SLAC-PUB-4570
March 1988
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# CHIRAL BOSONS*

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## ABSTRACT

The local Lagrangian formulation for chiral bosons recently suggested by Floreanini and Jackiw is analyzed. We quantize the system and explain how the unconventional Poincare generators of left and right chiral bosons combine to form the standard generators. The left-U(1) Kac-Moody algebra and the left-Virasoro algebra are shown to be the same as for left Weyl fermions. We compare the partition functions, on the torus, of a chiral boson and a chiral fermion. The left moving boson is coupled to gauge fields producing the same anomalies as in the fermionic formulation. It is pointed out that the unconventional Lorentz transformations are inapplicable for the coupled system and a set of different transformations is presented. A coupling to gravity is proposed. We present the theory of chiral bosons on a group manifold, the chiral WZW model. The (1,0) supersymmetric abelian and non-abelian chiral bosons are described.

Submitted to *Nucl. Phys.* **B**

* This work is supported in part by Dr. Chaim Weizmann Postdoctoral Fellowship and by the U.S. Department of Energy under contract #DE-AC03-76SF00515 (SLAC).

## 1. Introduction

Chiral bosons in two dimensions recently became an important phenomena mainly in the context of string theories. The Hamiltonian formulation of the chiral bosons is understood but the Lagrangian picture is much less transparent. As a result the bosonization of a free chiral fermion in terms of an action was missing.

An action proposed several years ago, by W.Siegel[1], for self-dual bosons described classically chiral bosons in two dimensions. The question was how to quantize this system or differently whether it corresponded to the Hamiltonian description mentioned above. This problem was addressed in two different approaches: a BRST quantization[2-4] based on the Siegel gauge symmetry[1] and a Dirac quantization[5] which was focused on the second class constraint of this action. In the first approach it was pointed out that the gauge symmetry was anomalous and it was suggested that it was possible to cancel the anomaly either by introducing a Liouville term[2-3] or by taking a system of 26 chiral bosons[4]. A careful BRST quantization revealed the fact that only a pair of chiral bosons could be consistently quantized[3]. It was also shown that the proposed coupling of the Liouville term to gravity encountered difficulties in staying Siegel anomaly free while producing the correct gravitational anomaly[4]. Meanwhile the second approach was abandoned because it was not able to cope with anomalies. Later on the action of Siegel was generalized to the group manifold case, the so called chiral WZW[4,6], action and the coupling to abelian and non-abelian gauge fields where introduced[6]. The (1,0) supersymmetrization of the "abelian"[7] and "non-abelian"[8] chiral bosons were also written down. Some application of the Lagrangian formulation of chiral bosons were given for the heterotic and 4-dimensional string theories[9].

R.Floreanini and R. Jackiw[10] proposed an apparently unrelated Lagrangian formulation for chiral bosons which was based on an unconventional Poincare symmetry. It turned out[5] that using the second approach in the path-integral quantization formalism[11] the Siegel action reduced to the local form of the action of ref. [10].* Moreover, recently it was argued[13] that the first approach is inapplicable for the quantization of the chiral boson system because the gauge generator is related to a first class constraint which is a square of a second class one. It is not clear to us at this stage whether this last statement is justified. In any event, we believe that the action of a single chiral boson can be properly quantized. Lastly we wish to comment on the powerful works on chiral bosonization that were recently published. [14]. These works present the bosonization of ordinary and higher spin fermions propagating on Reimann surfaces of higher genus. Note however that the actions involved are of unchiral bosons and the chirality property is achieved there by a holomorphic projection performed on the various correlation functions.

In this work we analyze the chiral boson theory given in the local Lagrangian formulation of ref. [10], supplemented with chiral boundary conditions. In the following section we present the classical chiral scalar[10]. The symmetries of the action are then discussed including an unconventional Poincare invariance, chiral symmetry, and modified affine Kac-Moody and Conformal transformations. In section 3, the chiral boson is quantized in canonical and path-integral formulations, treating it as a system with a second class constraint[15]. The equivalence between the theory of a complex chiral fermion and our chiral scalar is discussed in section 4. We prove that the Kac-Moody current and the energy momentum tensor of the two systems transform in the same representations of the corresponding affine algebras. In section 5, we explain how the theories of left and right chiral bosons combine to the theory of an ordinary scalar field. The Hamiltonian, the momentum, the Lorentz generator, and the Kac-Moody currents of

* Upon completing this work we received a preprint from L.D.Faddeev and R.Jackiw[12] in which the equivalence of the two actions is proven using the first order Lagrangian formulation.

the unchiral scalar are just the sum of the corresponding operators in the two opposite chiral bosons, even though the Lagrangian does not share this property. Considering possible application to string theories, we calculate the one loop partition function on the torus in section 6. We present the derivation in the Hamiltonian approach. The chiral boson is then coupled, in section 7, to abelian gauge fields. Following a similar analysis for the Siegel action[6)] we present the actions and anomalies in two formulations which correspond to the "vector conserving" and "left right" regularization schemes in the fermionic version. We point out that the unconventional Lorentz transformation are inapplicable for chiral bosons coupled to gauge fields and we discuss a way to overcome this difficulty. Section 8 is devoted to the coupling to two dimensional gravity. By gauging each of the "conformal symmetries" separately or both of them together we write down three actions. We show that the action with the two gauge symmetries, reproduce the gravitational anomaly of a chiral fermion. The quantization[5)] of the other two cases are summarized. The generalization of the previous results to chiral scalars on group manifolds-the chiral WZW theory-is derived in section 9. The analysis of the affine symmetries, the equivalent fermionic theories, and the coupling to non abelian gauge fields and gravity are briefly presented. In section 10 we combine our left moving boson with a left Weyl-Majorana fermion to form the (1,0) supersymmetric multiplet. The super Kac-Moody and superconformal symmetries are briefly discussed as well as the construction of (1,1) supersymmety theory and the supersymmetric chiral WZW. A short summary and some conclusions are included in the last section. In appendix A we present the "curved" Lorentz transformations and show that the actions for the uncoupled and coupled chiral bosons are invariant under them.

## 2. CLASSICAL ACTION

The Lagrangian formulation for a chiral boson recently suggested by Floreanini and Jackiw[10], is given by the following non-local form:

$$L = \frac{1}{2}\int dxdy\chi(x)\epsilon(x-y)\dot{\chi}(y) \quad - \int dx\chi^2(x) \tag{1}$$

where $\chi$ is a local bosonic field. The system can also be described in terms of the non-local bosonic field

$$\phi(x) = \frac{1}{2}\int dy\epsilon(x-y)\chi(y), \qquad \phi'(x) = \chi(x) \tag{2}$$

with a local Lagrangian density

$$\mathcal{L} = (\phi'\dot{\phi} - {\phi'}^2) \quad = \quad \partial_-\phi(\partial_+\phi - \partial_-\phi) \tag{3}$$

One way to interpret this action is as the action of a massless boson in a curved two dimensional background with $g^{00} = 0,\ g^{01} = 1,\ g^{11} = -2$. The classical equation of motion* which corresponds to (3) is $\partial_-\phi' = 0$. This equation has the general solution $\partial_-\phi = g(t)$, however by requiring $\partial_-\phi = 0$ at the spatial boundaries, we set $g = 0$ and get the chiral solution $\phi = \phi(x^+)$.

Even though the action (3) is not "manifestly" Lorentz invariant, it was shown in ref.[10]** that it is invariant under the time translation $\delta_T\phi = \dot{\phi}$, the space translation $\delta_S\phi = -\phi'$, and the unconventional Lorentz transformation $\delta_M\phi(x) = (t+x)\phi'(x)$. For the above invariances to exist we assume vanishing surface terms. The associated Noether charges are $H = \int dx{\phi'}^2$, $P = -H$ and $M = \int dx[(x+t)({\phi'}^2)]$. The system is, in fact, invariant under yet another unusual "Lorentz transformations": $\delta(\partial_-\phi) = x^+\partial_+(\partial_-\phi) - x^-\partial_-(\partial_-\phi) -$

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* The light-cone derivatives are given by: $\partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$.

** In ref. [10] the formulation with $\chi$ is recommended. Here we use the other one because the comparison to the ordinary bosonic action is more transparent.

$\partial_-\phi$, $\delta(\partial_1\phi) = x^+\partial_+(\partial_1\phi) - x^-\partial_-(\partial_1\phi) + \partial_1\phi$. These transformations, as shown in the Appendix, can be deduced as the Lorentz transformations in the curved metric mentioned above. They have an important role for chiral bosons coupled to gauge fields as will be explained in section 7.

In addition, eqn. (3) is invariant under the global axial transformation: $\phi \to \tilde{\phi} = \phi + \alpha$. The associated current has as its components $J_{(ax)-} = \partial_-\phi$, $J_{(ax)+} = \partial_+\phi - 2\partial_-\phi$. There is yet another conserved current, the "vector current"[6]: $J_{(v)-} = -\partial_-\phi$, $J_{(v)+} = \partial_+\phi$. From the vector and the axial currents we can write down the left and right currents $J_{\binom{l}{r}} = \frac{1}{2}[J_{(v)} \pm J_{(ax)}]$ respectively. They have the following expressions:

$$J_{(l)-} = 0, \quad J_{(l)+} = \phi'; \qquad J_{(r)-} = -\partial_-\phi, \quad J_{(r)+} = \partial_-\phi \tag{4}$$

Since upon using the equation of motion the axial current is conserved and since the vector current is topologically conserved, the left and right chiral current are therefore also conserved. Note however that only the left current is holomorphically conserved namely $\partial_- J_{(l)+} = 0$ ; while the right current is not antiholomorphically conserved. This property is related to the invariance of the lagrangian (3) under $\delta_\alpha\phi = \alpha(x^+)$ and not under $\delta_\alpha\phi = \alpha(x^-)$. In fact, we can rewrite the right current as $J_{(r)1} = \partial_-\phi$ and $J_{(r)0} = 0$ with $\partial_1 J_{(r)} = 0$, which correspond to the invariance of the action under $\delta\phi = \alpha(t)$. As will be explained in the section 5 only the left $U(1)$ Kac-Moody current exists in the quantum theory.

Similarly, the Lagrangian (3) is invariant under the conformal transformations $\delta_\epsilon\phi = \epsilon(x^+)\phi'$, $\delta_\epsilon\phi = \epsilon(t)\partial_-\phi$ but is not invariant under $\delta_\epsilon\phi = \epsilon(x^-)\phi'$. The associated Noether currents $T_{++}$ $T_{11}$ , the $++$ and 11 components of the energy momentum tensor, are given in a Sugawara form as :

$$T_{(l)++} = \phi'^2 = (J_{(l)+})^2 \qquad T_{(r)11} = (\partial_-\phi)^2 = (J_{(r)1})^2 \tag{5}$$

Upon using the equation of motion the energy- momentum is holomorphically conserved $\partial_- T_{(l)++} = 0$ and $\partial_1 T_{(r)11} = 0$.

## 3. THE QUANTIZED THEORY

A method for quantizing the system was introduced in ref. [10] using a prescription applicable to first order Lagrangians[12]. Here we briefly describe the canonical and path integral quantization of (3) treating it as a constrained system. The constraint $\varrho \equiv \pi - \phi' = 0$, where $\pi$ is the conjugate momentum of $\phi$, is a second class constraint since $\{\varrho(x), \varrho(y)\} = -2\delta'(x-y)$. The Hamiltonian density of the system takes the following form $\mathcal{H} = \mathcal{H}_C + v(\pi, \phi)\varrho$, where $\mathcal{H}_C = \phi'^2 = \pi^2 = \pi\phi'$. Using this form for $\mathcal{H}$ it is easy to verify that the condition $\dot{\varrho} = \{\varrho, H\} = 0$, does not lead to further constraints but fixes $v(\pi, \phi)$. The passage to the quantum theory is performed by using the Dirac brackets[15] for the definition of the commutator of any two operators $F$ and $G$ via:

$$
\begin{aligned}
&-i[F(x), G(y)] \equiv \{F(x), G(y)\}^* \\
&=\{F(x)G(y)\} - \int d\xi_1 d\xi_2 \{F(x), \varrho(\xi_1)\}[-\frac{1}{4}\epsilon(\xi_1 - \xi_2)]\{\varrho(\xi_2), G(y)\}.
\end{aligned} \tag{6}
$$

Following this definition, the operator algebra for $\pi$ and $\phi$ takes the form

$$
\begin{aligned}
[\pi(x), \phi(y)] &= \frac{1}{2i}\delta(x-y), \\
[\phi(x), \phi(y)] &= \frac{1}{4i}\epsilon(x-y), \\
[\pi(x), \pi(y)] &= \frac{i}{2}\delta'(x-y),
\end{aligned} \qquad (7).
$$

One also finds that, for an arbitrary operator $F$, $[\varrho(x), F(y)] = 0$ so that the constraint $\pi(x) - \phi'(x) = 0$, is now realized at the operator level. For example the Hamiltonian density can now be expressed in the three forms of $\mathcal{H}_C$ mentioned above. The system is instantaneously solved by Fourier transforming,

$$
\begin{aligned}
\phi(x) &= \int_0^\infty \frac{dk}{2\sqrt{\pi k}}[a_k e^{-ikx} + a_k^\dagger e^{+ikx}] \\
\pi(x) &= \phi'(x)
\end{aligned} \quad , \tag{8}
$$

with, as usual,

$$[a_k, a^\dagger_{k'}] = \delta(k - k') \qquad .$$

Note that only $k \geq 0$ appears in the decomposition of $\phi(x)$, which expresses the chiral nature of the field. The single-particle Hilbert space is then a continuum of states with energy $E = k$, $k \geq 0$. Hence the Hamiltonian formalism has correctly implemented the chirality constraint $\partial_- \phi = 0$. Furthermore, this property can also be deduced from the Hamiltonian equation of motion $\dot{\phi} = i[H, \phi] = \phi'$. Note that to get the chiral solution $\partial_- \phi = 0$ as a solution of the equation of the motion, we had to assume chiral boundary conditions. Here it looks at first that the chirality property was derived with no assumptions, but in passing from (6) to (7) we assumed that $(\pi - \phi')(x = \infty) = -(\pi - \phi')(x = -\infty)$, so together with choosing zero surface terms we in fact assumed chiral boundary conditions.

For the path integral quantization of the system we use the method developed for Hamiltonian systems with constraints. The generating functional is given by

$$\begin{aligned} Z[J] &= \int [d\phi][d\pi]\delta(\varrho) exp[i \int d^2x(\pi\dot{\phi} - \mathcal{H}_C - J\phi)] \\ &= \int [d\phi] exp[i \int d^2x \mathcal{L} - J\phi] \end{aligned} \tag{9}$$

where a normalization by $Z[0]^{-1}$ is implied. The Lagrangian density that emerges in (9) is clearly in the original form of eqn. (3). The functional integral (9) is not specified completely until we include the boundary conditions; thus by requiring $\partial_- \phi = 0$ we are in the same situation as in the canonical quantization.

Now by using the commutation relations (7), it is straightforward to verify that the Noether charges $H, P, M$ respectively generate the transformations $\delta_T \phi$, $\delta_S \phi$ and $\delta_M \phi$ given in section 2. It can be easily shown that they satisfy the Poincare algebra $[H, P] = 0, \quad [M, H] = iP \quad$ and $[M, P] = iH$.

## 4. ABELIAN BOSONIZATION OF A CHIRAL FERMION

The action of one left-handed complex chiral fermion, $i\int dx\psi^\dagger\partial_-\psi$, is classically invariant under both an affine chiral transformation $\delta\psi = i\alpha(x^+)\psi$ and the conformal transformation $\delta\psi = \epsilon^+(x^+)\partial_+\psi$. The associated Noether currents have the quantum form of $J_+ =: \psi^\dagger\psi :$ and $T_{++} = i : \psi^\dagger\partial_+\psi := \pi : J_+J_+ :$. They obey the left $U(1)$ Kac-Moody and Virasoro algebras respectively with the well known central charges $k=1$ and $c=1$.

Next we examine the generators of the symmetry transformation for the chiral boson theory. Using the operator algebra (7) it is easy to verify that the charges $J_{(l)} = \phi'$ and $T_{(l)} = \phi'^2$ generate the transformations $\delta_\alpha\phi$ * and $\delta_{\epsilon^+}\phi$, respectively. Moreover we can now evaluate the commutators of the chiral current and of the energy momentum tensor[10]. The results are the following:

$$[J_{(l)}(x), J_{(l)}(y)] = [\phi'(x), \phi'(y)] = \frac{i}{2}\delta'(x-y) \tag{10}$$

$$\begin{aligned}[T_{(l)}(x), T_{(l)}(y)] &= [: (\phi'(x))^2 :, : (\phi'(y))^2 :] \\ &= i\{: (\phi'(x))^2 : + : (\phi'(y))^2 :\}\delta'(x-y) - (\frac{i}{24\pi})\delta'''(x-y).\end{aligned} \tag{11}$$

These commutation relations correspond to central charges of $c = k = 1$. Hence our bosonic theory furnishes the same irreducible representation of the Kac-Moody and Virasoro algebras as one free left handed chiral fermion. Since for $k=1$ the Kac-Moody algebra has a unique representation the two theories on flat two-dimensional space-time are therefore equivalent[17]. In sections 7,8 it will be shown that the anomalies of the bosonized theory in coupling to gauge and gravitational background is the same as for the fermionic theory.

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* To get the standard form of the Kac-Moody algebra, one has to take the current $\tilde{j}_{(l)} = \frac{1}{\sqrt{\pi}} j_{(l)}$ and then the Sugawara construction is $T_{(l)} = \pi : \tilde{J}_{(l)}\tilde{J}_{(l)} :$. In what follows, we continue for simplicity to use $j_{(l)}$.

## 5. COMBINING LEFT AND RIGHT CHIRAL BOSONS

The canonical quantization procedure of the last section can be repeated for a right chiral boson. Let us rename the operators for the $\partial_-\phi = 0$ case as $\phi_L, \pi_L, \ldots$ and call the corresponding operators for the $\partial_+\phi = 0$ case $\phi_R, \pi_R, \ldots$ . The Lagrangian density for the right moving field has the form $\mathcal{L}_{right} = -\sqrt{2}\partial_+\phi_R\partial_1\phi_R$ Then, starting from the combined Lagrangian

$$\mathcal{L} = \mathcal{L}_{left} + \mathcal{L}_{right} \qquad , \tag{12}$$

one arrives at the Hamiltonian system

$$\mathcal{H} = \pi_L^2 + \pi_R^2 \qquad , \tag{13}$$

with the operator algebra

$$\begin{aligned} [\phi_L(x), \phi_L(y)] &= \frac{1}{4i}\epsilon(x-y) = -[\phi_R(x), \phi_R(y)], \\ [\pi_L(x), \pi_L(y)] &= \frac{i}{2}\delta'(x-y) = -[\pi_R(x), \pi_R(y)], \end{aligned} \tag{14)}$$

and the consistent operator constraints

$$\pi_L(x) - \phi_L'(x) = 0 = \pi_R(x) + \phi_R'(x) \qquad . \tag{15}$$

It is now straightforward to see that the definitions

$$\phi = \phi_L + \phi_R, \qquad \pi = \pi_L + \pi_R \qquad , \tag{16}$$

permit the above system to be recast as

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\pi^2 + \frac{1}{2}\phi'^2, \\ [\phi(x), \phi(y)] &= 0 = [\pi(x), \pi(y)] \qquad , \\ [\pi(x), \phi(y)] &= \frac{1}{i}\delta(x-y) \end{aligned} \tag{17}$$

which, is the Hamiltonian theory for a single, free, massless scalar field.

Let us now examine the Poincare algebra of the combined system. The Hamiltonian and momentum of the combined system are

$$\begin{aligned} H =& H_L + H_R = \int dx(\pi_L^2 + \pi_R^2) = \int dx(\phi_L'^2 + \phi_R'^2) = \int dx \frac{1}{2}(\pi^2 + \phi'^2), \\ P =& P_L + P_R = \int dx(-\pi_L^2 + \pi_R^2) = \int dx(-\phi_L'^2 + \phi_R'^2) = -\int dx(\pi\phi'), \end{aligned} \tag{18}$$

which guarantee the commutator $[H, P] = 0$. Similarly, the Lorentz generator for the combined system is given by:

$$\begin{aligned} M = M_L + M_R =& \int dx(x^+\phi_L'^2 - x^-\phi_R'^2) = \int dx[x(\phi_L'^2 + \phi_R'^2) + t(-\phi_L'^2 + \phi_R'^2) \\ =& \int dx(x\mathcal{H} - tP). \end{aligned} \tag{19}$$

The algebra of the combined system is therefore identical to the Poincare algebra associated with ordinary free massless scalar.

The affine currents of the combined system are also simply related to those of the left and right systems.

$$J_{\binom{l}{r}\pm} = \partial_\pm\phi = \pi \pm \phi' = (\pi_L + \pi_R) \pm (\phi_L' + \phi_R') = J_{\binom{l}{r}\binom{L}{R}\pm}. \tag{20}$$

The $\binom{l}{r}$ $U(1)$ Kac-Moody currents of the combined system are therefore given by the left current of the left system and the right current of the right system respectively. The central charges are $k = 1$ for the algebras in both sectors. Similarly, because of the Sugawara construction, $T_{++} = T_{++L}$ and $T_{--} = T_{--R}$ with $c = 1$ for the left and the right Virasoro algebras.

## 6. PARTITION FUNCTION.

We would now like to compare the one loop partition function of a chiral fermion and that of our chiral boson. We therefore pass to a two dimensional space-time domain with $1 \geq x \geq 0$. The mode expansion for $\phi$ previously given by eqn. (8) now takes the form:

$$\phi(x,t) = \phi_0 + p(x+t) + \sum_{n>0} \frac{1}{\sqrt{2n}} [a_n^\dagger e^{2\pi i n x^+} + a_n e^{-2\pi i n x^+}] \tag{21}$$

The one loop partition function which corresponds to this mode expansion[16] is

$$\mathcal{Z}(\tau) = Tr e^{-\tau_2 H + i\tau_1 P} = Tr q^{\frac{H+P}{2\pi}} \bar{q}^{\frac{H-P}{2\pi}} = \bar{q}^{\frac{H}{\pi}} \tag{22}$$

where $q = e^{i\tau\pi}$ and we used $H = -P$. Let us first calculate the contribution to the trace of the oscillation modes

$$Tr \bar{q}^{\frac{H}{\pi}} = Tr[\bar{q}^{(\sum n a_n^\dagger a_n - \frac{1}{12})}] = \prod_{n=1} \bar{q}^{-\frac{1}{12}} [1 - \bar{q}^{2n}]^{-1} = \eta^{-1}(\bar{\tau}) \tag{23}$$

where the factor $-\frac{1}{12}$ is the output of the normal ordering and $\eta$ is the Dedekin $\eta$ function. The Hamiltonian of the zero modes is $H = \frac{1}{4\pi} p^2$. If we now take the bosonic momenta[18] to lie on a shifted lattice such that the eigenvalues of p are $2\pi(m+\alpha)$ and introduce the twist operator $g = e^{i\beta p}$, we get the zero mode partition function to be equal to the Reimann theta function $\Theta[\binom{\alpha}{\beta}](\bar{\tau}|0)$ and so that altogether the full partition function is given by

$$\mathcal{Z}(\bar{\tau}) = \frac{\Theta[\binom{\alpha}{\beta}](\bar{\tau}|0)}{\eta(\bar{\tau})}. \tag{24}$$

This corresponds to the partition function of a Weyl fermion with the boundary conditions $\psi(x+1,t) = -e^{2\pi i \alpha}\psi(x,t)$; $\psi(x + Re\tau, t + Im\tau) = -e^{-2\pi i \beta}\psi(x,t)$. For the case of unshifted momenta, $\alpha = 0$, we may interpret the bosonic zero modes as those of a compactified scalar field with a radius $R = \frac{1}{\sqrt{2}}$ and winding numbers which are equal to the momentum eigenvalues[16)]. Note that in (22) we have just used $H = -P$ which is a property of the Lagrangian (3). The expansion (21), on the other hand, was based also on the boundary conditions which we imposed. Obviously, on the torus we cannot impose boundary conditions, so instead we have to assume that there is a point with fixed $x$ where $\partial_- \phi = 0$ holds at any time. This assumption leads to chiral solution of the equation of motion.

## 7. COUPLING TO ABELIAN GAUGE FIELDS

There are several ways to couple an abelian gauge field to a chiral boson corresponding to the various regularization schemes in the fermionic theory. We start by analyzing the bosonization in the "vector conserving" regularization scheme. The vector current is coupled to an abelian gauge field via

$$\mathcal{L}_{(V)} = \mathcal{L}_o + (J_{(v)-}A_+ + J_{(v)+}A_-) = \mathcal{L}_0 + (\partial_+ \phi A_- - \partial_- \phi A_+), \tag{25}$$

where the subscript $(V)$ stands for vector conserving scheme. The vector current is still obviously conserved, but the divergence of the axial current

$$\partial_- J_{(ax)+} + \partial_+ J_{(ax)-} = (\partial_- A_+ - \partial_+ A_-) = \epsilon^{\mu\nu} F_{\mu\nu}, \tag{26}$$

is now equal to the anomaly deduced in the fermionic theory from the one loop diagram.

Next we discuss the bosonization in the "left-right" scheme. As discussed in ref. [6] a term bilinear in the gauge fields has to be added to the $J_{(l)+}A_{(l)-}$ term. The lagrangian then takes the form:

$$\tilde{\mathcal{L}}_{(LR)} = \mathcal{L}_o + J_{(l)+}A_{(l)-} + \frac{\sqrt{2}}{4}[A_{(l)-}A_{(l)1}], \tag{27}$$

where $(LR)$ indicates the left right scheme. The divergence of the left current which is derived from $\frac{\partial \mathcal{L}_{(LR)}}{\partial A_{\pm}} = (J^{\pm}_{(l)})_{(LR)}$, has now the form:

$$\partial_\mu (J^\mu_{(l)})_{LR} = \partial_- (J^-_{(l)})_{LR} + \partial_+ (J^+_{(l)})_{LR} = \frac{1}{4}(\partial_- A_{(l)+} - \partial_+ A_{(l)-}) = \frac{1}{4}\epsilon^{\mu\nu} F_{\mu\nu}. \tag{28}$$

The Lagrangian (27) is therefore really the bosonized fermionic action regularized in the left right scheme. Obviously, a similar prescription for the (V) and (LR) schemes can be applied to the right chiral boson.

We want now to address the question of the Lorentz invariance of the coupling terms in (25) and (27). If we adopt for the variation of $A_-$ and $A_+$ the unconventional transformations $\delta A_- = x^+ \partial_1 A_-$ $\quad \delta A_1 = A_1 + x^+ \partial_1 A_1$ then (27) is invariant but (25) is not. Moreover, for any other coupling of the gauge fields to other matter fields (or even for $(F_{\mu\nu})^2$ ) the unconventional Lorentz symmetry will be broken. Under standard Lorentz transformations the coupling term in (25) is invariant but $\mathcal{L}_0$ and the coupling in (27) are not invariant. A possible way out is to use the "curved" Lorentz transformations discussed in the appendix. It is shown in the appendix that both (25) and (27) are invariant under these transformations. It is important to note that since $\delta A_-$ and $\delta A_+$ transform in the standard way under the "curved" transformations, any additional terms in the action which involve them are required to be standard Lorentz scalars.

## 8. Coupling to Gravitational Fields

To find the correct coupling to a gravitational field we start by gauging the "conformal transformations" discussed in section 2: $\delta_\epsilon \phi = \epsilon(x^+)\phi'$ and $\delta_\epsilon \phi = \epsilon(t)\partial_- \phi$. It is straightforward to check that the action

$$S[\hat{h}, \phi] = \int d^2x [\dot{\phi}\phi' + \hat{h}\phi'\phi'] \tag{29}$$

is invariant classically under the following gauge $\delta_\epsilon$ and affine $\delta_\epsilon$ symmetries:

$$\begin{aligned} \delta_{\epsilon^1}\phi &= \epsilon^1(x,t)\phi', \qquad & \delta_{\epsilon^1}\hat{h} &= -\dot{\epsilon}(x,t) + \epsilon(x,t)\hat{h}' - \epsilon'(x,t)\hat{h}, \\ \delta_{\epsilon^-}\phi &= \epsilon^-(t)\partial_-\phi, \qquad & \delta_{\epsilon^-}\hat{h} &= \partial_-\epsilon(t) + \epsilon(t)\partial_-\hat{h} + \partial_-\epsilon(t)\hat{h}. \end{aligned} \tag{30}$$

When $\hat{h}$ transforms as $h^{11}$, then the local transformation can be interpreted[6] as a combined coordinate transformation $x^1 \to x^1 - \epsilon^1(x,t)$ and a Weyl rescaling $\delta h_{\alpha\beta} = -\partial_1\epsilon^1 h_{\alpha\beta}$. Interchanging $\epsilon^1(x,t)$ with $\epsilon^-(t)$ and $\partial_1$ with $\partial_-$ leads to a similar interpretation for the second (affine) symmetry. However if we rewrite (29) as

$$S[h,\phi] = \int d^2x[\partial_-\phi\partial_1\phi + h\partial_1\phi\partial_1\phi] \tag{29a}$$

then the local transformation

$$\delta_{\epsilon^1}\phi = \epsilon^1(x,t)\phi', \qquad \delta_{\epsilon^1}h = -\partial_-\epsilon(x,t) + \epsilon(x,t)h' - \epsilon'(x,t)h, \tag{30a}$$

leaves the action invariant. This is the coordinate transformation mentioned above where now $h = h^{++}$. The Lorentz invariance of (29a) requires again, as for the gauge fields, to use the "curved" transformation "$\delta\partial_1\phi$" discussed in the Appendix. In fact it looks to us that performing the coordinate transformations on $\delta_\alpha\phi$ also in the "curved" form will reveal[20] an invariance under $x^+ \to x^+ - \epsilon^+$ rather the one with $x^1$.

In a complete analogy we can gauge $\delta\epsilon^-$ while maintaining $\delta\epsilon^1$ as an affine symmetry. This leads to the Siegel action

$$S[h^{--},\phi] = \int d^2x[\partial_+\phi\partial_-\phi + h^{--}\partial_-\phi\partial_-\phi] \tag{31}$$

with the following transformation of $h^{--}$:

$$\begin{aligned} \delta_{\epsilon^-}h^{--} &= -\partial_+\epsilon(x,t) + \epsilon(x,t)\partial_-h^{--} - \partial_-\epsilon(x,t)h^{--} \\ \delta_{\epsilon^1}h^{--} &= \partial_1\epsilon(x^+) + \epsilon(x^+)\partial_1h^{--} + \epsilon(x^+)'h^{--}. \end{aligned} \tag{32}$$

Note in passing that the two actions (29) and (31) are connected to each other by the coordinate transformation[19] $x^{\binom{0}{1}} \to x^{\binom{+}{-}}$. Therefore the interpretation of the transformations (32) are similar to those of (30).

There is yet another possible action in which both symmetries are gauged:

$$S_h = \frac{1}{2}\int d^2x\sqrt{h}[2h^{+-}\partial_1\phi\partial_-\phi - h^{--}\partial_-\phi\partial_-\phi + h^{++}\partial_1\phi\partial_1\phi]. \qquad (33)$$

This action can be derived in fact from the standard action of a scalar in two dimensional curved space-time by the transformation $\partial_- \to \partial_- \quad ; \partial_+ \to \partial_1$. The gauge symmetries exist provided $\tilde{h}^{\alpha\beta} = \sqrt{h}h^{\alpha\beta}$ transform as follows:

$$\begin{aligned}
&\delta_{\epsilon^1}\tilde{h}^{++} = -2\partial_-\epsilon\tilde{h}^{+-} + \epsilon\partial_1\tilde{h}^{++} - \epsilon'\tilde{h}^{++}, \qquad &&\delta_{\epsilon^-}\tilde{h}^{++} = \partial_-(\epsilon\tilde{h}^{++})\\
&\delta_{\epsilon^1}\tilde{h}^{+-} = -\partial_-\epsilon\tilde{h}^{--} + \epsilon\partial_1\tilde{h}^{+-}, &&\delta_{\epsilon^-}\tilde{h}^{+-} = -\partial_1\epsilon\tilde{h}^{++} + \epsilon\partial_-\tilde{h}^{+-}\\
&\delta_{\epsilon^1}\tilde{h}^{--} = \partial_1(\epsilon\tilde{h}^{--}), &&\delta_{\epsilon^-}\tilde{h}^{--} = -2\partial_1\epsilon\tilde{h}^{+-} + \epsilon\partial_-h^{--} - \partial_-\epsilon\tilde{h}^{--}.
\end{aligned} \qquad (34)$$

We expect that the classical local symmetries in eqns. (30),(32) and (34) have anomalies. Unlike the former case of coupling to a gauge field, here the anomaly would not be a tree level phenomena but rather, as usual, a quantum effect. To calculate the anomaly[4)] we treat the metric components as background and pass to the linearized theory $h^{\alpha\beta} = \eta^{\alpha\beta} + \bar{h}^{\alpha\beta}$ where $\eta^{\alpha\beta}$ is the "flat" metric, namely $\eta^{+-} = 1, \eta^{--} = \eta^{++} = 0$. It is clear that we cannot derive the anomaly from (31). We choose here to take the action (33) but the same result will emerge from (29a). The linearized action is therefore $S_{\bar{h}} = \frac{1}{2}\int d^2x[2\partial_1\phi\partial_-\phi - \bar{h}^{--}\partial_-\phi\partial_-\phi + \bar{h}^{++}\partial_1\phi\partial_1\phi]$ and the corresponding effective action $W[h^{\alpha\beta}]$ given by $e^{iW[h]} = \int[d\phi]e^{iS[\bar{h}]}$ is

$$W[\bar{h}^{\alpha\beta}] = \frac{1}{8}\int d^2x\int d^2y[i < T^*T_{\mu\nu}(x)T_{\rho\sigma}(y) > \bar{h}^{\mu\nu}(x)\bar{h}^{\rho\sigma}(y)] \qquad (35)$$

where $T^*$ denotes time ordered product, $T_{++}(x) = \partial_1\phi\partial_1\phi(x)$, $T_{--}(x) = \partial_-\phi\partial_-\phi(x)$ and $T_{+-} = 0$. Using the boundary condition discussed in section 2 and assuming that the chiral nature of $\phi$ will be preserved here too in the quantum level, then $T_{++} = \frac{1}{2}\partial_+\phi\partial_+\phi = T_{(l)++}$ of equation (5) and we get that the variation of the effective action has the following form:

$$\delta_\epsilon W[\bar{h}^{++}] = -\frac{1}{48\pi}\int d^2x \bar{h}^{++}\partial_+^3\epsilon^1. \tag{36}$$

This result is equivalent to eqn. (11). Hence, the system of one chiral boson coupled to gravity reproduces the same gravitational anomaly as that of a left Weyl fermion coupled to gravity.

The quantization of the systems can in principle be accomplished by the standard procedure of fixing the gauge, introducing ghost fields and applying the BRST prescription. Such an analysis was performed for the Siegel action[2,3]. From the simple relation between the two actions (29) and (31) it is obvious that the light cone quantization of (29) will be identical to the ordinary quantization of (31) mentioned above. As for (33), one can again adopt the "standard quantization". However it is not clear if this "standard quantization" applies to our cases because the gauge generators are related to a first class constraints which are the squares of second class constraints[13]. Alternatively, using the[5] second class constraint and following Dirac's quantization procedure for the Siegel action or following the method described in ref. [12] leads again to the system described in eqn.(3). Therefore the action (31) does not describe the coupling of our chiral boson to gravity. The same conclusion follows from the fact that at the linearized level the action does not have the form of $T_{++}$ coupled to a gauge field. The quantum picture of (29) is again given by (7 ) after interchanging[19] $x^{\binom{0}{1}} \rightarrow x^{\pm}$. Hence, this action describes a boson with $\phi' = 0$. The quantization of the action (33) following similar guide lines and some related topics will be discussed elsewhere[20]

## 9. CHIRAL WZW AND THE COUPLING TO NON-ABELIAN GAUGE FIELDS

The non-Abelian bosonization of Majorana (Dirac) fermions was proposed by Witten[17]. It is expressed in terms of the bosonic action, the so called WZW action, which has the form:

$$S[u] = \frac{1}{8\pi}\int d^2x Tr(\partial_\mu u \partial^\mu u^{-1}) + \frac{1}{12\pi}\int_B d^3y \varepsilon^{ijk} Tr(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u), \tag{37}$$

where $u$ is a group element $u \subset G$ ( $G = O(N)$ for Majorana and $G = U(N)$ for Dirac fermions). For the non-Abelian bosonization of left chiral fermions we propose to generalize the action (3) to the group manifold case. Therefore the action becomes :

$$S_+[u] = \frac{\sqrt{2}}{4\pi}\int d^2x Tr(\partial_- u \partial_1 u^{-1}) + \frac{1}{12\pi}\int_B d^3y \varepsilon^{ijk} Tr(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u) \tag{38}$$

In fact we can generalize the last action to the so called $k$-level chiral WZW namely $S_{+k}(u) = kS_+(u)$. The equation of motion which corresponds to the variation of this action with respect to the variation of $u$ can be expressed as:

$$\partial_-(u^{-1}\partial_1 u) = 0 \qquad or \qquad \partial_1(\partial_- u u^{-1}) = 0, \tag{39}$$

where each form can be obtained from the other.

The global, chiral transformations $u \to Au$, $u \to uB^{-1}$; $A, B \subset G$ leave the action (38) invariant. However, out of the two Kac-Moody invariances of the original WZW action $u \to uB(x^+)$ and $u \to A(x^-)u$, only the first survives . As for the abelian case the second Affine invariance is lost. The Noether currents associated with the left right transformations of (38) are

$$J_{(l)-} = 0 \quad J_{(l)+} = \frac{i\sqrt{2}k}{2\pi}u^{-1}\partial_1 u \quad J_{(r)-} = \frac{ik}{2\pi}u\partial_- u^{-1} \quad J_{(r)+} = -\frac{ik}{2\pi}u\partial_- u^{-1}. \tag{40}$$

The conservation of the left and right currents follow here simply from the equations of motion. However, unlike the ordinary WZW action only the left current is holomorphically conserved $\partial_- J_{(l)} = 0$; whereas $\partial_+ J_{(r)} \neq 0$. Obviously this is a manifestation of the invariance of the action only under the left Kac Moody transformation $\delta u = -iu\omega(x^+)$ discussed above. The left current transforms as follows $\delta_\omega J_{(l)} = [i\omega(x^+), J_{(l)}] + \frac{\sqrt{2}k}{2\pi}\omega'$, leading to the $O(N)(U(N))$ Kac-Moody algebra[21] with central charge equal to $k$.

The action $S_{+k}$ is invariant classically under the left conformal transformation $\delta_\epsilon = \epsilon(x^+)u'$. The energy momentum tensor has again a Sugawara form. Assuming that in the quantum theory this form holds, then the results of ref. [21] apply to the left sector of our case as well, namely:

$$T_{(l)} = \frac{2\pi}{(c_{ad} + k)} \sum_a : J^a_{(l)} J^a_{(l)} : \qquad c = \frac{kD_g}{(c_{ad} + k)}, \tag{41}$$

where $J_{(l)} = J^a_{(l)} T^a$, $T^a$ are hermitian matrices representing the algebra of the group, $c_{ad}$ is the second Casimir operator in the adjoint representation and $D_g$ is the dimension of the group. The classification of the non-chiral WZW actions which are equivalent to free fermions[22] will apply for the chiral non-abelian bosonization as well. The connection between the symmetry generators of the non-chiral theory and those of the chiral system follows similar lines as for the abelian case of eqns. (18-20).

The coupling of non-abelian gauge fields to a chiral WZW action, which is a generalization of the results of section 7, was derived in ref. [6] in the Siegel formulation. Here we summarize the results for the present formulation given by eqn.(38 ). Again there are several ways to couple gauge fields corresponding to the various regularization schemes in the fermionic theory. The bosonized action (for $k = 1$) related to the "vector conserving " regularization scheme is given by

$$\begin{aligned} S_V[u, A_-, A_+] = S_+[u] + \int d^2x Tr\{[J^-_{(v)} A_- + J^+_{(v)} A_+] \\ - \frac{\sqrt{2}}{2\pi}[(u^{-1} A_1 u A_-) - (A_- A_1)]\}, \end{aligned} \tag{42}$$

where $J_{(v)}$ and $J_{(ax)}$ correspond to the chiral currents (40). Using the equation of motion one finds the conservation of the vector current and the anomalous divergence of the axial current:

$$D_\mu(J^\mu_{(v)})_{(V)} = 0, \qquad D_\mu(J^\mu_{(ax)})_{(V)} = \frac{1}{\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{43}$$

So we have realized the covariant conservation of the vector current.

In coupling to a left non-abelian gauge field the action is

$$S_+[u] = \frac{i\sqrt{2}}{2\pi}\int d^2x Tr[(u^{-1}\partial_1 u + \frac{1}{2\pi}A_1)A_-]. \tag{44}$$

This action corresponds to the fermionic description in the left-right regularization scheme[23]. The associated current divergence is

$$D_\mu(J^\mu_L)_{LR} = D_-(J^-_L)_{LR} + D_+(J^+_L)_{LR} = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu A_\mu. \tag{45}$$

This expression for the anomalous divergence of the left current is identical to the result of the loop calculation in the fermionic version regularized in the left-right scheme. The coupling of the chiral WZW action to gravity can be achieved easily by generalizing (33) namely, we alter the sigma model term to the following term:

$$\begin{aligned} S_{+h} = \frac{1}{4\pi}\int d^2x\sqrt{h}Tr[&h^{1-}\partial_- u\partial_1 u^{-1} + h^{+-}\partial_- u^{-1}\partial_1 u + h^{++}\partial_1 u\partial_1 u^{-1} \\ &+ h^{--}\partial_- u\partial_- u^{-1}] \end{aligned} \tag{46}$$

## 10. (1,0) SUPERSYMMETRY

The supersymmetrization of the chiral boson theory was studied extensively in the context of the Siegel formulation[7]. Here we present the (1,0) supersymmetric version of (3). Naturally we add to (3) the action of a free Weyl-Majorana left fermion, $\beta$. The action

$$S_L = \int d^2x(\sqrt{2}\partial_-\phi\partial_1\phi + i\beta\partial_-\beta) \tag{47}$$

is invariant under the following supersymmetric transformation

$$\delta_\xi\phi = -\xi\beta \qquad \delta_\xi\beta = -i\sqrt{2}\xi\partial_1\phi \tag{48}$$

The comparison with the ordinary (1,0) supersymmetry suggests that the supersymmetric covariant derivative $D$ must satisfy $DD = i\sqrt{2}\partial_1$. The superfield $\Phi$ and the superspace action are given by

$$S_L = -i\int d^2\sigma d\varsigma^-[\partial_-\Phi D\Phi], \qquad \Phi\big| = \phi, \quad D\Phi\big| = \beta \tag{49}$$

where $\big|$ denotes the Grassmann independent part of a superfield and $\int \varsigma^- = D\big|$. The corresponding equation of motion $\partial_- D\Phi = 0$ leads, of course, to the equations $\partial_-\partial_1\phi = \partial_-\beta = 0$.

The action (49 ) is invariant under the global axial transformation $\Phi \to \Phi + A$ and the left Kac-Moody transformation $\delta\Phi = A(x^+, \varsigma^+)$. The corresponding super current $\mathcal{J} = D\Phi$ has the spin 1 component $J_{(l)+} = \phi'$ and the spin $\frac{1}{2}$ component $j_{(l)} = \beta$. It is holomorphically conserved, $\partial_-\mathcal{J} = 0$. Similarly, the super conformal transformation: $\delta\Phi = E\partial_1\Phi + \frac{1}{2}DED\Phi$, where $E$ is a superfield with $\partial_- E = 0$, leaves the action invariant .The Noether super current $\mathcal{T} = D\Phi\partial_1\Phi$ $(\partial_-\mathcal{T} = 0)$ has the spin 2 component $T_{(l)} = \phi'^2 + i\sqrt{2}\beta\partial_1\beta$ and the spin $\frac{3}{2}$ component $t_{(l)} = \beta\partial_1\phi$. Note that, on shell, the fermionic part of $T_{(l)}$ takes the ordinary form $i\beta\partial_+\beta$.

We can now construct the (1,1) theory as a combined left (1,0) action together with a right (0,1) action. The supersymmetic transformation is $\delta_{\xi^+}\Phi = i[\xi^+ Q_+, \Phi] = -\xi^+(\partial_{\varsigma^+}\Phi + i\varsigma^+\partial_+\Phi) = -\xi^+(\partial_{\varsigma^+}\Phi_L + i\varsigma^+\partial_+\Phi_L) = \delta_{\xi^+}\Phi_L$ Similarly, $\delta_{\xi^-}\Phi = \delta_{\xi^-}\Phi_R$, where we have used the decomposition $\Phi$ as $\Phi = \Phi_L + \Phi_R$ and $\partial_{\varsigma^+}\Phi_R = \partial_{\varsigma^-}\Phi_L = \partial_+\Phi_R = \partial_-\Phi_L = 0$

This supersymmetrization procedure can obviously be adopted also to the chiral level $k$ WZW model for a group G producing[8] the following action

$$S_{WZW}(\mathcal{U}) \;=\; -ik\tfrac{1}{16\pi}\int d^2\sigma d\varsigma^-\; Tr\{[\Pi_-\Pi] + \int_0^1 dy\; Tr[\tilde{\Pi}_y(D\tilde{\Pi}_- - \partial_-\tilde{\Pi})]\}. \quad (50)$$

Here, $\Pi \equiv \mathcal{U}^{-1}D\mathcal{U}$ and similarly for $\Pi_-$, $\tilde{\mathcal{U}} = \mathcal{U}(x^{(\stackrel{+}{-})}, y, \varsigma^+)$ and the components of $\mathcal{U}$ are $\mathcal{U}\big| \;\equiv\; u$ , $D\mathcal{U}\big| \;\equiv\; iu\beta$ with $u$ the WZW group element of (38) and $\beta$ denotes free Weyl-Majorana fermions in the adjoint representation of G. This action is in fact the sum of a WZW action given in (38) and the action of $\beta$. The discussion of the affine symmetries and the combined (1,1) supersymmetric theory follows the same lines as those for the ordinary supersymmetric chiral boson given above.

## 11. Summary and Conclusions

The controversy about the Lagrangian formulation of chiral bosons can be settled once the system is quantized as a constrained system or as a first order Lagrangian. In this approach the actions of refs. [1] and [10] are equivalent. In this paper we analyzed the theory of left moving bosons starting from the local form of the Lagrangian given in ref. [10] supplemented with chiral boundary conditions. We explained how the unconventional Poincare generators of a left and a right chiral boson combine to form the standard generators. We showed that the currents related to the left-U(1) Kac-Moody algebra and the left-Virasoro algebra transform in the same representations as the corresponding currents in the theory of a left Weyl fermion. We compared the partition functions on the torus of the theories of a chiral boson and a chiral fermion. Several possible actions for the left moving boson coupled to gauge and gravitational fields where introduced. The corresponding anomalies reproduced those of the fermionic formulation. We presented the theory of chiral bosons on a group manifold, the chiral WZW model. We summarized the associated Kac-Moody and Virasoro algebras and wrote down the coupling to non-abelian gauge fields and gravity. The (1,0) supersymmetric abelian and non-abelian chiral bosons which are based on an unconventional supersymmetry charge were described.

We intend to address in the future some related topics such as the quantization of chiral bosons coupled to gravity, the partition function in higher genus Reimann surfaces and possible applications to string theories.

**Acknowledgements**

The author thanks R.Brooks, M. Golteman for helpful discussions and for reading the manuscript. Conversations with S.Bellucci, P. Griffin, D. Lewellen and A. Sen are gratefully acknowledged.

## APPENDIX

We start by writing the Lagrangian (3) in the form of a scalar field on a "curved background" (the Siegel form) with $g^{--} = 2\lambda$ and latter we will substitute $\lambda = -1$.

$$\mathcal{L} = \frac{1}{2}[\partial_-\phi\partial^-\phi + \partial_+\phi\partial^+\phi] = \partial_-\phi(\partial_+\phi + \lambda\partial_-\phi) \qquad (A1)$$

where we substitute $\partial^-\phi = \partial_+\phi + 2\lambda\partial_-\phi$ and $\partial^+\phi = \partial_-\phi$. The Lorentz transformations of $\partial_-\phi$, $\partial_+\phi$ and $\lambda$ which correspond to this action are:

$$\delta(\partial_-\phi) = x^+\partial_+(\partial_-\phi) - x^-\partial_-(\partial_-\phi) - \partial_-\phi, \qquad (A2a)$$

$$\delta(\partial_+\phi) = x^+\partial_+(\partial_+\phi) - x^-\partial_-(\partial_+\phi) + \partial_+\phi \qquad (A2b)$$

$$\delta\lambda = x^+\partial_+\lambda - x^-\partial_-\lambda + 2\lambda \tag{A2c}$$

Now for $\lambda = -1$ we get:

$$\begin{aligned}\sqrt{2}\delta``(\partial_1\phi)'' &= \frac{1}{2}\delta(\partial^-\phi + \partial^+\phi) = \delta[\partial_+\phi + \lambda\phi]_{\lambda=-1} \\ &= x^+\partial_+(\partial_+\phi - \partial_-\phi) - x^-\partial_-(\partial_+\phi - \partial_-\phi) + (\partial_+\phi - \partial_-\phi)\end{aligned} \tag{A3}$$

The transformations (A2a) and (A3) that leave the action (A1) invariant are the "curved" Lorentz transformations mentioned in section 2. The origin of the unusual transformation of "$\partial_1\phi$" is now transparent. "$\partial_1\phi$" $= [\partial_+\phi + \lambda\partial_-\phi]_{\lambda=-1}$ transforms differently than $\partial_+\phi - \partial_-\phi$. Next we want to show the Lorentz invariance of the actions describing the coupling of chiral bosons to gauge fields (25), (27), (42) and (44). The gauge field components $A_+$, $A_-$ and $\sqrt{2}$"$A_1$" $=$ $[A_+ + \lambda A_-]_{\lambda=-1}$should transform as (A2a),(A2b) and (A3) respectively. With this transformation laws it is straightforward that the coupled actions are Lorentz invariant.

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