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CONTINUUM STRINGS FROM DISCRETE FIELD THEORIES*

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ABSTRACT

We demonstrate that one of the phases of a light-cone lattice gauge theory with an infinite number of colors exactly describes free fundamental strings. The lattice spacing does not have to be taken to zero. Thus, exact rotation and translation invariance can coexist with discrete space.

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1. Introduction

Many systems in statistical mechanics and field theory exhibit string-like excitations. Superconducting fluxoids, domain boundaries in 2+1 dimensional magnets, Nielsen-Olesen vortices and electric flux tubes in QCD are a few examples. In many respects these objects are similar to the idealized objects of string theory which have attracted a lot of interest recently. There are however very significant differences. In particular, the existence of massless gauge bosons and gravitons does not occur in these other systems. This raises the following interesting question: can ideal string behaviour occur in a more or less conventional field theoretic or statistical mechanical system. A related question is whether string theory can be simulated on a computer which stores information in terms of local degrees of freedom and evolves it according to near neighbor interactions on a discrete lattice, as in lattice gauge theory. The purpose of this paper is to show that the answer is positive. The basic premise is that for such a system to work it must have an instability which would ordinarily exclude it as a sensible theory. Indeed, almost everything one might think of calculating diverges except for the spectrum and the S-matrix. The nature of this instability is best understood by considering what the typical string or world sheet looks like in space-time. We will begin by considering the wave function of a string in the light-cone frame. The points of a closed string are parametrized by σ running from 0 to 1. The parameter σ is defined so that the total longitudinal momentum P^+ is uniformly distributed over σ. Thus the longitudinal momentum on an interval $d\sigma$ is $dP^+ = P^+ d\sigma$. In what follows we will need to regulate the string by chopping it into N segments each carrying longitudinal momentum P^+/N. Each segment is replaced by an indivisible 'parton'. The regulated string Hamiltonian is

$$H = \sum_i \frac{1}{2P^+(i)} \left\{ \vec{P}_\perp^2(i) + \left(\vec{X}(i+1) - \vec{X}(i)\right)^2 \right\} \tag{1.1}$$

where $\vec{P}_\perp(i)$ and $\vec{X}(i)$ are the transverse momentum and position of the i-th parton, and $P^+(i)$ is its longitudinal momentum. In the limit $N \to \infty$ the usual continuum limit is recovered. Replacing $\vec{P}_\perp(i)/\delta\sigma$ by the canonical momentum density $\vec{\Pi}_\perp(\sigma)$ and $\delta\vec{X}/\delta\sigma$ by $\partial\vec{X}/\partial\sigma$ we arrive at

$$H = \frac{1}{2P+}\int_0^1 d\sigma\left(\vec{\Pi}_\perp^2 + \left(\frac{\partial\vec{X}}{\partial\sigma}\right)^2\right) \tag{1.2}$$

The regulated string with $2N+1$ partons has $2N$ normal modes of oscillation. A rough way to represent the theory with finite N is to truncate the modes with wave numbers beyond N. Another way to describe the regularization procedure is to say that the string cannot be subdivided into pieces with longitudinal momentum less than $1/N$. Consider next the ground state wave function of the string with regulator N. It is a functional of $\vec{X}(\sigma)$, $\Psi(\vec{X}(\sigma))$. To construct a particular cut-off representation of the ground state, we decompose $X(\sigma)$ into normal modes:

$$X^i(\sigma) = X^i_{cm} + \sum_{n>0}\left(X^i_n\cos(2\pi n\sigma) + \bar{X}^i_n\sin(2\pi n\sigma)\right) \tag{1.3}$$

and take into account the modes with wave numbers from 1 to N. The probability distribution factorizes in the normal mode basis:

$$P(X^i_n) = \left(\frac{n}{2\pi}\right)^{1/2}\exp\left(-n(X^i_n)^2/2\right) \tag{1.4}$$

A typical string configuration can be studied numerically by first generating a set of $2N \times (D-2)$ random numbers X^i_n and $\bar{X}^i_n$ and then plotting the string as a parametrized curve in the transverse space. In the previous paper [1] we have constructed a large number of closed strings in this manner. In fact, an overwhelming majority of this statistical ensemble have similar qualitative features. In particular we would like to call attention to the following.

1. The string is smooth. As the number of modes increases there is no tendency to develop small-scale structure in space. Transverse line curvature is a particular quantitative measure of smoothness. In any number of transverse dimensions greater than two the expectation value of curvature is completely cut-off independent.

2. The total length of string grows linearly with N. The entire string consists of $O(N)$ 'loops'. The structure of a loop is roughly independent of N. Adding modes simply adds more loops. Furthermore, each loop occupies a fraction $\sim 1/N$ of the parameter space σ. Thus, in a fairly uniform manner the individual loops tend to carry a longitudinal momentum $\sim P^+/N$.

3. The size of the region in transverse space occupied by string grows slowly with N. In fact, it can be shown that the *rms* radius of string grows as $\sqrt{\log N}$.

4. The string becomes space-filling in the limit $N \to \infty$. Since the total length grows as N while the radius grows only as $\sqrt{\log N}$ the string tends to cover the same region of transverse space many times. As we remove the cut-off, the string passes arbitrarily close to any point in space.

In addition to observing all of the above properties numerically we have derived most of them analytically relying only on elementary probability theory [1]. Obviously, these properties are very peculiar. The string is unlike any ordinary object studied in a conventional field theory. Consider, for example, the distribution of longitudinal momentum in the transverse space. The density of longitudinal momentum is T^{++}, a component of the energy-momentum tensor. In conventional field theory there exists a sensible distribution of longitudinal momentum. On the contrary, in string theory the longitudinal momentum becomes uniformly smeared over all transverse space. This leads us to the following speculation. Whatever dynamics can lead a system to have ideal string behaviour must be unstable with respect to creation of more and more string bits of lower and lower longitudinal momentum so that eventually string fills all space.

A related point concerns the theorem of Weinberg and Witten [2] on the existence of massless spin-2 bosons. This theorem states that a theory with a well-behaved Lorentz invariant energy momentum tensor cannot generate gravitons dynamically. How do various theories get around this theorem? In Einstein gravity one actually cannot define a Lorentz invariant energy momentum tensor. As we have seen, the form factors of the fundamental string, although Lorentz-invariant by construction, are not at all well behaved. For example, as shown in ref. [1], in the light-cone frame the Fourier transform of T^{++} is non-zero only at the origin. Therefore, it is possible that string theory uses its infinite zero-point motion to allow the existence of gravitons.

2. Lattice Theory on the Light Cone.

In this section we describe a light-cone gauge theory due to Bardeen, Pearson and Rabinovici [3] which has free string-like excitations in the large-N_c limit. This theory has a parameter, μ, which governs the average length of string in the ground state. For μ greater than the critical value μ_c the average length of string is finite in lattice units. For $\mu = \mu_c$ the average length of string diverges. We will be interested in the range $\mu < \mu_c$ where the theory is unstable with respect to infinite growth of strings. This instability is similar to the behaviour of the fundamental strings reviewed in the previous chapter. In fact we will show that, for $\mu < \mu_c$, the strings in this light-cone lattice theory are identical to the fundamental strings. This exact equivalence does not require lattice spacing to be taken to zero.

We introduce the light-cone variables $x^+ = (x^0 + x^{D-1})/\sqrt{2}$, $x^- = (x^0 - x^{D-1})/\sqrt{2}$, and x^i, where i labels the $D-2$ transverse directions. Following Bardeen, Pearson and Rabinovici we replace the transverse space by a $(D-2)$-dimensional cubic lattice $\vec{n}$ with integer coordinates. On each directed link of the lattice L there is a unitary matrix-valued variable which satisfies

$$U_{ij}(L, x^-, x^+) = U^\star_{ji}(-L, x^-, x^+) \tag{2.1}$$

In the light-cone quantization the variable x^+ is treated as time. After passing to the light-cone gauge $A_- = 0$ the gauge theory light-cone Lagrangian becomes

$$L = \frac{1}{g^2} \int dx^- \{ \sum_{links} tr(\partial_\mu U(L) \partial^\mu U(-L)) + \sum_{plaq} tr(U(L_1)U(L_2)U(L_3)U(L_4)) + g^4 \int dx^{-\prime} \sum_{\vec{n}} |x^- - x^{-\prime}| J_-^m(\vec{n}, x^-) J_-^m(\vec{n}, x^{-\prime}) \} \tag{2.2}$$

In the above formula, oriented links $L_1, \ldots, L_4$ form an elementary plaquette of the cubic lattice, the index μ refers to the $+$ and $-$ directions and the lattice spacing has been set to 1 for convenience. The longitudinal momentum current at each lattice site is given by

$$J_-^m(\vec{n}) = \frac{i}{2g^2} \sum_L tr\{T^m(U(L)\partial_- U(-L) - (\partial_- U(L))U(-L))\} \tag{2.3}$$

where T^m are the $U(N_c)$ generators normalized to $tr(T^m T^n) = \delta^{mn}/2$ and the sum is taken over all directed links L beginning at the site $\vec{n}$. To make the theory tractable, ref. [3] relaxes the condition of unitarity. Thus the matrices U are replaced by gM where $M(L, x^-, x^+)$ are general $N \times N$ complex matrices which satisfy

$$M_{ij}(L, x^-, x^+) = M_{ji}^\star(-L, x^-, x^+). \tag{2.4}$$

Upon quantization this rule translates into

$$M_{ij}(L, x^-, x^+) = M_{ji}^\dagger(-L, x^-, x^+) \tag{2.5}$$

where $\dagger$ does not act on indices but has a purely quantum meaning. In order to restore unitarity, ref. [3] introduces the effective potential for M:

$$V(M) = \mu\ M_{ij} M_{ij}^\star + g^2 \lambda\ M_{ij} M_{ik}^\star M_{lk} M_{lj}^\star \tag{2.6}$$

The speculation of ref. [3] is that the continuum limit of the $U(N_c)$ gauge theory can be obtained by tuning μ and λ along a renormalization group trajectory, as the lattice spacing is being taken to zero. In this paper we are not interested in that. Instead we will argue that, with no fine tuning, there exists a broad range of parameters where this theory exactly reproduces fundamental strings. In terms of M the Lagrangian reduces to

$$\begin{aligned} L = \int dx^- \{ \sum_{links} tr\big(\partial_\mu M(L) \partial^\mu M(-L)\big) - V(M(L)) + \\ g^2 \sum_{plaq} tr\big(M(L_1)M(L_2)M(L_3)M(L_4)\big) + \\ g^2 \int dx^{-\prime} \sum_{\vec{n}} |x^- - x^{-\prime}| J^m_-(\vec{n}, x^-) J^m_-(\vec{n}, x^{-\prime}) \} \end{aligned} \tag{2.7}$$

For our purposes it is necessary to add other terms quartic in M which would be trivial if gM was unitary. These are the plaquette-like terms $tr(M^4)$ shown in figure 1. The trace is taken around all possible loops of length 4 and zero area. In fact the second term in (2.6) is of this type. All other terms of this type reside on pairs of links, L and K, beginning at the same vertex:

$$\delta L = -g^2 \lambda' \int dx^- \sum_{L,K} tr\big(M(L)M(-L)M(K)M(-K)\big) \tag{2.8}$$

Using standard methods we derive the light-cone Hamiltonian

$$\begin{aligned} P^- = g^2 \int dx^- \{ \sum_{links} tr\big(\frac{\mu}{g^2} M(L)M(-L) + \lambda\ M(L)M(-L)M(L)M(-L)\big) + \\ \lambda' \sum_{L,K} tr\big(M(L)M(-L)M(K)M(-K)\big) - \sum_{plaq} tr\big(M(L_1)M(L_2)M(L_3)M(L_4)\big) - \\ \int dx^{-\prime} \sum_{\vec{n}} |x^- - x^{-\prime}| J^m_-(\vec{n}, x^-) J^m_-(\vec{n}, x^{-\prime}) \} \end{aligned} \tag{2.9}$$

The link fields may be decomposed into creation and annihilation operators:

$$M_{ij}(x^-) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2k}} \big(A_{ij}(k) \exp(-ikx^-) + B^\dagger_{ij}(k) \exp(ikx^-)\big) \tag{2.10}$$

which obey the commutation relations

$$[A_{ij}(k), A^\dagger_{kl}(q)] = [B_{ij}(k), B^\dagger_{kl}(q)] = \delta_{ik}\delta_{jl}\delta(k-q) \qquad (2.11)$$

Each link is assigned a direction and $A^\dagger_{ij}(k)$ creates a string bit which carries longitudinal momentum k and points from index i to index j along the link's direction. Similarly, $B^\dagger_{ij}(k)$ creates a bit pointing from j to i and opposite to the assigned direction. An oriented string state is defined in the following way. Consider an oriented connected loop Γ consisting of links L_i, with i running from 1 to N. The Fock state associated with this loop is

$$tr\{O^\dagger(L_1, k_1)\dots O^\dagger(L_N, k_N)\}|0> \qquad (2.12)$$

where $O^\dagger_{ij} = N_c^{-1/2} A^\dagger_{ij}$ or $N_c^{-1/2} B^\dagger_{ji}$ depending on whether a given string bit points along or opposite the assigned direction.

An important feature of the $N_c = \infty$ theory is that the Hamiltonian maps the set of connected oriented closed strings into itself. This allows for the construction of a single-string Schroedinger equation. For example, the term

$$\mu \int dx^- \sum_L tr\big(M(L)M(-L)\big) \qquad (2.13)$$

gives rise to the string potential energy $\sim \mu \sum_1^N k_i^{-1}$. The value of μ gets renormalized by normal ordering the $\vec{J}\cdot\vec{J}$ term and the plaquette-like terms shown in figure 1. The other Hamiltonian terms are capable of acting as kinetic terms which locally displace bits of string. The reader can check this by applying various terms in the Hamiltonian to simple string configurations.

Below we are going to discuss in detail the lattice model in one transverse dimension. Then the transverse plaquette terms do not exist and the Hamiltonian consists of the potential terms, the current-current terms and the plaquette-like terms. Each lattice state can be mapped into a function $X(\sigma)$ used in the light-cone description of the fundamental strings. Let the string carry longitudinal momentum k_{tot}. Starting with an arbitrary lattice site, we identify it with $X(\sigma = 0)$. The first link is represented by a segment between $\sigma = 0$ and $\sigma = k_1/k_{tot}$. Thereafter, each link is represented by a segment with length in parameter space proportional to its longitudinal momentum. At the point σ_i the function $X(\sigma_i)$ is given by the lattice coordinates of the corresponding site. Between site σ_i and σ_{i+1}, $X(\sigma)$ can be defined by a linear interpolation.

It is clear that, for sufficiently large values of the string tension μ, the ground state will be dominated by strings with small number of links, each one carrying a significant fraction of the total longitudinal momentum. In this case we do not expect a behaviour similar to the fundamental strings. As we decrease μ it becomes energetically favorable to have a larger number of links with smaller k. In fact, it is clear that, for a sufficiently negative μ, an instability develops which favors strings of infinitely many links each one carrying infinitesimal k. Evidently, in this case the σ-axis becomes densely populated suggesting the possibility of a continuum description in σ-space. Indeed, from here on 'continuum limit' will always refer to the continuum limit in parameter space, and not in real space.

In order to discuss this phase we need a regulator in the form of a minimum allowed longitudinal momentum. Following ref. [4] we will think of the matrices $M(x^-)$ as anti-periodic in the x^- direction. Then the k-space is discretized and the zero-momentum links are excluded so that each link carries an odd integer multiple of the minimum momentum k_{min}. The total length of string cannot exceed $k_{tot}/k_{min} = N$ in lattice units. The limit $N \to \infty$ defines our lattice string theory.

3. Effective Hamiltonian and Equivalence to a Free Field.

We will begin studying this model in the limit $\mu \to -\infty$. In this limit the ground state and low-lying excitations consist of strings of maximum possible length $k_{tot}/k_{min} = N$. To solve for their wave functions we use degenerate perturbation theory in the 'kinetic terms' of the Hamiltonian. It turns out that the normal ordered current-current interaction has vanishing matrix elements between the states of maximum allowed length. The reason is the fact that all the links of the ground state string carry equal longitudinal momenta and the matrix elements of the current-current term are proportional to the differences of momenta between adjacent links. The only terms in the Hamiltonian that act to move the string in the transverse dimension are the plaquette-like terms introduced above. Fortunately, the model based on these terms is soluble exactly. Below we describe the necessary construction in some detail.

If the maximum allowed length is N then the string configurations that need to be included are labeled by series of N pluses and minuses subject to the closed string constraint that their sum is zero. This requires N to be even. It is convenient to think of these configuration as series of Ising spins $\sigma_3(i)$. There are two types of terms that need to be taken into account. The first one,

$$\lambda' \sum_n \int dx^- M_{ij}(n) M_{jk}(n+1) M^\dagger_{lk}(n+1) M^\dagger_{il}(n), \tag{3.1}$$

is shown in figure 1b). The second one,

$$\lambda \sum_n \int dx^- M_{ij}(n) M^\dagger_{kj}(n) M_{kl}(n) M^\dagger_{il}(n), \tag{3.2}$$

is shown in figure 1c). It is not hard to show that, to leading order in N_c, the action of (3.1) on states in the spin representation is equivalent to

$$\lambda' N \sum_n \left(\frac{1+\sigma_3(n)\sigma_3(n+1)}{2} + \sigma_+(n)\sigma_-(n+1) + \sigma_-(n)\sigma_+(n+1)\right) \tag{3.3}$$

where in terms of the standard Pauli matrices

$$\sigma_+ = \frac{\sigma_1 + i\sigma_2}{2}, \qquad \sigma_- = \frac{\sigma_1 - i\sigma_2}{2} \tag{3.4}$$

and we have set $g^2 N_c = 1$. Similarly, the term (3.2) is represented by

$$\lambda N \sum_n (1 - \sigma_3(n)\sigma_3(n+1)) \tag{3.5}$$

Let us choose temporarily $2\lambda = \lambda' = 1$. Then, up to an additive constant, the Hamiltonian is simply given by the quantum XY model:

$$H = N \sum_n (\sigma_+(n)\sigma_-(n+1) + \sigma_-(n)\sigma_+(n+1)) \tag{3.6}$$

Actually, since we have ignored the center of mass motion of strings, the above Hamiltonian is only applicable to states that are translation invariant on the transverse lattice (have zero lattice momentum). A generalization of the effective Hamiltonian to states of finite lattice momentum will be given below. But first let us show that the quantum XY model can be solved by introducing anti-commuting variables

$$\psi_+(n) = i^{-n} \prod_{m<n} \sigma_3(m)\sigma_+(n), \tag{3.7}$$

$$\psi_-(n) = i^{+n} \prod_{m<n} \sigma_3(m)\sigma_-(n). \tag{3.8}$$

These are the staggered fermions on the lattice in the parameter space of the string.* In terms of these variables the fermion number on a site $\frac{1}{2}\sigma_3(n)$ is given by $\frac{1}{2}[\psi_+(n), \psi_-(n)]$. The transverse spatial separation between any two points on the string is then given by twice the fermion number contained between these two

* These fermions have nothing to do with the world sheet fermions of superstrings.

points. Since the string is closed, we must always work in the sector where the total fermion number is zero. We define a fermion doublet

$$\psi(n) = \sqrt{\frac{N}{2}} \begin{pmatrix} \psi_-(2n-1) \\ \psi_-(2n) \end{pmatrix} \tag{3.9}$$

Then in the continuum limit $N \to \infty$ the Hamiltonian reduces to

$$H = i \int d\sigma \psi^\dagger \alpha \frac{\partial \psi}{\partial \sigma} \tag{3.10}$$

together with the boundary condition $\psi(\sigma = 1) = -\psi(\sigma = 0)$. α acts on Dirac indices as the Pauli matrix σ_1. It is well-known [5,6] that a Dirac fermion is equivalent to a periodic boson variable $\phi(\sigma)$ which is defined so that the fermion number density

$$\frac{1}{2}[\psi^\dagger, \psi](\sigma) = \frac{1}{\sqrt{\pi}} \frac{\partial \phi}{\partial \sigma} \tag{3.11}$$

From the previous discussion it follows that the separation between two points on the string is $\frac{2}{\sqrt{\pi}}(\phi(\sigma_1) - \phi(\sigma_2))$. Therefore the bosonized variable $\frac{2}{\sqrt{\pi}}\phi(\sigma)$ is a smeared version of the original lattice position $X(\sigma)$. The boson Hamiltonian equivalent to (3.10) is

$$\frac{1}{2}\Pi_{cm}^2 + 2\pi \sum_m m(a_m^\dagger a_m + \tilde{a}_m^\dagger \tilde{a}_m) \tag{3.12}$$

Since the field ϕ is defined on a circle of radius $\frac{1}{2\sqrt{\pi}}$ the center of mass momentum is restricted to integer multiples of $2\sqrt{\pi}$. This is connected with the fact that we are studying the theory in the sector of zero lattice momentum: the states are invariant under discrete shifts of 1 lattice unit. To introduce non-zero lattice momentum let us consider string states which pick up a phase $\exp(ip)$ when translated by one lattice unit. The effective Hamiltonian for this system is

$$H = N \sum_n (\sigma_+(n)\sigma_-(n+1)\exp(2ip/N) + \sigma_-(n)\sigma_+(n+1)\exp(-2ip/N)) \tag{3.13}$$

To justify the appearance of phases in the above formula we note that whenever $\sigma_+(n)\sigma_-(n+1)$ acts on a string state, the center of mass moves $2/N$ lattice units to the right. Similarly, the conjugate term moves the center of mass to the left. In fact the interaction (3.13) can be obtained from (3.6) by introducing constant vector potential along the string. Let us make the gauge transformation

$$\sigma_-(n) \to \sigma_-(n)\exp(-2ipn/N), \qquad \sigma_+(n) \to \sigma_+(n)\exp(2ipn/N) \qquad (3.14)$$

which reduces (3.13) to (3.6) at the expense of a non-trivial boundary condition

$$\sigma_-(N+1) = \sigma_-(1)\exp(-2ip), \qquad \sigma_+(N+1) = \sigma_+(1)\exp(2ip). \qquad (3.15)$$

In terms of the continuum fermionic variables,

$$\psi(\sigma = 1) = -\psi(\sigma = 0)\exp(-2ip). \qquad (3.16)$$

Upon bosonization, this condition changes the constraint on the values of the center of mass momentum:

$$\Pi_{cm} = 2\sqrt{\pi}(n + \frac{p}{\pi}), \qquad (3.17)$$

where n is an integer. It follows that, once we include string states with non-zero lattice momentum, the zero mode in the boson Hamiltonian acquires continuous spectrum. Therefore, (3.12) becomes the standard light-cone free string Hamiltonian. The physical reason for this is the fact that, as $N \to \infty$, the possible positions of the string center of mass become continuous.

Now we would like to argue that relaxing the constraint $\lambda' = 2\lambda$ on the coefficients of the plaquette-like terms does not in general violate the free boson behaviour demonstrated above. In fact, this adds a term

$$-c\sum_n \sigma_3(n)\sigma_3(n+1) \qquad (3.18)$$

to our effective Hamiltonian. This term can be interpreted as introducing dependence on extrinsic line curvature into the string Hamiltonian [7]. Depending on the sign of the coefficient c it favors alignment or anti-alignment of adjacent links. The continuum limit of this new Hamiltonian is the Thirring model. For any positive value of c, the Thirring model is equivalent to a free boson field. The only effect of changing c is rescaling of the string tension. Increasing c makes string wiggles look bigger on a fixed lattice scale. Therefore pairs of adjacent links become more aligned in agreement with the fact that positive c favors alignment. Small and negative c does not violate free boson behaviour either. However, there exists a critical negative value of c at which this behaviour collapses. Beyond this point the behaviour of lattice string is actually dominated by the lattice. The above discussion actually confirms the simple intuition that one should be allowed to take the spatial continuum limit in our lattice model without changing the equivalence with fundamental strings. However, if we increase the lattice spacing beyond the typical size of string wiggles in the 'fundamental' phase, a transition to the lattice-dominated phase takes place. Another implication of this discussion is that, for a broad range of parameters, introduction of extrinsic curvature terms does not alter the critical behaviour of strings.

We have demonstrated that the spatial behaviour and energy levels of our lattice string theory are completely determined by the equivalence with a massless free field in σ-space. For example, as shown in ref. [1] and reviewed in the appendix, this guarantees that the growth of the mean squared radius of string is logarithmic in the length of string. As in the usual string formalism, the ground state energy is quadratically divergent in the cut-off. However, in our system it has the negative sign. Conventionally, in the light-cone formalism this divergence is absorbed in renormalization of the speed of light.

Consider next the corrections to the limit $\mu \to -\infty$. Then the terms in the Hamiltonian which act to decrease the total length of string become important. For example, the plaquette-like terms can replace three links of longitudinal momentum k_{min} by one link of momentum $3k_{min}$. Such perturbations act locally on the string and in the continuum limit can be represented by a series of operators local in σ-space. The only operators that can affect the critical behaviour in σ-space are renormalizable and super-renormalizable. Let us for simplicity restrict ourselves to the case of zero lattice momentum. Then, the only operators that need to be considered are symmetric under $X \to X+1$ and $X \to -X$. In terms of the bosonic variable $\phi(\sigma)$ these operators can be written as

$$\int d\sigma \Pi^2(\sigma); \qquad \int d\sigma (\frac{\partial \phi}{\partial \sigma})^2; \qquad \int d\sigma F(\phi). \tag{3.19}$$

The first two simply renormalize the already existing terms in the Hamiltonian. Their only effect is rescaling of the string tension. The third operator is restricted to even functions of ϕ periodic under $\phi \to \phi + \frac{\sqrt{\pi}}{2}$. This narrows the choice down to $F(\phi) = \cos 4\sqrt{\pi} n\phi$ where n is an integer. Such operators have dimensions $4n^2$: for any n they are non-renormalizable and therefore irrelevant. The above arguments indicate that a range of μ must exist in which the effect of corrections is absorbed in renormalizing the string tension. Since for large and positive μ there exists another phase of the theory where strings have finite length, there must be a critical value $\mu = \mu_c$ where a phase transition occurs.

A straightforward generalization of the lattice field theory described above can be given in any number of dimensions. If the number of transverse dimensions is 24 then, in the phase where strings are infinite, all the finite energy spectrum is identical to the spectrum of the conventional bosonic string. The same equivalence applies to the expectation values of products of vertex operators.

4. Conclusions.

Perhaps the most striking result of this paper is that a discrete theory is completely equivalent to a free string in continuous space. No spatial continuum limit is required. This occurs because the instability only allows string bits with vanishingly small longitudinal momentum. Typically the light cone time scale for motion of the i-th string bit is $\sim P^+(i)$, i.e., the string bits move very rapidly. These rapid motions completely wash out any memory of the lattice. Presumably this is deeply connected with the critical behaviour of two-dimensional field theory and probably cannot occur for objects other than strings. It is evident that this equivalence does not depend on the details of any particular lattice structure. We expect that there exists a very large universality class of discrete theories which exhibit identical behaviour for a broad range of parameters. The important ingredient in all such theories is the existence of the instability.

Another interesting question is the connection between the theory discussed above and the $N_c = \infty$ QCD. The original work of ref. [3] suggested that QCD is a very special limit of this theory in which the spatial continuum limit is taken as the parameters are carefully tuned along some renormalization group trajectory. This should be contrasted with the fundamental string behaviour which occurs for a broad range of parameters, without necessity of taking the lattice spacing to zero.

The construction of a discrete field theory which exactly reproduces bosonic strings makes it clear that string theory has vastly fewer short distance degrees of freedom in space than a conventional quantum field theory: roughly one degree of freedom per Planck unit is sufficient. This was already suggested by the smoothness of the string pictures of ref. [1]. This probably underlies the exponential fall-off of fixed angle scattering amplitudes at high energy[8].

The model we have studied required the number of colors N_c to be taken to infinity. It is interesting to inquire what new effects are induced by a finite N_c. We find that the string can then split and join in a manner qualitatively similar to the conventional picture of string interactions. The string coupling constant is of order $1/N_c$. It should be possible to determine whether the $1/N_c$ expansion reproduces the bosonic string amplitudes. This is the subject of our current investigation.

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FIGURE CAPTIONS

1) The plaquette-like terms which would be trivial if the link variables were unitary.

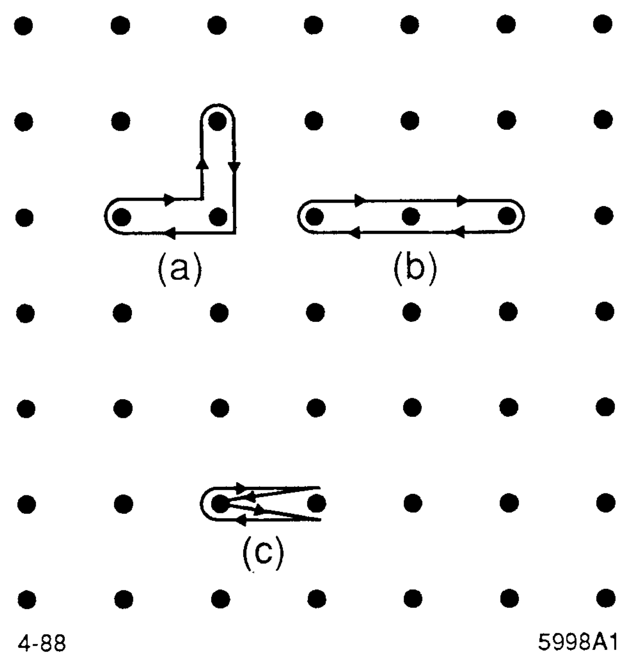

Fig. 1