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THE MECHANISM OF MASS GENERATION IN (2 + 1)–DIMENSIONAL PURE YANG–MILLS THEORY*

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ABSTRACT

In the temporal gauge formulation of a gauge theory, Gauss's Law is imposed as a constraint on the physical states. In the (2 + 1)–dimensional Abelian theory the states satisfying the constraint are wavefunctionals of the magnetic field; moreover, acting on such states, the Hamiltonian is a function of the magnetic field and its canonical conjugate. There is a parallel in the non–Abelian theory: physical states can be written as wavefunctionals of a nonlocal "magnetic coordinate," and the Yang–Mills Hamiltonian is a function of this coordinate and its conjugate momentum. In the non–Abelian case the conjugate field acquires a divergent mass. This divergence can be absorbed in a renormalization, leaving a finite but massive propagator for the conjugate field.

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I. Introduction

The subject of this note is the pure Yang–Mills theory in two spatial dimensions. We work in the temporal gauge. In that gauge, the quantum problem is defined by two elements.[1] These are the Hamiltonian, which is obtained in a straightforward way from the Lagrangian with the time component of the gauge field set to zero, and Gauss's Law, which restricts physical states to lie in the gauge-invariant sector of the Hilbert space. We will select as coordinates for configuration space a pair of fields, one being a spatial component of the gauge field, the other being a nonlocal "magnetic coordinate," which transforms only globally under local gauge transformations. We will see that physical states must lie in the subspace of wavefunctions that depend only on the magnetic coordinate. In this subspace the Hamiltonian can be written in terms of the magnetic coordinate and its conjugate momentum. There is a degree of symmetry between these conjugate variables, in that neither occurs to higher than second order in any term of the Hamiltonian. Therefore, if one represents the generating functional of Euclidean Green's functions as a simultaneous integral over coordinates and momenta, it is possible to pass to a Lagrangian form for the path integral by carrying out a Gaussian integral over either set of variables.[2] We will integrate out the magnetic coordinate. In the Abelian case (where the magnetic coordinate is the magnetic field) the Lagrangian of the conjugates and their time derivatives which then appears is that of a free massless field. However, in the non–Abelian case this procedure leads to the appearance in the Lagrangian of a mass term with a divergent coefficient. This divergence can be absorbed in a rescaling, which we exhibit explicitly. The principal conclusion, then, is that massless modes are absent from the non–Abelian problem.

The plan of the balance of the note is as follows. In the next section we carry out the procedure defined here for the Abelian problem, in order to fix notation and methods, and to identify the mechanism that will generate the mass term in the non–Abelian case. In the third section we show explicitly how this mechanism

acts in the non–Abelian theory. The final section contains a few remarks on the procedure.

II. The Abelian Problem

To begin, we fix dimensions. In order to avoid a dimensionful coupling in the covariant derivative, we fix the dimension of the gauge potential to be inverse length. A length λ appears in the Lagrangian, which in temporal gauge assumes the form:

$$L = \lambda \int d^2\vec{x} \left\{ \frac{1}{2}\, \dot{A}_\rho \dot{A}_\rho - \frac{1}{2}\, (\partial_1 A_2 - \partial_2 A_1)^2 \right\} \quad .$$

The corresponding Hamiltonian is

$$H = \int d^2\vec{x} \left\{ \frac{1}{2\lambda}\, P_\rho P_\rho + \frac{\lambda}{2}\, b^2 \right\} \quad ,$$

where $b = (\partial_1 A_2 - \partial_2 A_1)$ is the magnetic field strength, and $iP_\rho(\vec{x})$ is the functional derivative with respect to $A_\rho(\vec{x})$. The physical subspace consists of those states $\Phi[A_\rho]$ which do not change under gauge transformation of A_ρ.

The physical eigenstates of this system are states of noninteracting photons, with one particle mode per wavevector (corresponding to the single polarization available to the light wave in $2+1$ dimensions). We will recover this result by eliminating the constraint and examining the resulting Hamiltonian.

Given any gauge configuration A_ρ, we can write A_2 in terms of b and A_1 as

$$A_2(x_1, x_2) = \int_{-\infty}^{0} dy\, \{b(x_1 + y, x_2) + \partial_2 A_1(x_1 + y, x_2)\} \quad . \tag{1}$$

This shows that we can use the magnetic field strength b and the component A_1 of the gauge field as independent coordinates on the configuration space $[A_\rho]$. Any wavefunction $\Phi[A_\rho]$ has a unique representation as a function of b and A_1. In these coordinates, the constraint is that Φ depend on b alone: if two configurations

have the same magnetic field, they differ only by gauge transformation and Φ must assume the same value at both. Now we write the operator

$$\nabla^2 = -\int d^2\vec{x}\; P_\rho(\vec{x})P_\rho(\vec{x})$$

in terms of functional derivatives with respect to b and A_1. The first derivative is

$$\frac{\delta}{\delta A_\rho(\vec{x})} = \epsilon_{\rho\sigma}\,\frac{\partial}{\partial x^\sigma}\,\frac{\delta}{\delta b(\vec{x})} + \delta_{\rho 1}\,\frac{\delta}{\delta A_1(\vec{x})} \quad . \tag{2}$$

By squaring this operator, and discarding terms which annihilate states that satisfy the constraint, we find that the Hamiltonian acting in the physical subspace is

$$H = \int d^2\vec{x}\left\{\frac{1}{2\lambda}\,\vec{\nabla}\pi\cdot\vec{\nabla}\pi + \frac{\lambda}{2}\,b^2\right\} \quad , \tag{3}$$

where

$$i\pi(\vec{x}) = \frac{\delta}{\delta b(\vec{x})} \quad .$$

We now determine the Hilbert space inner product in the physical subspace. Nominally, the overlap of two states $\Phi[A_\rho]$ and $\psi[A_\rho]$ is given by the integral

$$\int \mathcal{D}[A_1]\mathcal{D}[A_2]\;\Phi^*\psi \quad .$$

If Φ and ψ satisfy the constraint, this is necessarily divergent by a factor of the volume of the group of time-independent gauge transformations; this volume is to be factored by asserting a Faddeev–Popov prescription. Before doing so we may choose what coordinates we please on the configuration space. Choose (b, A_1). The Jacobian determinant of the transformation from (A_1, A_2) to (b, A_1) is field independent, as is the Faddeev–Popov determinant induced by the gauge-fixing

condition $A_1 = 0$. The inner product of two states $\Phi[b]$ and $\psi[b]$ in the physical subspace is

$$\langle \Phi | \psi \rangle = \int \mathcal{D}[b]\ \Phi^* \psi \quad . \tag{4}$$

The inner product, Eq. (4), and the Hamiltonian, Eq. (3), specify an unconstrained quantum system, which is exactly soluble. We proceed along the line to be followed in the non–Abelian problem, where exact solution is impossible: we give the generating functional for the Euclidean Green's functions of the field π. Define the operator

$$\pi(\vec{z}, \tau) = e^{\tau H}\ \pi\ (\vec{z})\ e^{-\tau H}$$

for τ real. Then we can write the Green's function

$$\langle 0 | T_\tau \pi(\vec{z_1}, \tau_1) \cdots \pi(\vec{z_n}, \tau_n) | 0 \rangle \tag{5a}$$

as a path integral over trajectories $[\pi(\vec{z_n}, \tau), b(\vec{z}, \tau)]$ in phase space:

$$\int \mathcal{D}[\pi] \mathcal{D}[b] \pi_1 \cdots \pi_n \exp\{S(\pi, b)\} \quad . \tag{5b}$$

The action $S(\pi, b)$ is

$$\int d\tau \int d^2\vec{x} \left\{ i\pi\ \frac{\partial b}{\partial \tau} - \frac{1}{2\lambda}\ \vec{\nabla}\pi \cdot \vec{\nabla}\pi - \frac{\lambda}{2}\ b^2 \right\} \quad , \tag{5c}$$

and $\mathcal{D}[\pi]\mathcal{D}[b]$ is normalized so that $\langle 0 | 0 \rangle = 1$. We may carry out the integral over the magnetic field. Defining $\ell_E(\dot{\pi}, \pi)$ by

$$\exp\left\{ - \int d\tau\ \ell_E \right\} = \int D[b]\ \exp\{S\}$$

we find that

$$\ell_E(\dot{\pi}, \pi) = \frac{1}{2\lambda} \int d^2\vec{x} \left\{ \dot{\pi}^2 + \vec{\nabla}\pi \cdot \vec{\nabla}\pi \right\} \quad . \tag{6}$$

In other words, the Green's functions of π are those of a massless free scalar.

We have come a long way for a meager conclusion. However, the steps we have followed can be paralleled almost exactly in the non–Abelian problem. We pause to consider how the conclusion of a massless state might be averted in that case. This can be done without considering the actual Yang–Mills Hamiltonian, which the next section treats in detail. Instead reconsider the toy problem which we have just worked. Suppose that f is a function on an N–dimensional space, $f = f(q_1 \ldots q_n)$, and $V(q_1 \ldots q_n)$ is a potential in the form $V = V(f)$. We seek the eigenstates of the Hamiltonian

$$\frac{-\partial^2}{\partial q_\kappa \, \partial q_\kappa} + V$$

subject to the constraint that the wavefunction depend on f alone. (Such a problem may or may not have a solution, but solutions exist when, for example, f is the radial coordinate.) Acting on a wavefunction which satisfies the constraint, the Hamiltonian is

$$- \frac{\partial^2 f}{\partial q_\kappa \partial q_\kappa} \frac{\partial}{\partial f} - \frac{\partial f}{\partial q_\kappa} \frac{\partial f}{\partial q_\kappa} \frac{\partial^2}{\partial f^2} + V(f) \quad .$$

In the electromagnetic problem, f is the magnetic field, q_κ is the vector potential, and $\partial/\partial f$ is $i\pi$. The term proportional to $f_{\kappa\kappa}$ does not appear in that problem, because the field strength is linear in the potential. But let us suppose that, owing to some nonlinearity, the $f_{\kappa\kappa}$ term had not vanished, and that instead of Eq. (3) we had generated the Hamiltonian

$$\int d^2\vec{x} \, \frac{1}{2\lambda} \, \vec{\nabla}\pi \cdot \vec{\nabla}\pi + \frac{\lambda}{2} \, b^2 + i \, \frac{c_1}{\lambda} \, b\pi \quad ,$$

where c_1 is a real, dimensionless constant. This is not an Hermitian operator, but for a few breaths we may hold our nose and proceed. Again, we can obtain a Euclidean Lagrangian $\ell_E(\dot{\pi}, \pi)$. But now instead of Eq. (6) the answer (up to a total time derivative) is

$$\frac{1}{2\lambda} \int d^2\vec{x} \left\{ \dot{\pi}^2 + \vec{\nabla}\pi \cdot \vec{\nabla}\pi + \frac{c_1^2}{\lambda^2} \, \pi^2 \right\} \quad .$$

That is, we have generated a mass term.

Can we get this same effect by an Hermitian modification to the Hamiltonian? In the instance that $f_{\kappa\kappa}$ does not vanish, we have to conclude that f_κ is not constant. That opens the possibility that the $f_\kappa f_\kappa$ term gives a contribution of the sort

$$\frac{c_2}{\lambda}\int d^2\vec{z}\; d^2\vec{w}\; b(\vec{z})\; b(\vec{w})\; \pi(\vec{z})\; \pi(\vec{w}) \quad .$$

This term is not Hermitian either. Neither is its sum with the quadratic non–Hermitian term, no matter what choice of the constants c_1 and c_2 we make. The mass term seems to follow from an illegal ploy. However, the non–Abelian problem that we study below is subtler than these *ad hoc* modifications. We will find that the nonlinearities inherent in that problem do introduce into the unconstrained Hamiltonian two non-Hermitian terms, one quadratic and one quartic. The sum of these two terms is indeed Hermitian, and treated consistently they inevitably introduce a mass term into the non–Abelian analog of (6). But to see this, we must turn to the non–Abelian problem itself.

III. The Non–Abelian Problem

We turn to the quantum theory of the pure Yang–Mills field in $2+1$ dimensions. Again we work in the temporal gauge. The gauge field and its conjugate are

$$\begin{aligned} A_\rho(\vec{x}) &= A^a_\rho(\vec{x})\; t^a \\ iP_\rho(\vec{x}) &= \frac{\delta}{\delta A^a_\rho(\vec{x})}\; t^a \end{aligned} \quad ,$$

where t^a are traceless Hermitian matrices satisfying

$$\begin{aligned} [t^a, t^b] &= if^{abc}\; t^c \\ Tr[t^a t^b] &= \delta^{ab} \end{aligned}$$

(The normalization of the trace is not conventional, but here it will prove convenient.) The Hamiltonian in this instance is

$$H = \int d^2\vec{x} \left\{ \frac{1}{2\lambda} P_\rho P_\rho + \frac{\lambda}{2} b^2 \right\} \quad ,$$

where

$$b = \partial_1 A_2 - \partial_2 A_1 + i[A_1, A_2] \quad .$$

If $u(\vec{x})$ is a matrix in the representation of the gauge group generated by t^a, then under gauge transformation by $u(\vec{x})$

$$\begin{aligned} A_\rho &\to u A_\rho u^\dagger - iu\partial_\rho u^\dagger \\ b &\to ubu^\dagger \end{aligned}$$

The constraint on physical states $\Phi[A_\rho]$ is that Φ be invariant when A_ρ undergoes gauge transformation.[3]

It is possible to pass by gauge transformation from any configuration A_ρ to another, $\mathcal{A}_\rho$, in which $\mathcal{A}_1$ vanishes. We use the gauge transformation

$$Y(\vec{x}) = P_s \exp\left\{ i \int_0^\infty ds\ A_1(x_1 - s, x_2) \right\} \tag{7a}$$

to eliminate $\mathcal{A}_1$:

$$Y A_1\ Y^\dagger - iY\ \partial_1\ Y^\dagger = 0 \quad .$$

We define a gauge-transformed magnetic coordinate

$$\begin{aligned} \mathcal{B} &= Y b\ Y^\dagger \\ &= \mathcal{B}^n\ t^n \end{aligned} \tag{7b}$$

In order to use this transformation, we limit the configuration space to fields A_ρ which vanish sufficiently rapidly at infinity for Y to be defined. This condition

cannot alter the dynamics, since any finite-energy configuration can be gauge transformed to satisfy it. At the same time, we restrict the gauge group to elements which leave this condition intact: i.e., those which tend to constants at infinity. Then we have

$$\mathcal{A}_2(x_1, x_2) = \int_{-\infty}^{x_1} \mathcal{B}(s, x_2)\, ds \quad , \tag{8a}$$

$$A_2 = Y^\dagger \mathcal{A}_2\, Y - iY^\dagger\, \partial_2\, Y \quad . \tag{8b}$$

Since Y is a functional of A_1 alone, we conclude that $(\mathcal{B}, A_1)$ can be used as coordinates on the configuration space (A_ρ).

We want to consider the nature of Gauss's Law in $(\mathcal{B}, A_1)$ coordinates. It is clear from Eq. (8) that any two configurations A_ρ and A'_ρ with the same $\mathcal{B}$ are gauge-equivalent. [The function $u(\vec{x})$ which transforms one into the other will satisfy $u(\infty) = 1$]. In these coordinates, physical wavefunctions depend on $\mathcal{B}$ alone. In contrast to the Abelian problem, the subspace $\Phi = \Phi[\mathcal{B}]$ contains the physical subspace but is not equal to it. Under the local gauge transformation $u(\vec{x})$ the field $\mathcal{B}$ has a global transformation law

$$\mathcal{B} \to u(\infty)\, \mathcal{B} u^\dagger(\infty) \quad .$$

The physical states are invariant under this global transformation. The arguments concerning the Hilbert space inner product are not different from those in the Abelian case: 1) since states which satisfy the constraint are not normalizable with respect to the measure $\mathcal{D}[A_1]\mathcal{D}[A_2]$ we must assert a Faddeev–Popov prescription in order to define the inner product; 2) the measure $\mathcal{D}[\mathcal{B}]\mathcal{D}[A_1]$ differs from $\mathcal{D}[A_1]\mathcal{D}[A_2]$ by only a constant factor; 3) the Faddeev–Popov determinant induced by the condition $A_1 = 0$ is constant. The inner product is

$$\langle \psi | \Phi \rangle = \int \mathcal{D}[\mathcal{B}]\, \psi^* \Phi \quad .$$

We proceed to determine the form of the Hamiltonian acting in the subspace $\Phi[\mathcal{B}]$. In this subspace, the operator

$$\nabla^2 = -\int d^2\vec{x}\ P_\rho(\vec{x})P_\rho(\vec{x})$$

has the form

$$\int d^2\vec{z}\ ip^n(\vec{z})\ \pi^n(\vec{z}) - \int d^2\vec{z}\ d^2\vec{w}\ p^{ns}(\vec{z},\vec{w})\ \pi^n(\vec{z})\ \pi^s(\vec{w}) \tag{9}$$

where we use the notation

$$p^n(\vec{z}) = \int d^2\vec{x}\frac{\delta^2\mathcal{B}^n(\vec{z})}{\delta A^a_\rho(\vec{x})\delta A^a_\rho(\vec{x})}$$

$$p^{ns}(\vec{z},\vec{w}) = \int d^2\vec{x}\frac{\delta\mathcal{B}^n(\vec{z})}{\delta A^a_\rho(\vec{x})}\ \frac{\delta\mathcal{B}^s(\vec{w})}{\delta A^a_\rho(\vec{w})} \quad .$$

$$i\pi^n(\vec{z}) = \frac{\delta}{\delta\mathcal{B}^n(\vec{z})}$$

If we were to compute ∇^2 in $(\mathcal{B}, A_1)$ coordinates and then drop those terms which annihilate states $\Phi[\mathcal{B}]$, the result would be the same as (9). To find p^n and p^{ns} we need the functional derivative of $\mathcal{B}^n(\vec{z})$ with respect to $A^a_\rho(\vec{x})$. There are two preliminaries. The first is the functional derivative of the gauge transformation matrix Y. From Eq. (7a) it is obvious that

$$\frac{\delta Y(\vec{z})}{\delta A^a_\rho(\vec{x})} \tag{10}$$

is nonvanishing only on the line segment $(x_1 < z_1, x_2 = z_2)$. Define $\omega(\vec{y_1},\vec{y_2})$ to be the parallel transportation matrix in the gauge field A_ρ along the straight line path from $\vec{y_2}$ to $\vec{y_1}$. On the line segment where (10) is nonvanishing, we can write

$$Y(z_\rho) = Y(x_\rho - \epsilon\delta_{\rho 1})\omega(x_\rho - \epsilon\delta_{\rho 1}, x_\rho + \epsilon\delta_{\rho 1})Y^\dagger(x_\rho + \epsilon\delta_{\rho 1})Y(z_\rho) \tag{11}$$

for any $0 < \epsilon < z_1 - x_1$. Applying the functional derivative with respect to $A_\rho(\vec{x})$ to Eq. (11) affects only the ω factor. In this factor it suffices to consider the leading order in A_ρ, since we will let $\epsilon \to 0_+$. We find

$$\frac{\delta Y(\vec{z})}{\delta A^a_\rho(\vec{x})} = i\delta_{\rho 1}\ \delta(z_2 - x_2)\ \theta(z_1 - x_1)\ Y(\vec{x})\ t^a\ Y^\dagger(\vec{x})\ Y(\vec{z}) \quad . \tag{12}$$

We will write Eq. (12) more compactly by defining

$$\Delta_\rho(\vec{z},\vec{x}) = \delta_{\rho 1}\delta(z_2 - x_2)\ \theta(z_1 - x_1) \quad , \tag{13a}$$

$$\mathcal{T}^a(\vec{x}) = Y(\vec{x})\ t^a\ Y^\dagger(\vec{x}) \quad . \tag{13b}$$

Then

$$\frac{\delta Y(\vec{z})}{\delta A^a_\rho(\vec{x})} = i\Delta_\rho(\vec{z},\vec{x})\ \mathcal{T}^a(\vec{x})\ Y(\vec{z}) \tag{14a}$$

$$\frac{\delta Y^\dagger(\vec{z})}{\delta A^a_\rho(\vec{x})} = -i\Delta_\rho(\vec{z},\vec{x}) Y^\dagger(\vec{z})\ \mathcal{T}^a(\vec{x}) \quad . \tag{14b}$$

The other preliminary is familiar. For any field $h(z)$ in the adjoint representation

$$\frac{\delta}{\delta A^a_\rho(\vec{x})}\int d^2\vec{z}\ \mathrm{Tr}[h(\vec{z})b(\vec{z})] = \epsilon_{\rho\sigma}\ \mathrm{Tr}\ t^a\ (\partial_\sigma h + i[A_\sigma, h]) \quad . \tag{15}$$

Using Eq. (15) we find

$$\int d^2\vec{z}\ \mathrm{Tr}\left[\frac{\delta b(\vec{z})}{\delta A^a_\rho(\vec{x})} Y h Y^\dagger\right] = \epsilon_{\rho\sigma}\ \mathrm{Tr}\ \mathcal{T}^a(\vec{x})\,(\partial_\sigma h + i[\mathcal{A}_\sigma, h]) \quad . \tag{16}$$

We note that on the RHS of Eq. (16) it is $\mathcal{T}^a$ and $\mathcal{A}_\rho$, rather than t^a and A_ρ, which appear. Combining Eqs. (14)–(16) gives

$$\begin{aligned}\frac{\delta \mathcal{B}^n(\vec{z})}{\delta A^a_\rho(\vec{x})} = \mathrm{Tr}[\mathcal{T}^a(\vec{x})t^q]\ \Big\{&\delta^2(\vec{x}-\vec{z})\ \delta^{qn}\ \epsilon_{\rho\sigma}\frac{\partial}{\partial z_\sigma} \\ &- \delta^2(\vec{x}-\vec{z})\ f^{\ell nq}\ \epsilon_{\rho\sigma}\ \mathcal{A}^\ell_\sigma(\vec{z}) - \Delta_\rho(\vec{z},\vec{x})\ f^{\ell nq}\ \mathcal{B}^\ell(\vec{z})\Big\}\end{aligned} \quad . \tag{17}$$

We make a few observations about this rather complicated expression. The trace factor in front depends only on the gauge-variant coordinate A_1. Since by Eq. (8a)

$\mathcal{A}_2$ depends on $\mathcal{B}$ alone, the terms inside the bracket depend only on $\mathcal{B}$. The first two of these are a covariant derivative which arises from the variation of b in $YbY^\dagger$. The third term in the bracket is a commutator and arises from the variation of Y and $Y^\dagger$. Referring to Eqs. (8a) and (13a) we see that Eq. (17) can be rewritten in the more compact form

$$\frac{\delta \mathcal{B}^n(\vec{z})}{\delta A^a_\rho(\vec{x})} = \mathrm{Tr}[\mathcal{T}^a(\vec{x})t^q]\Big\{\delta^2(\vec{x}-\vec{z})\ \delta^{qn}\ \epsilon_{\rho\sigma}\frac{\partial}{\partial z_\sigma} + \frac{\partial}{\partial z_1}\Big[f^{n\ell q}\Delta_\rho(\vec{z},\vec{x})\ \mathcal{A}_2^\ell(\vec{z})\Big]\Big\} \quad . \tag{18}$$

It is unfortunate that this expression is not more transparent. However its various pieces have a perfectly comprehensible origin as we have seen. Equation (18) is the building block from which to construct the ∇^2 operator. First consider the term in ∇^2 that is quadratic in π. Since

$$\sum_a \mathrm{Tr}[\mathcal{T}^a(\vec{x})t^p]\ [\mathcal{T}^a(\vec{x})t^q] = \delta^{pq}$$

the coefficient $p^{ns}(\vec{z},\vec{w})$ is independent of A_1, and therefore gauge-invariant. The term quadratic in π then has three gauge-invariant pieces: one zeroth order in $\mathcal{B}$, one linear in $\mathcal{B}$, and one quadratic in $\mathcal{B}$. We call these B, C, and D, respectively. The B and C terms are

$$B = -\int d^2\vec{z}\ \vec{\nabla}\pi^n \cdot \vec{\nabla}\pi^n \quad , \tag{19a}$$

$$C = -2\int d^2\vec{z}d^2\vec{w}\epsilon_{\rho\sigma}\frac{\partial\pi^s(\vec{w})}{\partial w^\sigma}\pi^n(\vec{z})\frac{\partial}{\partial z_1}[\Delta_\rho(\vec{z},\vec{w})f^{n\ell s}\mathcal{A}_2^\ell(\vec{z})] \quad . \tag{19b}$$

The B term is familiar from the Abelian example. It is obviously Hermitian with respect to the inner product. The C term is Hermitian owing to the antisymmetry of the structure constants. The D term, which is evaluated below, is the higher

order non–Hermitian term conjectured in the previous section. From Eq. (18) it is clear that D contains a factor

$$\int d^2\vec{x}\ \Delta_\rho(\vec{z},\vec{x})\ \Delta_\rho(\vec{w},\vec{x}) \quad . \tag{20}$$

Referring to the definition of Δ_ρ, one finds (20) to be

$$\delta(z_2-w_2)\left[\theta(z_1-w_1)\int_{-\infty}^{w_1} dx+\theta(w_1-z_1)\int_{-\infty}^{z_1} dx\right] \quad .$$

We assign a value $(z+L)$ to $\int_{-\infty}^{z} dx$, where L is some length we will send to infinity at the end of our calculations. Then (20) is

$$L\delta(z_2-w_2)\left[1+\frac{w_1}{L}\ \theta(z_1-w_1)+\frac{z_1}{L}\ \theta(w_1-z_1)\right] \quad .$$

Since L will tend to infinity we discard the θ-terms, and assign

$$\int d^2\vec{x}\ \Delta_\rho(\vec{z},\vec{x})\Delta_\rho(\vec{w},\vec{x}) = L\delta(z_2-w_2) \quad . \tag{21}$$

With this identification we write the D term:

$$D=-L\int d^2\vec{z}d^2\vec{w}\ \delta(z_2-w_2)\ f^{n\ell q}\ f^{sjq}\ b^\ell(\vec{z})\ b^j(\vec{w})\ \pi^n(\vec{z})\ \pi^s(\vec{w}) \quad . \tag{22}$$

Next consider the term in ∇^2 that is linear in π. To compute it we iterate the functional derivative given by Eq. (18). Since

$$\frac{\delta\mathcal{T}^a(\vec{x})}{\delta A^a_\rho(\vec{y})} = i\Delta_\rho(\vec{x},\vec{y})\ [\mathcal{T}^a(\vec{x}),\mathcal{T}^a(\vec{y})]$$

vanishes when $\vec{y}=\vec{x}$, the functional derivative of the trace factor does not contribute. Addressing the factor inside the bracket and applying Eq. (21) we find:

$$\int d^2\vec{x}\ \frac{\delta^2\ \mathcal{B}^n(\vec{z})}{\delta A^a_\rho(\vec{x})\delta A^a_\rho(\vec{x})} = -2cL\ \delta(0)\ \mathcal{B}^n(\vec{z}) \quad .$$

(c is the usual quadratic Casimir: $f^{aij}f^{bij}=2c\delta^{ab}$. The factor of two arises from our nonstandard normalization of t^a). The corresponding term in ∇^2, which we

label A, is

$$A = -2icL\,\delta(0)\int d^2\vec{z}\;\mathcal{B}^n(\vec{z})\;\pi^n(\vec{z}) \quad . \tag{23}$$

This is the term introduced *ad hoc* in the previous section to generate a mass for the π field; it will indeed play that role here. We will see below that its divergent coefficient can be absorbed into the definition of λ. It is amusing to note that this calculation establishes that $\mathcal{B}^n(\vec{z})$ is an eigenfunction of the operator ∇^2, albeit with a divergent eigenvalue. From Eqs. (22) and (23) it is immediate that

$$A^\dagger = -A + 2cL\,\delta(0)\;\delta^2(0)\;\delta^{ss}\int d^2\vec{z}$$
$$D^\dagger = D + 2A - 2cL\,\delta(0)\;\delta^2(0)\;\delta^{ss}\int d^2\vec{z}$$

The sum $(A + D)$ is Hermitian. The remaining term in the Hamiltonian is the potential term

$$\frac{\lambda}{2}\int d^2\vec{x}\;b^2 = \frac{\lambda}{2}\int d^2\vec{x}\;\mathcal{B}^2 \quad .$$

In this formulation all interactions have been relocated to the kinetic term. This completes the evaluation of the Hamiltonian.

This analysis has been sufficiently protracted that we pause to summarize it. We found coordinates $\mathcal{B}$ with the property that the gauge-invariant states of the theory are those wavefunctions $\Phi[\mathcal{B}]$ invariant against global gauge transformations of $\mathcal{B}$. We found $\mathcal{D}[\mathcal{B}]$ to be the appropriate integration measure for the Hilbert space inner product. We constructed the form of the Hamiltonian acting on the subspace of states $\Phi[\mathcal{B}]$, in terms of $\mathcal{B}$ and its conjugate, π. The result,

$$H(\pi,\mathcal{B}) = -\frac{1}{2\lambda}(A + B + C + D) + \frac{\lambda}{2}\int d^2\vec{x}\;\mathcal{B}^2$$

can be understood in the following way. The functional derivative of $\mathcal{B}$ with respect to A_ρ has a constant piece (as in the Abelian theory) and a piece that is linear in $\mathcal{B}$. This means that $p^{ns}(\vec{z},\vec{w})$ has three terms, which are zeroth, first,

and second order in $\mathcal{B}$; these generate B, C, and D, respectively. By iterating the functional derivative on $\mathcal{B}$ we find that $p^n(\vec{z})$ is nonzero, and linear in $\mathcal{B}$; this generates A.

We return to the track of our argument. The object is to compute $\ell_E(\dot{\pi},\pi)$, defined by

$$\exp\left[-\int d\tau\ \ell_E(\dot{\pi},\pi)\right] = \int \mathcal{D}[\mathcal{B}]\ \exp\{S(\pi,\mathcal{B})\} \quad ; \tag{24a}$$

where

$$S(\pi,\mathcal{B}) = \int d\tau \left\{-h(\pi,\mathcal{B}) + i\int d^2\vec{z}\pi^n(\vec{z})\frac{\partial\mathcal{B}^n(\vec{z})}{\partial\tau}\right\} \quad , \tag{24b}$$

and we normalize so that

$$\int \mathcal{D}[\mathcal{B}]\mathcal{D}[\pi]\ \exp\{S\} = 1 \quad .$$

The c-number function $h(\pi,\mathcal{B})$ is chosen to insure that Eq. (5b) gives the Euclidean Green's functions (5a). If $|\pi\rangle$ and $|\mathcal{B}\rangle$ are eigenstates of π and $\mathcal{B}$, we must have[4]

$$h(\pi,\mathcal{B})\ \langle\pi|\mathcal{B}\rangle = \langle\pi|H(\pi,\mathcal{B})|\mathcal{B}\rangle$$

Since $H = H^\dagger$ we can substitute $H^\dagger$ for H in the equation above. But it is obvious how to write $H^\dagger$ with all π operators to the left of all $\mathcal{B}$ operators. The consequence is that we obtain h from H with this prescription: 1) replace every field operator in H by its eigenvalue; 2) reverse the sign on the A term.

For purposes of carrying out the integral (24), we make a change of variable to dimensionless quantities:

$$\begin{aligned} L' &= \frac{L}{\lambda} \\ \delta(0)' &= \lambda\delta(0) \\ \tau' &= \frac{\tau}{\lambda} \\ z'_\rho &= \frac{z_\rho}{\lambda} \\ \mathcal{B}' &= \lambda^2\mathcal{B} \end{aligned} \quad .$$

This simply has the effect of setting $\lambda = 1$ in all expressions. At the end returning to the original variables restores the correct powers of λ. Also, it suffices to carry out the integral at a fixed time slice.

The B term in the action factors out of the integral over $\mathcal{D}[B]$ and for the moment we needn't consider it. The type of integral with which we have to deal is

$$\int dq_\kappa \exp\{-q^n E_{nm} q^m + J^n q^n\}$$

for E a symmetric matrix with positive eigenvalues. If E is N by N, the answer is

$$\pi^{\frac{N}{2}} \exp\left\{\frac{1}{4}\, J^n\, E^{-1}_{nm}\, J^m - \frac{1}{2}\, Lndet(E)\right\} \quad . \tag{25}$$

In the problem at hand, the matrix E is determined by the potential term and the D term:

$$E_{\ell j}(\vec{z},\vec{w}) = \frac{1}{2}\,\delta_{\ell j}\,\delta^2(\vec{z}-\vec{w}) + \frac{1}{2}\,L\,\delta(z_2-w_2)\,f^{n\ell q}\,f^{sjq}\,\pi^n(\vec{z})\,\pi^s(\vec{w}) \quad .$$

(The variables are dimensionless; to avoid clutter we do not write the primes.) Since we cannot obtain E^{-1} or $\det(E)$ exactly, we obtain ℓ_E to second order in π. This will retain the mass term generated by A. Since the source has no part zeroth order in π, we can use the zeroth order piece of E^{-1} in $JE^{-1}J$. The C term makes a contribution to J that is second order in π and so does not contribute to ℓ_E in this order. In (25) we substitute

$$J^n(\vec{z}) = -i\dot{\pi}^n(\vec{z}) + icL\,\delta(0)\,\pi^n(\vec{z})$$

$$E^{-1}_{\ell j}(\vec{z},\vec{w}) = 2\delta_{\ell j}\,\delta^2(\vec{z}-\vec{w})$$

$$Lndet(E) = (\text{constant}) + 2cL\,\delta(0)\int d^2\vec{z}\,\pi^n(\vec{z})\,\pi^n(\vec{z})$$

Restoring powers of λ, and the contribution of B, we find that, up to an irrelevant additive constant,

$$\ell_E(\dot{\pi},\pi) = \frac{1}{2\lambda}\int d^2\vec{z}\Big\{\Big[\dot{\pi} - \frac{cL\delta(0)}{\lambda}\pi\Big]^2 + \vec{\nabla}\pi\cdot\vec{\nabla}\pi + \frac{2cL\delta(0)}{\lambda^2}\pi^2\Big\} \tag{26}$$

If in Eq. (26) we set

$$\lambda = \frac{cL\delta(0)}{M}$$

then, discarding a total (Euclidean) time derivative,

$$\ell_E(\dot{\pi},\pi) = \Big(\frac{M}{cL\delta(0)}\Big)\int d^2\vec{z}\Big\{\frac{1}{2}\,\dot{\pi}^2 + \frac{M^2}{2}\,\pi^2 + \frac{1}{2}\,\vec{\nabla}\pi\cdot\vec{\nabla}\pi + \frac{2M^2}{cL\delta(0)}\,\pi^2\Big\} \tag{27}$$

The last term above vanishes as L goes to infinity with M held fixed. The overall factor $(1/cL\delta(0))$ on the surviving terms can be absorbed in a wavefunction renormalization of π. Evidently, the conjecture of the preceding section has been verified: the A term generates a mass in the Euclidean Lagrangian of the π field, while the D term maintains the Hermiticity of the total Hamiltonian.

If we had computed ℓ_E to all orders in π, we could evaluate the $\pi\pi$ Green's function in saddle point approximation by locating the global minimum of the Euclidean action. What we have done, rather, is to show that the extremum at $\pi = 0$ is (at least) locally stable, and, in contradistinction to the Abelian case, massive.

IV. Remarks

We have investigated the $(2+1)$–dimensional pure Yang–Mills theory in temporal gauge, by passing to coordinates in which Gauss's Law is a global, not a local, constraint. In these coordinates the potential term of the Hamiltonian is quadratic, and all interactions stem from alterations to the kinetic energy term brought about by the geometry of the constraint. We identified the mechanism by which one of the interaction terms renders the theory massive. A few additional remarks are in order.

1. In a constrained quantum system, the physical states lie in a subspace of the *a priori* Hilbert space. Since the Hamiltonian does not couple physical to nonphysical states, it can be written in a form intrinsic to the physical subspace. In this form, it can be used to construct a path integral representation of the Green's functions, and thereby generate a Lagrangian for the system. But this Lagrangian will, in general, be radically different from the one that defined the system originally.

2. Because π transforms globally under the gauge symmetry, none of the states, whether massless or not, which it creates from the vacuum are physical—i.e., gauge invariant. But since in the coordinates we have chosen the $\pi\pi$ Green's function is the only propagator in the theory, one would not expect a massless mode in any channel if this propagator is massive.

3. Consider the possibility that $\pi = 0$ is the global minimum of the Euclidean action. Then by extending the calculations of ℓ_E to higher order in the π field, we could compute in saddle point approximation the multipoint Green's functions of this field. That is, we could find all vertices "at tree level." But we have already committed ourselves to a choice of λ and to a wavefunction renormalization for π; do these choices render the multipoint vertices finite, even at tree level? This is not clear. Nor, *a fortiori*, is the renormalization prescription for corrections to the saddle point approximation. Lastly, although the higher order corrections to the saddle point approximation are (at least formally) defined for this problem, we have not shown any sense in which they might be suppressed.

References

1. R. Jackiw, Rev. Mod. Phys. **49** (1977) 681.

2. K. Huang, *Quarks, Leptons and Gauge Fields*, World Scientific, Singapore (1982), pp. 122–125.

3. In $2 + 1$ dimensions the group of time-independent gauge transformations is simply connected, and therefore Gauss's Law is equivalent to the strict gauge-invariance of physical states.

4. K. Huang, ibid.