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FIELD–GRADIENT EFFECT IN QUANTUM BEAMSTRAHLUNG*

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ABSTRACT

The correction terms to the Sokolov–Ternov radiation formula due to variation of the magnetic field strength along the electron trajectory are calculated up to the second order in the power expansion of $\dot{B}\tau/B$, where τ is the formation time of radiation. It is found that the field-gradient effect reduces radiation intensity in the classical regime, and enhances it in the quantum regime. This is then applied to quantum beamstrahlung with Gaussian variation in e^+e^- bunch currents. The correction is shown to be substantial for beam parameters suggested by Himel and Siegrest.

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For future e^+e^- linear colliders, radiation induced by beam-beam collision is expected to be very strong.[1] This radiation, called beamstrahlung, would cause a substantial loss of energy and the degradation on energy resolution. Due to these concerns, the study of the subject has been intensive during recent years.[2] In the calculations done so far, the field was typically treated as locally uniform. The following question arises: On what scale must the field be uniform for this treatment to be valid?

Recently one of us (PC) initiated the investigation[3] on the corrections to the uniform field treatment proportional to $\dot{B}^2/B^2$ and $\ddot{B}/B$ in the field variation. However, due to the deficiency of the mathematical techniques employed, the result was inconclusive. Here we present an improved calculation that gives a definitive evaluation of quantum beamstrahlung that includes the field-gradient effect.

Our aim is to evaluate the average energy loss of the entire beam. To achieve this we want to derive the radiation intensity of one electron that sees the local field and its gradient arising from the oncoming positron beam. Our approach follows the spirit of Schwinger,[4] and Baier and Katkov[5] on quantum synchrotron radiation. It has been shown[5] that, to the accuracy of $1/\gamma$, the radiation intensity in the Coulomb gauge,

$$I = \alpha \int \frac{d^3k}{(2\pi)^2} \langle i| \int dt_1 \int dt_2 \; e^{i\omega(t_1-t_2)} M^*(t_2) M(t_1) |f\rangle \quad , \tag{1}$$

can be approximated by

$$I = \alpha \int dt_1 dt_2 \int \frac{d^3k}{(2\pi)^2} \left[\left(1 + u + \frac{u^2}{2}\right)(\vec{v}_1 \cdot \vec{v}_2 - 1) + \frac{u^2}{2\gamma^2} \right] e^{iu\mathcal{E}\tau(\hat{n}\cdot\tilde{v}-1)} \quad , \tag{2}$$

where $\alpha = 1/137$ is the fine structure constant, $\mathcal{E}$ and $\mathcal{E}\,'$ the initial and final energies, $(t_1, \vec{r_1})$ and $(t_2, \vec{r_2})$ the initial and final coordinates, of the electron, respectively, $(\omega, \vec{k})$ the four-momentum of the photon, $u = \omega/\mathcal{E}\,' = \omega/(\mathcal{E} - \omega)$, $\tau = t_2 - t_1$, $\tilde{v} = (\vec{r}_2 - \vec{r}_1)/\tau$, and $\hat{n} = \vec{k}/\omega$. Let $t = (t_1 + t_2)/2$, then $\int dt_1 dt_2 = \int d\tau dt$.

The photon angle in the phase can be easily integrated to obtain $\exp\{iu\mathcal{E}\tau|\tilde{v}|-1\}/(u\mathcal{E}\tau|\tilde{v}|-i0)$. Next we Taylor expand $\vec{r_1}$ and $\vec{r_2}$ (thus $\tilde{v}$), and $\vec{v}_1$ and $\vec{v}_2$ around t, to the order τ^4, which gives $\vec{v}_1\cdot\vec{v}_2 = \vec{v}^2 - \tau^2\dot{\vec{v}}^2/2 - \tau^4\dot{\vec{v}}\cdot\dddot{\vec{v}}/24$ and $|\tilde{v}| = 1 - 1/2\gamma^2 - \tau^2\dot{\vec{v}}^2/24 - \tau^4(\ddot{\vec{v}}^2 + 3\dot{\vec{v}}\cdot\dddot{\vec{v}})/1440$. Now we make one more approximation by bringing the highest order term in $|\tilde{v}|$ down from the phase:

$$e^{iu\mathcal{E}\tau(|\tilde{v}|-1)} \simeq \left[1 - i\frac{u\mathcal{E}\tau^5}{1440}\left(\ddot{\vec{v}}^2 + 3\dot{\vec{v}}\cdot\dddot{\vec{v}}\right)\right] e^{-iu\mathcal{E}\tau(\frac{1}{2\gamma^2}+\frac{\tau^2}{24}\dot{\vec{v}}^2)} \quad . \tag{3}$$

Define $x \equiv \gamma\dot{v}\tau/2$, and $y \equiv 2u\mathcal{E}/3\dot{v}\gamma^3$, Eq. (2) can then be rewritten as

$$\begin{aligned}\frac{dI}{dt} = \frac{i\alpha}{(2\pi)}\int_0^\infty \frac{\omega^2 d\omega}{u\mathcal{E}\gamma^2}\int_{-\infty}^\infty dx \exp\left\{-i\frac{3}{2}y\left(x+\frac{1}{3}x^3\right)\right\} \\ \times\left\{\frac{1+u}{x-i0} + 2\left(1+u+\frac{u^2}{2}\right)\left(x + \frac{1}{3\gamma^2}\frac{\dot{\vec{v}}\cdot\dddot{\vec{v}}}{\dot{\vec{v}}^4}x^3\right)\right. \\ \left. - i\frac{y}{30\gamma^2}\left(\frac{\ddot{\vec{v}}^2}{\dot{v}^4} + 3\frac{\dot{\vec{v}}\cdot\dddot{\vec{v}}}{\dot{v}^4}\right)\left[(1+u)x^4 + 2\left(1+u+\frac{u^2}{2}\right)x^6\right]\right\} \quad .\end{aligned} \tag{4}$$

It can be seen that the dominant contribution corresponds to the region where $x \sim 1$, or $\tau \sim 1/\dot{v}\gamma$. We shall therefore define a "radiation formation length," $\ell_R \equiv 1/\dot{v}\gamma$, which characterizes the length that an electron travels during an emission process. Integrating over x, we get Bessel functions of fractional order with argument y. For convenience we write $dI/dt = dI_0/dt + dI_2/dt$, with

$$\frac{dI_0}{dt} = \frac{\alpha}{\sqrt{3}\pi}\int_0^\infty \frac{\omega^2 d\omega}{u\mathcal{E}\gamma^2}\left[2\left(1+u+\frac{u^2}{2}\right)K_{2/3} + (1+u)\int_y^\infty K_{5/3}(y')dy'\right] \quad , \tag{5}$$

and

$$\frac{dI_2}{dt}=\frac{\alpha\ell_R^2}{30\sqrt{3}\pi}\int\limits_0^\infty\frac{\omega^2 d\omega}{u\mathcal{E}\gamma^2}\left\{\frac{\dot{B}^2}{B^2}\left[(1+u)\left(yK_{1/3}-\frac{4}{3}K_{2/3}\right)+2\left(1+u+\frac{u^2}{2}\right)\right.\right.$$
$$\left.\times\left[-\left(y+\frac{16}{9y}\right)K_{1/3}+4K_{2/3}\right]\right]$$
$$+\frac{\ddot{B}}{B}\left[3(1+u)\left(yK_{1/3}-\frac{4}{3}K_{2/3}\right)+2\left(1+u+\frac{u^2}{2}\right)\right.$$
$$\left.\left.\times\left[\left(-3y+\frac{4}{3y}\right)K_{1/3}+2K_{2/3}\right]\right]\right\} \quad . \tag{6}$$

In the above expression the vector products $\dot{\vec{v}}\cdot\ddot{\vec{v}}$ and $\dot{\vec{v}}\cdot\dddot{\vec{v}}$ have been replaced by $B\dot{B}$ and $B\ddot{B}$. This is because the only components that $\ddot{\vec{v}}$ and $\dddot{\vec{v}}$ contribute are proportional to $\vec{v}\times\dot{\vec{B}}$ and $\vec{v}\times\ddot{\vec{B}}$, respectively. The term dI_0/dt corresponds to the synchrotron radiation intensity in uniform fields derived by Sokolov and Ternov,[6] and many others.[5] The term dI_2/dt corresponds to the correction to the leading behavior arising from the local field gradient.

Next we proceed to integrate over u in Eqs. (5) and (6). Recall that $u=\omega/(\mathcal{E}-\omega)$, thus $\omega^2 d\omega=\mathcal{E}^3u^2du/(1+u)^4$. For this purpose it is convenient to introduce the representation

$$\frac{1}{(1+u)^k}=\frac{1}{2\pi i}\int\limits_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma(-s)\Gamma(k+s)}{\Gamma(k)}u^s ds \quad , \tag{7}$$

where $-k<\lambda<0$, and Γ's are the gamma functions. We then straightforwardly obtain

$$\frac{dI_0}{dt}=\frac{\alpha m^2}{24\sqrt{3}\pi}\frac{1}{2\pi i}\int\limits_{\lambda-i\infty}^{\lambda+i\infty}ds(3\Upsilon)^{s+2}(s^2+2s+8)\Gamma(-s)\Gamma(s+2)\Gamma\!\left(\frac{s}{2}+\frac{4}{3}\right)\Gamma\!\left(\frac{s}{2}+\frac{2}{3}\right) \ , \tag{8}$$

where the Lorentz invariant parameter $\Upsilon\equiv\dot{v}\gamma^3/\mathcal{E}=\gamma B/B_c$ ($B_c\equiv m^2c^3/e\hbar$) is intoduced, and

$$\frac{dI_2}{dt} = \frac{\alpha m^2 \ell_R^2}{1440\sqrt{3}\pi} \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} s ds (3\Upsilon)^{s+2} \left[\frac{\dot{B}^2}{B^2}(-s^3 + 6s + 28) \right.$$

$$\left. - \frac{\ddot{B}}{B}(3s^3 + 10s^2 + 32s + 36) \right] \Gamma(-s)\Gamma(s+2)\Gamma\left(\frac{s}{2} + \frac{1}{3}\right)\Gamma\left(\frac{s}{2} + \frac{2}{3}\right) \quad , \tag{9}$$

with the constraint $-2/3 < \lambda < 0$.

The asymptotic forms of the above equations can be derived by closing the contour to the right for $\Upsilon \ll 1$, and to the left for $\Upsilon \gg 1$. Thus we find

$$\frac{dI_0}{dt} = \begin{cases} \frac{2}{3}\alpha m^2 \Upsilon^2 + \mathcal{O}(\Upsilon^3) \quad , & \Upsilon \ll 1 \quad , \\ \\ \frac{32}{243}\Gamma(\frac{2}{3})\alpha m^2 (3\Upsilon)^{2/3} + \mathcal{O}(1) \quad , & \Upsilon \gg 1 \quad . \end{cases} \tag{10}$$

The above expression for $\Upsilon \ll 1$ is the well-known formula for classical synchrotron radiation, and that for $\Upsilon \gg 1$ is the so called Sokolov-Ternov formula for quantum synchrotron radiation.

The classical limit of dI_2/dt has relevant poles at $s = n = 1, 2, 3, \ldots$, i.e.,

$$\begin{aligned} \frac{dI_2}{dt} &= \frac{\alpha m^2 \ell_R^2}{1440\sqrt{3}\pi} \sum_{n=1}^{\infty} (-1)^n (3\Upsilon)^{n+2} n(n+1)\Gamma\left(\frac{n}{2} + \frac{1}{3}\right)\Gamma\left(\frac{n}{2} + \frac{2}{3}\right) \\ &\qquad \times \left[\frac{\dot{B}^2}{B^2}(-n^3 + 6n + 28) - \frac{\ddot{B}}{B}(3n^3 + 10n^2 + 32n + 36) \right] \\ &= \frac{\sqrt{3}}{80}\alpha m^2 \ell_R^2 \left[-11\frac{\dot{B}^2}{B^2} + 27\frac{\ddot{B}}{B} \right] \Upsilon^3 + \mathcal{O}(\Upsilon^4) \quad , \quad \Upsilon \ll 1 \quad . \end{aligned} \tag{11}$$

For the quantum limit, the relevant poles are at $s = -n/3(n = 2, 3, \ldots$, but excluding $n = 5, 7, 11, 13, \ldots$ (mod. 6)). So we have

$$
\begin{aligned}
\frac{dI_2}{dt} &= \frac{\alpha m^2 \ell_R^2}{1440\sqrt{3}\pi} \sum_{n=2}^{\infty} (-1)^{n+1} \frac{4\pi^2}{243} \left(1 - 2\cos\frac{n\pi}{3}\right) \frac{n(n-3)(3\Upsilon)^{2-n/3}}{\Gamma\left(\frac{1}{3}+\frac{n}{6}\right)\Gamma\left(\frac{2}{3}+\frac{n}{6}\right)} \\
&\qquad \times \left[\frac{\dot{B}^2}{B^2}(n^3 - 54n + 756) - \frac{\ddot{B}}{B}(-3n^3 + 30n^2 - 288n + 972)\right] \\
&= \frac{41}{3^8}\Gamma\left(\frac{1}{3}\right) \alpha m^2 \ell_R^2 \left[\frac{4}{5}\frac{\dot{B}^2}{B^2} - \frac{3}{5}\frac{\ddot{B}}{B}\right] (3\Upsilon)^{4/3} + O(\Upsilon) \quad , \quad \Upsilon \gg 1 . \qquad (12)
\end{aligned}
$$

Comparing Eqs. (11) and (12) with Eq. (10), we see that the field-gradient correction does not contribute at the leading order ($\sim \Upsilon^2$) in the classical limit, and scales as $\ell_R^2 \Upsilon^{4/3}$ in the quantum limit, which confirm qualitatively our previous findings in Ref. 3.

To appreciate the field-gradient effect in quantum beamstrahlung, let us consider a Gaussian variation of the field strength along the electron trajectory provided by the oncoming beam: $B(t) = B_0 \exp(-2t^2/\sigma_z^2)$, where the time of flight in the CM frame is $t = z/2$. Since Υ is proportional to B, it varies the same way, i.e., $\Upsilon(t) = \Upsilon_0 \exp(-2t^2/\sigma_z^2)$. It can also be seen that $\ell_R = 1/\dot{v}\gamma = \lambda\!\!\!^{-}_c \gamma/\Upsilon(t)$.

The leading behavior of the fractional energy loss of the electron, from Eq. (10), is

$$
\epsilon_0 = \frac{I_0}{\mathcal{E}} = \begin{cases} \dfrac{\sqrt{\pi}}{3} \dfrac{\alpha\sigma_z}{\lambda\!\!\!^{-}_c\gamma} \Upsilon_0^2 \quad , & \Upsilon_0 \ll 1, \\[2ex] \dfrac{16\sqrt{\pi}}{81\sqrt{3}} \Gamma(\tfrac{2}{3}) \dfrac{\alpha\sigma_z}{\lambda\!\!\!^{-}_c\gamma} (3\Upsilon_0)^{2/3} \quad , & \Upsilon_0 \gg 1 \quad . \end{cases} \qquad (13)
$$

The integration of the correction term dI_2/dt is not easy, particularly when $\Upsilon_0 \gg 1$, because the leading term in the series expansion of dI_2/dt does not dominate the integration. If one insists on applying the asympotic form of dI_2/dt in Eq. (12), one is forced to introduce a cutoff in the integration over time, which renders the evaluation of ϵ_2 inconclusive. This was the situation of our previous calculation. The cutoff symptom, however, can be avoided if one integrates over t before the s-integration in Eq. (9).

First we notice that for a Gaussian field, $\dot{B}$ vanishes at $t = \pm\infty$. Thus integration by parts in t turns $\ddot{B}/B$ into $\dot{B}^2/B^2$, and we get

$$\epsilon_2 = \frac{I_2}{\mathcal{E}} = \frac{\alpha m}{1440\sqrt{3}\pi\gamma}\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} sds(3s^4+6s^3+22s^2+10s-8)\Gamma(-s)\Gamma(s+2)$$

$$\times\Gamma\left(\frac{s}{2}+\frac{1}{3}\right)\Gamma\left(\frac{s}{2}+\frac{2}{3}\right)\int_{-\infty}^{\infty} dt\frac{\dot{B}^2}{B^2}\ell_R^2(3\Upsilon)^{s+2} \equiv \frac{\alpha\gamma}{m}\int_{-\infty}^{\infty}\frac{\dot{B}^2}{B^2}Q(\Upsilon)dt \quad . \qquad (14)$$

The integration over time can be further developed by changing the variable to Υ:

$$\int_{-\infty}^{\infty} dt\frac{\dot{B}^2}{B^2}\ell_R^2(3\Upsilon)^{s+2} = 108\frac{\ell_{R0}^2}{\sigma_z}\Upsilon_0^2\sqrt{2\ell n\Upsilon_0}\left[\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} d\Upsilon(3\Upsilon)^{s-1} + \mathcal{O}\left(\frac{1}{\ell n\Upsilon_0}\right)\right] \quad , \qquad (15)$$

where $\ell_{R0} = \lambda\!\!\!^{-}_c\gamma/\Upsilon_0$. Since Re$s = \lambda < 0$, the above integral gives $[\Upsilon^s]_\varepsilon^\infty = -\varepsilon^s$. Thus we have

$$\epsilon_2 = \left[\frac{1}{360\sqrt{3}\pi}\left(\frac{\alpha\sigma_z}{\lambda\!\!\!^{-}_c\gamma}\right)\left(\frac{\ell_{R0}}{\sigma_z}\right)^2\sqrt{2\ell n\Upsilon_0}\;\Upsilon_0^2\right]\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} ds(3s^4+6s^3+22s^2$$

$$+10s-8)\Gamma(-s)\Gamma(s+2)\Gamma\left(\frac{s}{2}+\frac{1}{3}\right)\Gamma\left(\frac{s}{2}+\frac{2}{3}\right)\cdot 3^{s+2}\varepsilon^s \quad . \qquad (16)$$

where $-2/3 < \lambda < 0$. For small ε, we close the contour to the right, where the leading pole is at $s = 0$, and all higher poles vanish. So

$$\epsilon_2 = \frac{2}{15}\frac{\alpha\sigma_z}{\lambda\!\!\!^{-}_c\gamma}\left(\frac{\ell_{R0}}{\sigma_z}\right)^2\Upsilon_0^2\sqrt{2\ell n\Upsilon_0}\left[1+\mathcal{O}\left(\frac{1}{\ell n\Upsilon_0}\right)\right], \quad \Upsilon_0 \gg 1 \quad . \qquad (17)$$

On the other hand, ϵ_2 for $\Upsilon_0 \ll 1$ can be trivially obtained as

$$\epsilon_2 = -\frac{11\sqrt{6\pi}}{40}\frac{\alpha\sigma_z}{\lambda\!\!\!^{-}_c\gamma}\left(\frac{\ell_{R0}}{\sigma_z}\right)^2\Upsilon_0^3 \quad , \quad \Upsilon_0 \ll 1 \quad . \qquad (18)$$

For intermediate values of Υ_0, it is difficult to find an analytic expression for ϵ_2.

A numerical integration of Eq. (14) is plotted in Fig. 1 for Υ_0 from 10^{-2} to 10^4. We see that the effect due to the field gradient is to reduce the fractional energy loss for $\Upsilon_0 \ll 1$, and to enhance it for $\Upsilon_0 \gg 1$. The transition occurs at around $\Upsilon_0 = 2/3$, which corresponds to the situation where the initial electron energy equals to the critical energy of synchrotron radiation.

As an example, consider the Himel–Siegrest parameters[7] for a conceptual 5+5 TeV linear collider, where $\gamma = 10^7$, number of particles per bunch $N = 1.2 \times 10^8$, and beam size $\sigma_r^* = 2.5$ Å, $\sigma_z = 0.4$ μm. The beamstrahlung parameter Υ_0 corresponds to twice the local field strength (i.e., $|\vec{B}| \simeq |\vec{E}|$), and varies with radius. Thus the evaluation of the mean value of ϵ_2/ϵ_0 for the entire beam involves an average over the transverse distribution, which is rather intricate. Instead, we assume uniform transverse density profile, and look at a typical electron that has an impact parameter $r = \sigma_r^*$. In this case $\Upsilon_0 = 5094 \gg 1$, and we find $\epsilon_2/\epsilon_0 \simeq 30\%$. This is indeed a substantial effect. In comparison, $\epsilon_2/\epsilon_0 \simeq 45\%$ at $r = \sigma_r^*/2$ and $\epsilon_2/\epsilon_0 \simeq 20\%$ at the beam edge $r = 2\sigma_r^*$. For the next generation of e^+e^- linear colliders at around 1 TeV in the center-of-mass energy, the effective beamstrahlung paramter Υ_0 is expected to be of order unity. Therefore, the field-gradient effect would not be a concern, as can be seen from Fig. 1.

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FIGURE CAPTION

Fig. 1. Numerical plots of functions in Eq. (14). The solid curve is the integrand $Q(\Upsilon)$, which is independent of specific variations of the field B. The dashed curve is the correction to the fractional energy loss, ϵ_2, as a function of Υ_0, with Gaussian variation of B assumed.

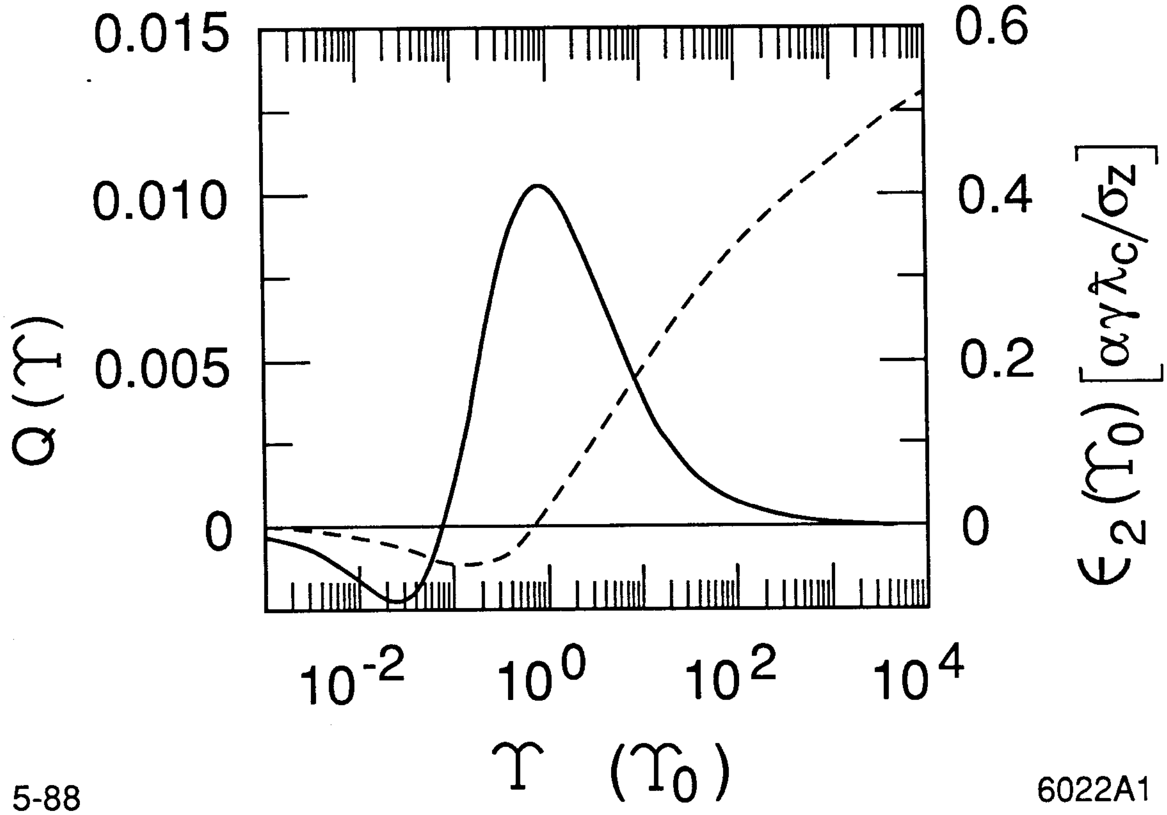

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Fig. 1