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THE KOBAYASHI–MASKAWA MIXING MATRIX*

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In the standard model with SU(2)×U(1) as the gauge group of electroweak interactions, both the quarks and leptons are assigned to be left-handed doublets and right-handed singlets. The quark mass eigenstates are not the same as the weak eigenstates, and the matrix connecting them has become known as the Kobayashi-Maskawa[1] (K-M) matrix, since an explicit parametrization in the six-quark case was first given by them in 1973. It generalizes the four-quark case, where the matrix is parametrized by a single angle, the Cabibbo angle.[2]

By convention, the three charge 2/3 quarks (u, c, and t) are unmixed, and all the mixing is expressed in terms of a 3×3 unitary matrix V operating on the charge $-1/3$ quarks (d, s, and b):

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} . \tag{1}$$

The values of individual K-M matrix elements can in principle all be determined from weak decays of the relevant quarks, or, in some cases, from deep inelastic neutrino scattering. Using the constraints discussed below (in the full-sized edition only), together with unitarity, and assuming only three generations, the 90% confidence limits on the magnitude of the elements of the complete matrix are:

$$\begin{pmatrix} 0.9748 \text{ to } 0.9761 & 0.217 \text{ to } 0.223 & 0.003 \text{ to } 0.010 \\ 0.217 \text{ to } 0.223 & 0.9733 \text{ to } 0.9754 & 0.030 \text{ to } 0.062 \\ 0.001 \text{ to } 0.023 & 0.029 \text{ to } 0.062 & 0.9980 \text{ to } 0.9995 \end{pmatrix} . \tag{2}$$

The ranges shown are for the individual matrix elements. The constraints of unitarity connect different elements, so choosing a specific value for one element restricts the range of the others.

There are several parametrizations of the K-M matrix. In view of the need for a "standard" parametrization in the literature, we advocate the form:

$$V =$$

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \tag{3}$$

in the notation of Harari and Leurer[3] for a form generalizable to an arbitrary number of "generations" and also proposed by Fritzsch and Plankl.[4] The choice of rotation angles follows that of Maiani,[5] and the placement of the phase follows that of Wolfenstein.[6] The three-"generation" form was proposed earlier by Chau and Keung.[7] Here $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, with i and j being "generation" labels, $\{i,j = 1,2,3\}$. In the limit $\theta_{23} = \theta_{13} = 0$ the third generation decouples, and the situation reduces to the usual Cabibbo mixing of the first two generations with θ_{12} identified with the Cabibbo angle.[2] The real angles θ_{12}, θ_{23}, θ_{13} can all be made to lie in the first quadrant by an appropriate redefinition of quark field phases. Then all s_{ij} and c_{ij} are positive, and $|V_{us}| = s_{12}c_{13}$, $|V_{ub}| = s_{13}$, and $|V_{cb}| = s_{23}c_{13}$. As c_{13} deviates from unity only in the fifth decimal place (from experimental measurement of s_{13}), $|V_{us}| = s_{12}$, $|V_{ub}| = s_{13}$, and $|V_{cb}| = s_{23}$ to an excellent approximation. The phase δ_{13} lies in the range $0 \leq \delta_{13} < 2\pi$, with nonzero values generally breaking CP invariance for the weak interactions. This parametrization can be easily generalized to the n-generation case where there are $n(n-1)/2$ angles and $(n-1)(n-2)/2$ phases.[3,4] The range of matrix elements in Eq. (2) corresponds to 90% CL limits on the angles of s_{12} = 0.217–0.223, s_{23} = 0.030–0.062, and s_{13} = 0.003–0.010.

(Continuation of this discussion, and all references, may be found in the full-sized edition of the Review of Particle Properties only.)

Kobayashi and Maskawa[1] originally chose a parametrization involving the four angles, θ_1, θ_2, θ_3, δ:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3-s_2s_3e^{i\delta} & c_1c_2s_3+s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3+c_2s_3e^{i\delta} & c_1s_2s_3-c_2c_3e^{i\delta} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} , \tag{4}$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ for $i = 1,2,3$. In the limit $\theta_2 = \theta_3 = 0$, this reduces to the usual Cabibbo mixing with θ_1 identified (up to a sign) with the Cabibbo angle.[2] Slightly different forms of the Kobayashi-Maskawa parametrization are found in the literature. The K-M matrix used in the 1982 Review of Particle Properties is obtained by letting $s_1 \rightarrow -s_1$ and $\delta \rightarrow \delta + \pi$ in the matrix given above. An alternative[8] is to change Eq. (4) by $s_1 \rightarrow -s_1$ but leave δ unchanged. With this change in s_1, the angle θ_1 becomes the usual Cabibbo angle, with the "correct" sign (i.e., $d' = d\cos\theta_1 + s\sin\theta_1$) in the limit $\theta_2 = \theta_3 = 0$. The angles θ_1, θ_2, θ_3 can, as before, all be taken to lie in the first quadrant by adjusting quark field phases. Since all these parametrizations are referred to as "the" Kobayashi-Maskawa form, some care about which one is being used is needed when the quadrant in which δ lies is under discussion.

Other parametrizations, mentioned above, are due to Maiani,[5] to Chau and Keung,[7] and to Wolfenstein.[6] The latter emphasizes the relative sizes of the matrix elements by expressing them in powers of the Cabibbo angle. Still other parametrizations[9] have come into the literature in connection with attempts to define "maximal CP violation." No physics can depend on which of the above parametrizations (or any other) is used as long as a single one is used consistently and care is taken to be sure that no other choice of phases is in conflict.

Our present knowledge of the matrix elements comes from the following sources:

(1) Nuclear beta decay, when compared to muon decay, gives[10,11]

$$|V_{ud}| = 0.9747 \pm 0.0011 . \tag{5}$$

This includes refinements over the past few years in which leading log radiative corrections have been summed using the renormalization group and structure-dependent $O(\alpha)$ terms analyzed and estimated[10] (thereby lowering the value of $|V_{ud}|$); and, more importantly, the order $Z\alpha^2$ Coulomb corrections have been revised[11] to bring the ft-values from low- and high-Z Fermi transitions into better agreement (thereby raising the value of $|V_{ud}|$).

(2) Analysis of K_{e3} decays yields[12]

$$|V_{us}| = 0.2196 \pm 0.0023 \tag{6}$$

The isospin violation between K_{e3}^+ and K_{e3}^0 decays has been taken into account, bringing the values of $|V_{us}|$ extracted from these two decays into agreement at the 1% level of accuracy. The analysis of hyperon decay data has larger theoretical uncertainties because of first-order SU(3) symmetry-breaking effects in the axial-vector couplings, but due account of symmetry breaking gives a consistent value[13] of $0.220 \pm 0.001 \pm 0.003$. We average these two results to obtain:

$$|V_{us}| = 0.2197 \pm 0.0019 . \tag{7}$$

(3) The magnitude of $|V_{cd}|$ may be deduced from neutrino and antineutrino production of charm off valence d quarks. When the dimuon production cross sections of the CDHS group[14] are supplemented by more recent measurements of the semileptonic branching fractions and the production cross sections in neutrino reactions of various charmed hadron species, the value[15]

$$|V_{cd}| = 0.21 \pm 0.03 \tag{8}$$

is extracted.

(4) Values of $|V_{cs}|$ from neutrino production of charm are dependent on assumptions about the strange-quark density in the parton sea. The most conservative assumption, that the strange-quark sea does not exceed the value corresponding to an SU(3) symmetric sea, leads to a lower bound,[14] $|V_{cs}| > 0.59$. It is more advantageous to proceed analogously to the method used for extracting $|V_{us}|$ from K_{e3} decay; namely, we compare the

experimental value for the width of D_{e3} decay with the expression[16] that follows from the standard weak interaction amplitude:

$$\Gamma(D \rightarrow \bar{K}e^+\nu_e) = |f_+^D(0)|^2 |V_{cs}|^2 (1.54\times 10^{11}\ \mathrm{sec}^{-1})\ . \quad (9)$$

Here $f_+^D((p_D - p_K)^2)$ is the form factor for $D_{\ell 3}$ decay which is the analogue of $f_+((p_K - p_\pi)^2)$ for $K_{\ell 3}$ decay; its variation has been taken into account with the parametrization $f_+^D(t)/f_+^D(0) = M^2/(M^2 - t)$, where $M = 2.1$ GeV, the mass of the D_s^*, a form and mass consistent with Mark-III measurements.[17] Combining data on branching ratios for $D_{\ell 3}$ decays[17,18] with accurate values[19] for τ_{D^+} and τ_{D^0}, gives the value $(0.78 \pm 0.11)\times 10^{11}\ \mathrm{sec}^{-1}$ for $\Gamma(D \rightarrow \bar{K}e^+\nu_e)$. Therefore

$$|f_+^D(0)|^2 |V_{cs}|^2 = 0.51 \pm 0.07\ . \quad (10)$$

With sufficient confidence in a theoretical calculation of $|f_+^D(0)|$, a value of $|V_{cs}|$ follows,[20,21] but even with the very conservative assumption that $|f_+(0)| < 1$ it follows that

$$|V_{cs}| > 0.66\ . \quad (11)$$

The constraint of unitarity when there are only three generations gives a much tighter bound (see below).

(5) The ratio $|V_{ub}/V_{cb}|$ can be obtained from the semileptonic decay of B mesons by fitting to the lepton energy spectrum as a sum of contributions involving $b \rightarrow u$ and $b \rightarrow c$. The relative overall phase space factor between the two processes is calculated from the usual four-fermion interaction with one massive fermion (c quark or u quark) in the final state. The value of this factor depends on the quark masses, but is roughly one-half. The lack of observation of the higher momentum leptons characteristic of $b \rightarrow u\ell\bar{\nu}_\ell$ as compared to $b \rightarrow c\ell\bar{\nu}_\ell$ has resulted thus far only in upper limits which depend on the lepton energy spectrum assumed for each decay.[21,22,23] Using the lepton momentum region near the end-point for $b \rightarrow c\ell\bar{\nu}_\ell$ and taking the calculation[23] of the lepton spectrum that gives the least restrictive limit results in[24]

$$|V_{ub}/V_{cb}| < 0.20\ . \quad (12)$$

A lower bound on $|V_{ub}|$ can be established from the observation[25] of exclusive baryonic B decays into $p\bar{p}\pi$ and $p\bar{p}\pi\pi$ which involve $b \rightarrow u + d\bar{u}$ at the quark level. A chain of assumptions on the relative phase space, the fraction of the quark-level process which hadronizes into baryonic channels, and the fraction of those that occur in the observed modes is required. No other channels that reflect $b \rightarrow u$ at the quark level have been observed.[26] Given the branching fractions of the two observed modes, a reasonable lower limit is[25]

$$|V_{ub}/V_{cb}| > 0.07\ . \quad (13)$$

(6) The magnitude of V_{cb} itself can be determined if the measured semileptonic bottom hadron partial width is assumed to be that of a b quark decaying through the usual $V-A$ interaction:

$$\Gamma(b \rightarrow c\ell\bar{\nu}_\ell) = \frac{\mathrm{BF}(b \rightarrow c\ell\bar{\nu}_\ell)}{\tau_b} = \frac{G_F^2 m_b^5}{192\pi^3} F(m_c/m_b) |V_{cb}|^2\ , \quad (14)$$

where τ_b is the b lifetime and $F(m_c/m_b)$ is the phase space factor chosen as 0.45. Using an average semileptonic branching fraction BF measured in the continuum of[27] $12.1 \pm 0.8\%$ (which from Eq. (12) is $\mathrm{BF}(b \rightarrow c\ell\bar{\nu}_\ell)$ to within 10%), a world-average bottom hadron lifetime[28] of $(1.18 \pm 0.14)\times 10^{-12}$ sec, and m_b between 4.8 and 5.2 GeV, we get:

$$|V_{cb}| = 0.046 \pm 0.010\ . \quad (15)$$

Most of the error quoted in Eq. (15) is not from the experimental uncertainty in the value of the b lifetime, but in the theoretical uncertainties in choosing a value of m_b and in the use of the quark model to represent inclusively semileptonic decays which, at least for the B meson, are dominated by a few exclusive channels. We have made the error bars larger than they are sometimes stated to reflect these uncertainties. They include the central values obtained for $|V_{cb}|$ by using a model for the exclusive final states in semileptonic B decay and extracting $|V_{cb}|$ from the absolute width for one or more of them.[21,23,29]

The results for three generations of quarks, from Eqs. (5), (7), (8), (11), (12), (13), and (15), plus unitarity, are summarized in the matrix in Eq. (2). The ranges given there are different from those given in Eqs. (5)–(15) (because of the inclusion of unitarity), but are consistent with the one-standard-deviation errors on the input matrix elements.

The data do not preclude there being more than three generations. Moreover, the entries deduced from unitarity might be altered when the K-M matrix is expanded to accommodate more generations. Conversely, the known entries restrict the possible values of additional elements if the matrix is expanded to account for additional generations. For example, unitarity and the known elements of the first row require that any additional element in the first row have a magnitude $|V_{ub'}| < 0.07$. When there are more than three generations, the allowed ranges (at 90% CL) of the matrix elements connecting the first three generations are

$$\begin{pmatrix} 0.9729 \text{ to } 0.9760 & 0.217 \text{ to } 0.223 & 0.003 \text{ to } 0.010 & \cdots \\ 0.162 \text{ to } 0.230 & 0.65 \text{ to } 0.98 & 0.030 \text{ to } 0.062 & \cdots \\ 0 \text{ to } 0.15 & 0 \text{ to } 0.71 & 0 \text{ to } 0.9995 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \quad (15)$$

where we have used unitarity (for the expanded matrix) and Eqs. (5), (7), (8), (11), (12), (13), and (15).

Further information on the angles requires theoretical assumptions. For example, $B_d - \bar{B}_d$ mixing, if it originates from short-distance contributions to ΔM_B dominated by box diagrams involving virtual t quarks, gives information on $V_{tb}\ V_{td}^*$ once hadronic matrix elements and the t quark mass are known.[30] A similar comment holds for $V_{tb}\ V_{ts}^*$ and $B_s - \bar{B}_s$ mixing.

CP-violating processes will involve the phase in the K-M matrix, assuming that the observed CP violation is solely related to a nonzero value of this phase. This allows additional constraints to be brought to bear. More specifically, a necessary and sufficient condition for CP violation with three generations can be formulated in a parametrization-independent manner in terms of the determinant of the commutator of the mass matrices for the charge $2e/3$ and charge $-e/3$ quarks.[31] CP-violating rates or differences of rates all are proportional to a single quantity which is the product of factors $s_{12}s_{13}s_{23}c_{12}c_{13}^2c_{23}s_{\delta_{13}}$ in the explicit parametrization of Refs. 3 and 4, and is $s_1^2 s_2 s_3 c_1 c_2 c_3 s_\delta$ in that of Ref. 1. While hadronic matrix elements whose values are imprecisely known now enter, the constraints from CP violation in the neutral kaon system are tight enough that there may be no solution at all for certain quark masses, values of the phase, etc.[30]

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