) contributes only to the imaginary part of the impedance.

The real part of the impedance per cell is now given by the sum of residues of the roots $-\pi \le t_m(k) \le \pi$ of Eq. (79). For Re $Z/M$ we get:

$$\text{Re}\,\frac{Z}{M} = -\frac{Z_0}{2ka^2}\sum_n |V_n|2 \sum_m \frac{F(t_m, M)}{(\partial S/\partial t)_{t=t_m}} \quad , \tag{80}$$

where the factor

$$F(t, M) = \frac{\sin^2(Mt/2)}{M\sin^2(t/2)} \tag{81}$$

is introduced.

Formula (80) has also an equivalent integral form:

$$\text{Re}\,\frac{Z}{M} = \frac{Z_0}{2ka^2}\sum_n |V_n|^2 \int_{-\pi}^{\pi} dt F(t, M)\delta(S(t, k)) \quad . \tag{82}$$

The impedance in this form can be averaged over the frequency interval $\Delta k = \pi/2g$ [see Eq. (40)] in the same way as it is done in Sec. 3 for a single cavity:

$$\left\langle \text{Re}\,\frac{Z}{M} \right\rangle = -\frac{Z_0 g}{\pi k a^2}\sum_n |V_n|^2 \sum_{l=0}^{l_{max}} \int_{-\pi}^{\pi} dt \frac{F(t, M)}{(\partial S/\partial k)_{k=k_{nl}}} \quad , \tag{83}$$

where the second summation is performed over the roots $k_{nl}$ of Eq. (79) up to $l_{max}(k)$ which is given in Eq. (42).

Substituting expression (49) for $y(k)$ into Eq. (79), we obtain the following equation which defines the roots $k_{nl}$:

$$\cot x_{nl} = \frac{a}{\pi g(b-a)} x_{nl} \text{Re}\,(P_n(t, k)) \quad , \tag{84}$$

where $x_{nl} = (b-a)\sqrt{k_l^2 - \lambda_n{}^2}$.

Evaluation of the roots $k_{nl}$ is facilitated by the following considerations. On the left-hand side of Eq. (84) there is one cotangent function rapidly changing with $k$. On a given interval $\Delta k$ $\cot x_{nl}$ has a large number ($\sim \sqrt{k}$) of branches going from $+\infty$ to $-\infty$. On the right-hand side of this equation there is a sum of many cotangent functions changing slowly. The period of variation of the cotangent terms in the sum defining Re $P_n(t,k)$ in Eq. (78) is much longer than the considered interval $\Delta k$. It is of the order of $L$ or even smaller for small $l$ (of the order of $kL$). For most intervals of averaging this means that $P_n(t,k)$ is approximately constant in the narrow interval $\Delta k \ll k$. For large $P_n(t,k)$ the roots $k_{nl}$ lay close to the values which are defined by the equation $x_l = \pi l$. Those exceptional intervals where one of the cotangents in Re $P_n$ occasionally is very large will be considered separately below.

Hence, to find $(\partial S/\partial k)_{k=k_{nl}}$ it is sufficient to differentiate the most rapidly varying term $\cot x_{nl}$ in Eq. (79):

$$(\partial S/\partial k)_{k=k_{nl}} = \frac{g^2(b-a)\xi^2}{a}\frac{l^2+(l_{max}/\xi)^2}{l^2} \quad , \tag{85}$$

where the function

$$\xi(t,k) = \frac{a l_{max} \text{Re } P_n(t,k)}{g(b-a)} \tag{86}$$

is approximately independent of $l$. Performing now the summation over $l$ in Eq. (83) and using the estimate $|V_n|^2 \approx (g/2)^2$ for the most important values $n \simeq n_0$ we get:

$$\left\langle \text{Re } \left(\frac{Z}{M}\right) \right\rangle = \frac{2Z_0}{(ka)^{3/2}}\left(\frac{2L}{\pi a}\right)^2 \sqrt{\frac{\pi a}{g}}\Phi(k,M) \quad , \tag{87}$$

where

$$\Phi(k,M) = k\pi g\left(\frac{a}{4L}\right)^2 \int_{-\pi}^{\pi} \frac{dt}{2\pi}\frac{F(t,M)}{\xi^2}\left(1-\frac{\arctan\xi}{\xi}\right) \quad . \tag{88}$$

To evaluate function $\xi(t,k)$ we notice that each root defined by Eq. (84) lies between such frequencies for which one of $\cot x_{nl}$ is infinite. For our purpose it is sufficient to estimate Re $P_n(t,k)$ for $t \ll 1$. For such values of $t$, substantial contribution to the sum give terms for which $L(k-u_l) \sim t \ll 1$. Using the estimate $|V_n|^2 \approx (g/2)^2$ and retaining the first terms in the expansions of cot $\psi_l/2$ in small values $\psi_l < 1$ and of $u_l$ in small values $\nu_l/ka$ we obtain:

$$\text{Re } P_n(t,k) \simeq \frac{\pi g^2}{L} \sum_{l=0} \frac{1}{\nu_l^2 + 2kta^2/L} \quad . \tag{89}$$

Here the summation over $l$ can be extended to infinity since the terms with large $l$ do not change the result. For $\xi(t,k)$ in Eq. (86) we obtain:[9]

$$\xi(t,k) = \frac{a}{2L}\sqrt{k\pi g}\frac{J_1(q)}{qJ_0(q)} \quad , \tag{90}$$

where

$$q = \sqrt{-\frac{2ka^2}{L}\theta(t)}, \quad \theta(t) = \begin{cases} t & \text{for} \quad t<0 \\ \\ t-2\pi & \text{for} \quad t>0 \end{cases} \quad . \tag{91}$$

In Fig. 2 we compare the function Re $P_n(t,k)$ calculated by using the exact formula of Eq. (78) and the approximate expression obtained by substituting expression (90) for $\xi$ into Eq. (86). As one can see the approximate expression reproduces all the features of the exact formula.

For an infinite periodic array of cavities

$$\lim_{M\to\infty} F(t,M) = 2\pi\delta(t) \tag{92}$$

and, according to Eq. (82), the real part of the impedance for such a structure can be expressed as a sum of $\delta$-functional contributions from each eigenmode. This is similar to a pill box cavity, since modes of an infinite array of cavities are stationary.

For the function $\Phi(k, M)$ we obtain in this case

$$\Phi(k, \infty) = 1 - \frac{2\pi L}{a\sqrt{\pi k g}} \quad , \tag{93}$$

and the average real part of the impedance per one cell is:

$$\left\langle \text{Re } \left(\frac{Z}{M}\right) \right\rangle = \frac{2Z_0}{(ka)^{3/2}} \left(\frac{2L}{\pi a}\right)^2 \sqrt{\frac{\pi a}{g}} + O(k^{-2}) \quad . \tag{94}$$

It decreases with frequency as $(ka)^{-3/2}$.

To evaluate the impedance for finite $M \gg ka$ we split the integral in Eq. (88) into two parts $\Phi = \Phi_1 + \Phi_2$, such that in the first part $q$ defined by Eq. (91) is in the region $|q| < \nu_{01}, J_1(\nu_{01}) = 0$, and in the second part $\nu_{01} < |q| < \sqrt{2\pi k a^2/L}$. In the first integral $\xi$ is constant: $\xi \simeq (a/4L)\sqrt{\pi k g}$ and the main contribution comes from the interval $0 < |t| < 1/M$, where the factor $F(t, M)$ is large: $F \simeq M$. As a result $\Phi_1 = 1$. In the second integral the main contributions come from the vicinities of the roots of $J_1(q)$: $q_m = \nu_{1m}$, where $\xi \to 0$. Near the root $q_m$ the function $\xi$ can be approximated by

$$\xi(t) = \frac{a\sqrt{\pi k g}}{2L\nu_{1m}}(q - \nu_{1m}) \equiv \xi'(q - \nu_{1m}) \quad ,$$

from which follows a good approximation for the last factor in Eq. (88):

$$\frac{1}{\xi^2}\left(1 - \frac{\arctan \xi}{\xi}\right) = \frac{(1/3)}{1 + [\xi'(q - \nu_{1m})^2]} \quad .$$

This expression has the correct behavior in the vicinity of the roots $q = \nu_{1m}$ of $\xi(q)$ and decreases as $\xi^2$ far from them. The vicinity of the root $q_m$ contributes to the integral Eq. (88) with the weight $F(t, M) \simeq 4/Mt^2$. The magnitude of this contribution decreases with $m$ as $\nu_{1m}^{-2}$. Therefore, the main contribution comes from the roots $q_m$ which are found in the region $t \leq 1$. That justifies the way the estimate Eq. (90) for $\xi$ is obtained. The second integral $\Phi_2$ is equal to the sum of the contributions of the roots $q_m$:

$$\Phi_2 \approx \frac{1}{48M}\left(\frac{ka^2}{L}\right)^{3/2}\sqrt{\frac{\pi g}{L}} \quad , \tag{95}$$

where we used the formula[9]

$$\sum_m \frac{1}{\nu_{1m}^2} = \frac{1}{8} \quad .$$

Therefore, the factor $\Phi$ for $M \gg ka$ is

$$\Phi = 1 + \frac{1}{48M}\left(\frac{ka^2}{L}\right)^{3/2}\sqrt{\frac{\pi g}{L}} \quad . \tag{96}$$

For $M \to \infty$ $\Phi = 1$ as is shown above. The same is true as far as the second term in Eq. (96) is small:

$$\chi_{3/2} \equiv \frac{1}{48M}\left(\frac{ka^2}{L}\right)^{3/2}\sqrt{\frac{\pi g}{L}} \ll 1 \quad . \tag{97}$$

If the inequality (97) is fulfilled, the real part of the average impedance decreases with frequency as $(ka)^{-3/2}$. For a given large $M$ the transition from the regime $(ka)^{-1/2}$ Eq. (77) to the regime $(ka)^{-3/2}$ Eq. (94) takes place in the range

$$\frac{ka^2}{L} \ll M \ll \left(\frac{ka^2}{L}\right)^{3/2}\sqrt{\frac{\pi g}{L}} \quad . \tag{98}$$

In other words, the real part of the average impedance per cell decreases with frequency as $(ka)^{-3/2}$ if

$$ka \ll M^{2/3}\left(\frac{L}{a}\right)^{4/3}\left(\frac{a}{\pi g}\right)^{1/3} \tag{99}$$

and, as $(ka)^{-1/2}$ for very high frequency,

$$ka \gg M\left(\frac{L}{a}\right) \quad . \tag{100}$$

The intermediate region is the transition area. The transition from one regime to another is illustrated in Fig. 3. The curves represent the function $\Phi$ versus $ka^2/ML$ for different values $M$ and are obtained by numerical integration of Eq. (88) with $\xi$ defined by Eq. (90). The data are in agreement with the analytical estimates given above.

Now we are ready to discuss two exceptional situations which can arise in calculation of the roots Eq. (84). One, mentioned above, happens when the function Re $P_n(k,t)$ reaches a very large value inside an averaging interval. Our evaluation of the roots of Eq. (84) is not valid in this case. The value of the derivative $\partial S/\partial k$ at such a root, however, also becomes very large.Since the derivative enters the denominator of the expression for the average impedance, the contribution from such roots is very small. Evaluation of that contribution shows that the frequency dependence of the average impedance stays the same $\sim k^{-3/2}$. The coefficient in Eq. (94) is defined with the accuracy of the factor of order 1.

Another special case arises when $(\partial S/\partial k)_{k_{nl}}$ becomes very small at the same point where $S(k_{nl}) = 0$. At such point a peak in Re $Z$ can be produced. Such an event, however, when both $S(k_{nl})$ and $(\partial S/\partial k)_{k_{nl}}$ are equal to zero, is rare and requires a realization of a special combination of the system parameters. In general, these resonances might restore the magnitude of the total energy loss to that for a single cavity. Such rearrangement of the emission spectrum is a well-known phenomenon for the radiation in periodic structures. We have not investigated this possibility and leave this question open for future analysis.

## CONCLUSION

The explicit expression for the longitudinal impedance of an array with an arbitrary number of identical cylindrical cavities connected by side pipes is obtained for the high-frequency region $\gamma \gg ka \gg 1$ . Our result is based on the solution of the exact system Eq. (26), derived from the Maxwell equations with appropriate boundary conditions. To obtain the solution we have done two approximations.

Firstly, we have shown that there are only a few modes in the cavities which substantially interact with a relativistic particle in the high-frequency limit. For a single cavity the correct result, within a factor of order of one, is easily obtained in the diagonal approximation when the interaction with only a single mode is taken into account. This approximation is independent of the number of cavities in the array. Secondly, we neglected the interaction of a particle with waves travelling in the opposite direction taking into account only the interaction with waves which travel in the same direction as the particle. That reduces the infinite set of Eq. (26) to the recurrence equations in the form of Eq. (57). They are solved explicitly with the result given by Eq. (63). The interference and phase difference of the waves, generated in different cells, is taken into account. Since we are interested only in the impedance averaged over frequency, there is no need to calculate the exact frequencies of the eigenmodes for the array.

The explicit expression for the impedance in Eq. (63) is valid for an arbitrary number $M$ of cells in the array. Averaging this expression we obtained that the real part of impedance for a small number of cavities decreases with frequency as $k^{-1/2}$. For a large number of cavities the asymptotic frequency region is divided into two. For an extremely high frequency, the real part of the impedance depends on frequency similar to that for a single cavity, i.e., as $k^{-1/2}$. For moderate (but still large) frequencies satisfying the criterion of Eq. (97), the decrease of the impedance is much faster $\sim k^{-3/2}$ due to the interference of the radiated waves emitted from different cavities. There is a continuous transition from one regime to another in the range of values of the parameter $M$ given in Eq. (98).

This result agrees well both with numerical calculations fulfilled for a small number of cavities[7] and with the optical resonator model.[2]

The fast decrease of the real part of the impedance as $k^{-3/2}$ has a direct implication on the design of a short bunch accelerator. Indeed, had the asymptotic decrease of the longitudinal impedance followed the law $k^{-1/2}$, the main contribution to the total energy loss would be given by the high-frequency tail of the impedance and the total energy loss would depend on the longitudinal rms of the bunch $\sigma$ as $\sigma^{-1/2}$. The situation is quite different when the impedance falls off as $k^{-3/2}$. In this case, the total energy loss is defined by the low-frequency range of the impedance and, in general, is smaller than in the first case.

The appropriate parameters for two accelerators, the Stanford Linear Collider (SLC) and the TeV Linear Collider (TLC), are given in Table 1. If the total number of cavities in the accelerator is assumed as the parameter $M$ of the criterion in Eq. (99), the impedance falls off as $k^{-3/2}$ for both designs. The parameter $M$, however, could be smaller than the total number of cavities for different reasons (sectioning of the accelerator, changes in its geometry, production errors, etc.). If that is the case, the longitudinal impedance would fall off slower in the frequency range around the typical bunch frequency $f_\sigma = c/2\pi\sigma$ and the total energy loss would be large. For the impedance to decrease as $k^{-3/2}$, the accelerator should be designed to ascertain the inequality $M \gg M_\sigma$, where the minimal number $M_\sigma$ is:

$$M_\sigma \simeq (a/\sigma)^{3/2}(a/L)^2\sqrt{\pi g/a} \quad .$$

For the considered accelerators, $M_\sigma$ is given in the last row of the table.

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**Table 1. Relevant parameters of two accelerators.**

| N | Parameter | SLC | TLC |
|---|---|---|---|
| 1 | $f_0$ (GHz) | 3.0 | 11.0 |
| 2 | $a$ (cm) | 1.163 | 0.52 |
| 3 | $b$ (cm) | 4.134 | 1.12 |
| 4 | $g$ (cm) | 2.915 | 0.729 |
| 5 | $L$ (cm) | 3.499 | 0.875 |
| 6 | $\sigma$ (mm) | 1.0 | 0.040 |
| 7 | $f_\sigma$ (GHz) | 47.7 | $1.194 \cdot 10^3$ |
| 8 | $M_\sigma$ | 12.3 | $1.1 \cdot 10^3$ |

## FIGURE CAPTIONS

1. Geometry and the coordinate system.
2. Comparison of the exact Eq. (78) (solid curve) and the approximate Eqs. (86) and (90) (dashed curve) formulae for Re$P_n(t,k)$ : $ka = 100.0$.
3. Illustration of the transition from the dependence $\omega^{-3/2}$ to the dependence $\omega^{-1/2}$ for the real part of the longitudinal impedance (see text). Function $\Phi(k,M)$ obtained by a numerical evaluation of the integral in Eq. (88) is plotted versus the parameter $ka^2/ML$ for different numbers $M$ of cavities. a) Blow-up of the region $0 < ka^2/ML < 0.5$, b) region $0.5 < ka^2/ML$. Curves are labeled by numbers corresponding to: 1. M=500; 2. M=1,000; 3. M=3,000; 4. M=10,000; 5. M=30,000.

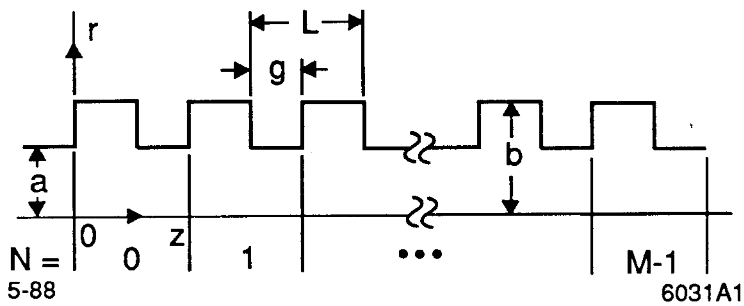


Fig. 1

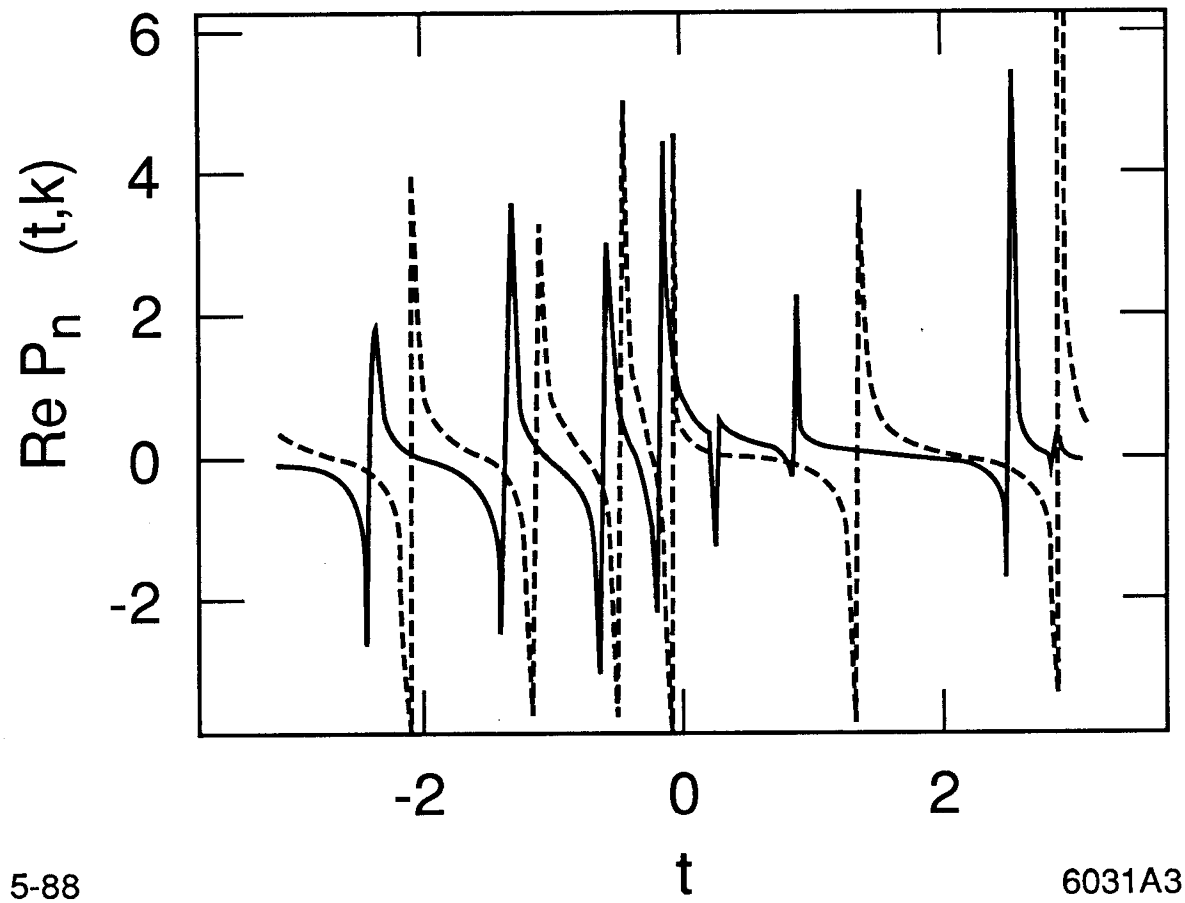


Fig. 2

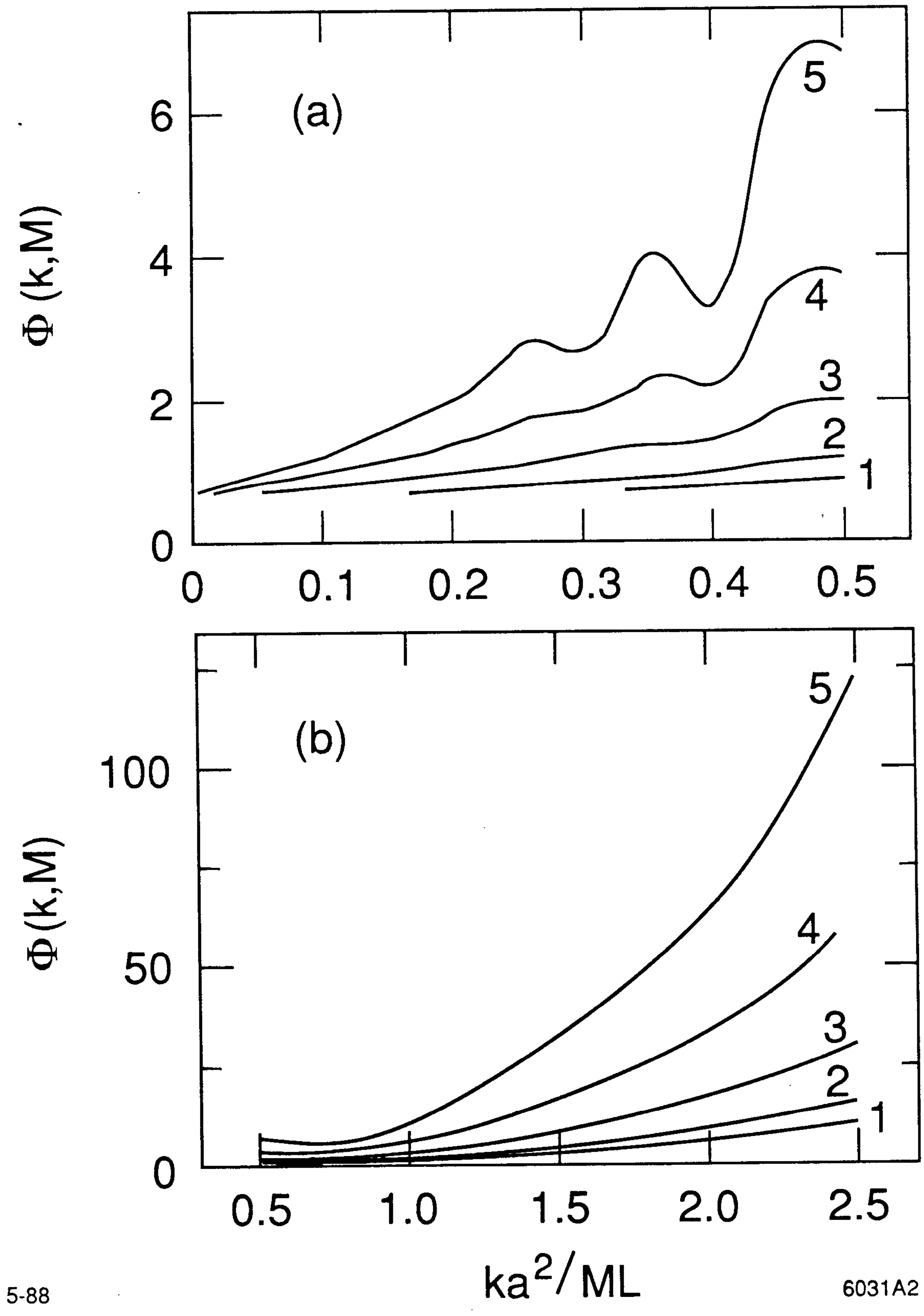
(a)
Φ (k,M)
0
2
4
6
0
0.1
0.2
0.3
0.4
0.5
1
2
3
4
5
(b)
Φ(k,M)
0
50
100
0.5
1.0
1.5
2.0
2.5
ka²/ML
1
2
3
4
5
5-88
6031A2

Fig. 3