SLAC–PUB–4543
UTHEP–88–0101
April 1988
(T/E)

ERRATA

EXPONENTIATION OF SOFT PHOTONS IN THE MONTE CARLO: THE CASE OF BONNEAU AND MARTIN*

STANISLAW JADACH

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

and

Institute of Physics, Jagellonian University,
Cracow, ul. Reymonta 4, Poland‡

and

B. F. L. WARD

Department of Physics and Astronomy,
The University of Tennessee, Knoxville, Tennessee 37996-1200

and

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

Please note the following:

1. In Eq. (1), $\int_{x_0}^{1}$ should be $\frac{2\alpha}{\pi}[\ell n(s/m_e^2) - 1]\int_{x_0}^{1}$.
2. In Eq. (3), $\pi^2/6$ should be $2\pi^2/3$.
3. In Eq. (5), $+(2\alpha/\pi)[\ell n(s/m_e^2) - 1]\ell n x_0$ is missing from the RHS.

We apologize for this.

---

*Work supported in part by the Department of Energy, contracts DE–AC03–76SF00515 and DE–AS05–76ER03956.

‡Permanent address.

Submitted to Physical Review D

SLAC–PUB–4543
UTHEP–88–0101
January 1988
(T/E)

EXPONENTIATION OF SOFT PHOTONS IN THE MONTE CARLO: THE CASE OF BONNEAU AND MARTIN*

STANISLAW JADACH

Stanford Linear Accelerator Center,
Stanford University, Stanford, California 94309

and

Institute of Physics, Jagellonian University,
Cracow, ul. Reymonta 4, Poland‡

and

B. F. L. WARD

Department of Physics and Astronomy,
The University of Tennessee, Knoxville, Tennessee 37996-1200

and

Stanford Linear Accelerator Center,
Stanford University, Stanford, California 94309

ABSTRACT

It is shown, explicitly, how to proceed in the Monte Carlo program in order to include multiple soft photon emission. The method is based on the rigorous theory for summing infrared contributions to the respective cross section by Yennie, Frautschi and Suura. Procedures are illustrated on the example of the initial state bremsstrahlung. One photon is allowed to be hard and an arbitrary number of real soft additional photons are confined to the neighborhood of the infrared point.

Submitted to Physical Review D

---

* Work supported in part by the Department of Energy, contracts DE-AC03-76SF00515 and DE-AS05-76ER03956.

‡ Permanent address.

I. Introduction

The currently used Monte Carlo (MC) programs for the calculation of QED bremsstrahlung effects in high energy lepton-lepton and lepton-hadron processes (see, for example, Ref. 1) are based on the single bremsstrahlung calculations. They include, typically, an emission of a single hard real photon while the infrared point (photon energy = zero) is excluded from the phase space by means of a traditional cutoff on the photon energy in the center-of-mass-system. Events without a photon are also generated and they populate phase space precisely at the infrared point, i.e., they are distributed within a reduced phase space with one particle (three dimensions) less. Their cross section includes contributions from virtual and real photon emission, the result being infrared finite. On the other hand, there was in the past a variety of the calculations based on the summation of the contributions from the infinite number of the soft photons, i.e., on the so-called exponentiation procedure. The most extensive and complete discussion of exponentiation was exposed in the paper of Yennie, Frautschi and Suura (YFS).[2] It provides a rigorous framework for the calculation in which one may improve the precision of the calculation step by step as in the traditional perturbative expansion. In most of the practical applications the common procedure was not to apply the YFS scheme precisely but rather to make an educated guess related to the YFS scheme. Typically that was done by an ad hoc modification of an analytical formula for the partly integrated cross section resulting from the single bremsstrahlung (one loop) calculation. An example of such a procedure may be found in the paper of Jackson and Scharre,[3] where the calculation of Bonneau and Martin[4] is "exponentiated." This sort of procedure is regarded as a relatively easy method of introducing higher order effects in the QED calculation. In fact, when the double bremsstrahlung (two loop) result is compared[5] with that of the "exponentiated" single bremsstrahlung (one loop), one finds that they are rather close.

The question which we address in this paper is the following: is it possible to find a corresponding procedure of introducing multiple soft photons in the MC event generators? Our ambition is also not to rely on ad hoc procedures but rather to refer to the original YFS scheme. The answer is generally positive and the first complete recipe of how one answers our question (proposed examples of MC algorithms) was given in Ref. 6. Here we shall work out an example of adding in the MC generator multiple soft photons in addition to the one hard photon. All photons are emitted from the initial state beams in the e^+e^- annihilation. This will be roughly analogous to the "exponentiation" made on the integrated cross section in Ref. 3. It should be stressed, however, that the procedure used in our MC calculation is based on the rigorous prescriptions of Ref. 2 whereas Ref. 3, and numerous other works related to it, involve various departures from rigor.[5,7] There will be no major obstacle in improving our calculations by inclusion of a second hard photon in the future. In some preliminary form it was done even in this work. It is needless to mention that, in addition to the necessity of calculating/correcting cross sections due to QED effects, there is another reason for including multiple soft and hard photon emission in the MC generators. They may be seen in the detector and it is essential to include them in the MC sample for apparatus acceptance studies.

The plan of the paper is the following. In the next section we consider the Bonneau–Martin cross section and its naive exponentiation in the spirit of the original work of Jackson and Scharre. (We consider Jackson and Scharre's work purely for pedagogical and historical reasons. We are aware[5,7] that their original work has been improved recently by several groups although, from the standpoint of rigor, these improved results are not complete. Accordingly, we feel that Ref. 3 is representative enough of the naive "exponentiation" procedure that it will not misrepresent the pedagogical relationship between the naive procedure and our rigorous methods.) In Section III, we review the relevant aspects of the YFS program from the standpoint of our MC methods. In Section IV, we describe the essential ingredients in our MC realization of the YFS program for the Bonneau–Martin case. Section V contains the numerical results which we use to illustrate the effects of including the multiple photons in the respective final state. It also contains our concluding remarks.

II. Bonneau–Martin Cross Section and Its Exponentiation

The effect of the initial state bremsstrahlung in the e^+e^- annihilation on the total cross section can be summarized in a simple formula usually referred to as a Bonneau–Martin formula. It includes an integral over the photon energy spectrum convoluted with the lowest order cross section at the reduced c.m. system energy:

$$\sigma_{BM}(s) = \sigma^B(s)[1 + \delta_{SX}(s/m_e^2, x_0)] + \int_{x_0}^{1} dx \frac{[1 + (1-x)^2]}{2x} \sigma^B[(1-x)s] \quad , \tag{1}$$

where s is c.m.s. energy squared, x is photon energy in units of $E_{\text{beam}} = \sqrt{s}/2$ and

$$\delta_{SX}(s/m_e^2, x_0) = 2\alpha\widetilde{B}(s/m_e^2, x_0) + 2ReF_1(s/m_e^2) \tag{2}$$

consists of the virtual photon (vertex) correction

$$\begin{aligned} 2ReF_1(s/m_e^2) = \frac{\alpha}{\pi}\Bigg(& [\ell n(s/m_e^2) - 1]\ell n(m_\gamma^2/m_e^2) \\ & - \frac{1}{2}\ell n^2(s/m_e^2) + \frac{3}{2}\ell n(s/m_e^2) - 2 + \frac{\pi^2}{6}\Bigg) \quad , \end{aligned} \tag{3}$$

and the real soft photon contribution ($p_{1(2)}$ is the four-momentum of $e(\bar{e})$ and $P \equiv p_1 + p_2$)

$$\begin{aligned} 2\alpha\widetilde{B}(s/m_e^2, x_0) &= -(\alpha/4\pi^2) \int_{|\vec{k}|<x_0\sqrt{s}/2} \frac{d^3k}{(\vec{k}^2 + m_\gamma^2)^{1/2}} \left(\frac{p_1}{kp_1} - \frac{p_2}{kp_2}\right)^2 \\ &\equiv \int_{|\vec{k}|<x_0\sqrt{s}/2} \frac{d^3k\widetilde{S}(k)}{(\vec{k}^2 + m_\gamma^2)^{1/2}} = \frac{\alpha}{\pi}\Bigg([\ell n(s/m_e^2) - 1]\ell n\left(\frac{m_\gamma^2 x_0^2}{m_e^2}\right) \\ & + \frac{1}{2}\ell n^2(s/m_e^2) - \frac{\pi^2}{3}\Bigg) \quad . \end{aligned} \tag{4}$$

Here m_γ is a photon mass introduced temporarily in order to regulate the infrared singularity. It drops out in the sum as it is seen from the explicit expression below

$$\delta_{SX}(s/m_e^2, x_0) = \frac{\alpha}{\pi}\left(\frac{3}{2}\ell n(s/m_e^2) - 2 + \pi^2/3\right) \quad . \tag{5}$$

The exponentiated formula of Jackson and Scharre (neglecting the contribution from the vacuum polarization) reads

$$\sigma_{JS}(s) = \delta_{SX}\sigma^B(s) + t\int_0^1 dx[x^{t-1} - (1 - x/2)]\sigma^B[(1-x)s] \quad , \tag{6}$$

where

$$t = \frac{2\alpha}{\pi}[\ell n(s/m_e^2) - 1] \quad , \tag{7}$$

and it is obtained by means of the replacement

$$1 + t\left(\frac{1}{x}\right)_+ \to t x^{t-1} \quad . \tag{8}$$

Note that both distributions when integrated in the range from 0 to 1 give precisely one.

As an introductory numerical exercise we plot in Fig. 1 the result from the Bonneau–Martin and Jackson-Scharre formulas for the Z° resonance near the top of the cross section. It is worth mentioning that the result is not very sensitive to the way the exponentiation is done. For example, one gets practically the same curve from another exponentiation ansatz:

$$\sigma'_{JS}(s) = \delta_{SX}\sigma^B(s) + t\int_0^1 dx x^{t-1}[1 - x(1 - x/2)]\sigma^B[(1-x)s] \quad . \tag{9}$$

As it was mentioned this result is not far from the result of the exact second order calculation.

III. Yennie–Frautschi–Suura Expansion

In the following we review briefly the essential ingredients of the YFS formalism which are necessary for further discussion of our MC calculation. Let us start with the YFS expansion truncated on the first two $\overline{\beta}$ terms:

$$\sigma_{\rm YFS}(s) = \exp(2Re\alpha B)\left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tau_{n+2}(P; q_1, q_2, k_1, \ldots, k_n) \left(\prod_{\ell=1}^{n} \widetilde{S}(k_\ell) \right) \overline{\beta}_0(q_1, q_2) \right.$$
$$\left. + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=1}^{n} \int d\tau_{n+2}(P; q_1, q_2, k_1, \ldots, k_n) \left(\prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} \widetilde{S}(k_\ell) \right) \overline{\beta}_1(q_1, q_2, k_j) \right\} \quad , \tag{10}$$

where $q_{1(2)}$ is the four-momentum of $f(\bar{f})$ and where

$$d\tau_n(P; \bar{p}_1, \ldots, \bar{p}_n) = \prod_{\ell=1}^{n} \frac{d^3\bar{p}_\ell}{\bar{p}_\ell^0} \delta^4 \left(P - \sum_{i=1}^{n} \bar{p}_i \right) \quad . \tag{11}$$

In the above formula, soft virtual photon contributions are sitting in the $\exp(2Re\alpha B)$ factor where

$$2Re\alpha B = Re \left[\frac{i\alpha}{4\pi^2} \int \frac{d^4k}{k^2 - m_\gamma^2 + i\epsilon} \left(\frac{2p_1 + k}{k^2 + 2kp_1} - \frac{2p_2 + k}{k^2 + 2kp_2} \right)^2 \right]$$
$$= \frac{\alpha}{\pi} \left([\ell n(s/m_e^2) - 1] \ell n(m_\gamma^2/m_e^2) - \frac{1}{2} \ell n^2(s/m_e^2) \right. \tag{12}$$
$$\left. + \frac{1}{2} \ell n(s/m_e^2) - 1 + \pi^2/6 \right) \quad ,$$

and the real photon emission cross section is rearranged in such a way that the formula as a whole is infrared finite. (The distributions $\overline{\beta}_i$ are finite and will be discussed below.) To see this more clearly, let us introduce temporarily a photon mass regulator and take advantage of the explicit factorizability of the infrared part of the formula (10):

$$\sigma_{\mathrm{YFS}}(s) = \exp(2Re\alpha B + 2\alpha\widetilde{B})\left\{ \int d\tau_2'(P;q_1,q_2)\overline{\beta}_0(q_1,q_2) \right. \\ \left. + \int d\tau_3'(P;q_1,q_2,k)\overline{\beta}_1(q_1,q_2,k) \right\} \tag{13}$$

where we define

$$d\tau_n'(P;\bar{p}_1,\ldots,\bar{p}_n) = \prod_{i=1}^{n} \frac{d^3\bar{p}_i}{\bar{p}_i^0} \frac{\int d^4y}{(2\pi)^4} \exp\left[iy(P - \sum_{i=1}^{n} \bar{p}_i) + D \right] \tag{14}$$

for

$$D \equiv \frac{\int d^3k}{k} \left(e^{-iyk} - \theta(K_{\max} - k) \right) \widetilde{S}(k) \tag{15}$$

so that $\widetilde{B}$ depends on $K_{\max}$. The sum in the exponent is finite (m_γ cancels out) and, assuming $K_{\max} \cong \sqrt{s}/2$, i.e., $x_0 = 1$, we get

$$2\alpha(\widetilde{B} + ReB) = \frac{\alpha}{\pi}\left(\frac{1}{2}\ell n(s/m_e^2) - 1 + \pi^2/3 \right) \quad . \tag{16}$$

For the purpose of the MC we repeat this exercise once more but this time we split the integral over $x = 2|\vec{k}|/\sqrt{s}$ from 0 to 1 into two parts: the first from 0 to δ and the second from δ to 1. The first contribution we include in the exponent

$$\exp\left(2\alpha[ReB + \widetilde{B}(\delta)] \right) = \exp\left(\frac{\alpha}{\pi}[(\ell n(s/m_e^2) - 1)\ell n\delta \right. \\ \left. + \frac{1}{2}\ell n(s/m_e^2) - 1 + \pi^2/3] \right) \quad , \tag{17}$$

and the second we leave where it was, i.e., in the phase space integral. In this way, we split the phase space into two regions: below $x = \delta$ where virtual and real soft photons are combined analytically to yield an infrared finite result and above where we have only real photons which will be generated in the MC. The energy limit $k_\delta = \delta\sqrt{s}/2$ which separates virtual and soft photons in the phase space may be set arbitrarily low. The resulting differential cross section reads [here, D in (14) is now $\int^{k\le k_\delta}(d^3k/k)\widetilde{S}(k)(e^{-iyk}-1) \to 0$ for $k_\delta \to 0$]

$$\sigma_{\rm YFS}(s) = e^{2\alpha[ReB+\widetilde{B}(\delta)]}\Bigg\{\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{k_\ell^0>k_\delta} d\tau'_{n+2}(P;q_1,q_2,k_1,\ldots,k_n)$$
$$\left(\prod_{\ell=1}^{n}\widetilde{S}(k_\ell)\right)\overline{\beta}_0(q_1,q_2) + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{j=1}^{n}\int\limits_{k_\ell^0>k_\delta} d\tau'_{n+2}(P;q_1,q_2,k_1,\ldots,k_n)$$
$$\left(\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}\widetilde{S}(k_\ell)\right)\overline{\beta}_1(q_1,q_2,k_j)\Bigg\} \quad . \tag{18}$$

The reader may worry that the above expression looks as if the four momentum was not conserved. For example the phase space element $d\tau'_3$ includes $\delta^4(P-q_1-q_2-k_j)$ and it seems that only one photon was included in the four-momentum conservation which would determine $\overline{\beta}_1$. Let us now clarify this point. In fact, for the sake of simplicity, some simplification in the notation was tacitly done. Our master formula should be better written in the following way:

$$\sigma_{\rm YFS}(s) = e^{2\alpha[ReB+\widetilde{B}(\delta)]}\Bigg\{\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{k_\ell^0>k_\delta} d\tau'_{n+2}(P;q_1,q_2,k_1,\ldots,k_n)$$
$$\left(\prod_{\ell=1}^{n}\widetilde{S}(k_\ell)\right)\overline{\beta}_0(\mathcal{R}q_1,\mathcal{R}q_2) + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{j=1}^{n}\int\limits_{k_\ell^0>k_\delta} d\tau'_{n+2}(P;q_1,q_2,k_1,\ldots,k_n)$$
$$\left(\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}\widetilde{S}(k_\ell)\right)\overline{\beta}_1(\mathcal{R}q_1,\mathcal{R}q_2,k_j)\Bigg\} \quad . \tag{19}$$

In the framework of the YFS scheme one performs certain manipulations on the differential cross sections in which infrared singularities for real soft photons are extracted in singular $\tilde{S}(k_\ell)$ factors and $\overline{\beta}_i$ functions are the *finite residua* in this procedure at the singular point, i.e., at the point reached by putting the momenta of some photons to zero. The related fact is that, strictly speaking, $\overline{\beta}_0$ is defined within two body phase space and $\overline{\beta}_1$ is defined inside three body phase space. The operation $\mathcal{R}$ is defined such that in $\overline{\beta}_0$ q_i obeying $q_1 + q_2 + \sum_i k_i = P$ are transformed into "reduced momenta" $\mathcal{R}q_i$ which obey $\mathcal{R}q_1 + \mathcal{R}q_2 = P$. Similarly, in $\overline{\beta}_1$, reduced momenta obey $\mathcal{R}q_1 + \mathcal{R}q_2 + k_j = P$ instead of $q_1 + q_2 + \sum_i k_i = P$. This corresponds exactly to going to the residue position. It amounts in practice to some manipulations on momenta in which momenta of some photons are excluded from the four momentum balance. There is a certain degree of freedom on how it is actually done but there are also some restrictions. The previous formula and the actual master formula are numerically equivalent in the sense that in the previous one the momenta q_i should be really treated as new integration variables $q_i^! = \mathcal{R}q_i$ used instead of the original ones. This can be true provided that the Jacobian of the transformation $\mathcal{R}$ is equal to 1 (otherwise it is included properly in the formula). In practice, one may take advantage of the Lorentz invariance of the phase space element $d\tau_n$ under boosts and rotations and use these transformations as the building blocks in the reduction operation $\mathcal{R}$. In general, one has to do at least one rescaling of the momenta and it turns out, not surprisingly, that the best is to do that in the rest frame of $q_1 + q_2$ where the corresponding Jacobian is equal to one (for almost massless fermions).

Finally let us write our master formula once again in a form which will be useful for further discussion of the MC algorithm:

$$\sigma_{\rm YFS}(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{k_\ell^0 > k_\delta} d\tau'_{n+2}(P; q_1, q_2, k_1, \ldots, k_n) \left(\prod_{\ell=1}^{n} \tilde{S}(k_\ell) \right) b_n(q_1, q_2, k_1, \ldots, k_n) \quad , \tag{20}$$

where

$$b_n(q_1,q_2,k_1,\ldots,k_n) = e^{2\alpha[ReB+\widetilde{B}(\delta)]}\Big\{\overline{\beta}_0(\mathcal{R}q_1,\mathcal{R}q_2) + \sum_{j=1}^{n}\overline{\beta}_1(\mathcal{R}q_1,\mathcal{R}q_2,k_j)/\widetilde{S}(k_j)\Big\} \quad . \tag{21}$$

IV. The Monte Carlo

For the purpose of the further discussion we need not only the Bonneau–Martin distribution which is integrated over angles but also the differential cross section. It may also be cast[8] into a semifactorizable form which involves the differential lowest order distributions $d\sigma^B/d\Omega(s',\cos\theta)$ at the reduced c.m. energy squared $s'=(1-x)s$ multiplied by certain functions dependent on the photon momentum:

$$\begin{aligned}\sigma_{BM}(s) &= \sigma^B(s)[1+\delta_{SX}(s/m_e^2,x_0)]\\ &+\int_{x_0}^{1}dx\int d\Omega_\gamma\Big\{\int d\Omega_1 g_1(x,\cos\theta_\gamma)\frac{d\sigma^B}{d\Omega_1}(s',\cos\theta_1)\\ &+\int d\Omega_2 g_2(x,\cos\theta_\gamma)\frac{d\sigma^B}{d\Omega_2}(s',\cos\theta_2)\Big\} \quad ,\end{aligned} \tag{22}$$

where $i=1,2$ in the parametrization of the final state fermion direction $d\Omega_i$ correspond to two well-defined choices of the z-axis in the rest frame of the final state fermions (rcms). In the first case ($i=1$) it points in the direction of the first beam (in rcms) and in the second case ($i=2$) it points opposite to the second beam (also in rcms). Coefficient functions are given by the following expressions:

$$\begin{aligned}g_i(x,\cos\theta_\gamma) &= \frac{\alpha}{2\pi^2}\left[1-\frac{1}{2}x\delta_i\right]^2\left\{\frac{1}{\delta_1\delta_2}-\frac{2m_e^2}{s}\frac{1-x}{1+(1-x)^2}\left(\frac{1}{\delta_1^2}+\frac{1}{\delta_2^2}\right)\right\} \quad ,\\ \delta_1 &= 1-\cos\theta_\gamma\sqrt{1-4m_e^2/s},\delta_2 = 1+\cos\theta_\gamma\sqrt{1-4m_e^2/s} \quad .\end{aligned} \tag{23}$$

The Bonneau–Martin formula is easily recovered using the identity

$$\int_0^1 dx[g_1(x,\cos\theta_\gamma) + g_2(x,\cos\theta_\gamma)] = \frac{1+(1-x)^2}{2x}\frac{2\alpha}{\pi}[\ell n(s/m_e^2) - 1] \quad . \tag{24}$$

Having in hand the above distributions we may proceed to constructing the functions $\overline{\beta}_0$ and $\overline{\beta}_1$ which are necessary to complete our master formula for the MC calculation. The relevant definitions may be found in Ref. 7:

$$\begin{aligned}\overline{\beta}_0(q_1,q_2) &= \frac{d\sigma^B}{d\tau_2(P;q_1,q_2)}(1+2ReF_1-2Re\alpha B)\\ &= \frac{2}{\beta'}\frac{d\sigma^B}{d\Omega}(s,\cos\theta)\left[1+\frac{\alpha}{\pi}(\ell n(s/m_e^2)-1)\right] \quad ,\end{aligned} \tag{25}$$

where $\beta' = (1-4m_f^2/s)^{1/2}$ and

$$\begin{aligned}\overline{\beta}_1(q_1,q_2,k)/\widetilde{S}(k) = \frac{2}{\beta'}\Bigg\{ &\frac{g_1(x,\cos\theta_\gamma)}{g_0(x,\cos\theta_\gamma)}\frac{d\sigma^B}{d\Omega}(s',\cos\theta_1)\\ &+\frac{g_2(x,\cos\theta_\gamma)}{g_0(x,\cos\theta_\gamma)}\frac{d\sigma^B}{d\Omega}(s',\cos\theta_2) - \frac{1}{2}\left(\frac{d\sigma^B}{d\Omega}(s,\cos\theta_1)\right.\\ &\left.\left.+\frac{d\sigma^B}{d\Omega}(s,\cos\theta_2)\right)\right\} \quad ,\end{aligned} \tag{26}$$

where

$$g_0(x,\cos\theta_\gamma) = \frac{\alpha}{2\pi^2}\left\{\frac{1}{\delta_1\delta_2} - \frac{m_e^2}{s}\left(\frac{1}{\delta_1^2}+\frac{1}{\delta_2^2}\right)\right\} \tag{27}$$

is up to a normalization constant equal to $\widetilde{S}(k)$. In $\overline{\beta}_1$, above, the last two terms represent $\overline{\beta}_0(\mathcal{R}q_1,\mathcal{R}q_2)$. The $\mathcal{R}$ procedure in that case amounts to taking $\cos\theta_i$ in the rest frame of q_1+q_2 system and the average over i is taken in order to have a symmetric solution. It should be stressed also that in these two terms s is taken in contrast to the first two where s' is used instead. In the presence of the additional photons one has to provide for the $\mathcal{R}$ procedure to produce $\mathcal{R}q_1, \mathcal{R}q_2$, and k momenta to be plugged in as arguments in the above $\overline{\beta}_1$ expression. In the actual Monte Carlo it is done in the following way. One transforms q_i to the $\vec{q}_1 + \vec{q}_2 = \vec{0}$ frame, then $\vec{q}_1$ and $\vec{q}_2$ are rescaled by the factor which corresponds to exclusion of additional photons and the momenta are boosted back to the cms system, taking again a boost parameter which takes the exclusion of additional photons into account. The resulting momenta obey $\mathcal{R}q_1 + \mathcal{R}q_2 + k = P$ where k is the momentum of the only one photon which actually was not touched in the $\mathcal{R}$ reduction procedure. (Using entirely analogous procedures we have also constructed the leading logarithmic approximation to $\overline{\beta}_2$.)

The MC algorithm is organized in such a way that there are two distinct levels in it. There is a low level MC generator which generates events according to our master formula with a simplified b_n function. It is simply

$$b_n'(q_1, q_2, k_1, \ldots, k_n) = \frac{1}{2\pi\beta'}\sigma^B[(q_1 + q_2)^2] \quad . \tag{28}$$

The events are generated on the four-momentum level using this simplified differential cross section and next the real distribution is recovered with help of the rejection procedure with the rejection weight

$$w = \frac{b_n'(q_1, q_2, k_1, \ldots, k_n)}{b_n(q_1, q_2, k_1, \ldots, k_n)} \quad . \tag{29}$$

The advantage of this arrangement is that the low-level MC part will remain the same even if more $\overline{\beta}$'s are included in the future in the YFS expansion in b_n. One will need to change only the model-dependent part of the program. In a sense, the low-level MC part is a sort of universal phase space MC program for QED initial state bremsstrahlung.

The question is now, however, how events are generated in the low-level MC according to our simplified distribution. The solution is quite similar to that proposed in Ref. 6. The integral may be written as follows

$$\sigma'_{\mathrm{YFS}} = \int_0^1 dv \sigma^B[(1-v)s] \sum_{n=0}^{\infty} \frac{1}{n!} \int_{k_\ell^0 > k_\delta} \left(\prod_{\ell=1}^{n} \frac{d^3 k_\ell}{k_\ell^0} \tilde{S}(k_\ell) \right) \delta \left(v - \frac{2KP - K^2}{P^2} \right) \quad , \tag{30}$$

where $K = \sum_i k_i$. Photon momenta are generated quite similarly as in the algorithm no. 2 in Ref. 6. In this algorithm the energy conservation is obtained by rescaling momenta of all photons by a certain factor. Here the method is the same but the condition to be fulfilled, that inside the delta function in the above expression, is slightly more complicated. Because of that one picks up a Jacobian factor in the integrand which has to be removed again by the rejection method. The details on that will be given elsewhere.

V. Numerical Results and Conclusions

In Fig. 1 we plot the total cross section in R-units for τ pair production at the vicinity of the Z° resonance. Included are the Born cross section, the cross section from the Bonneau–Martin formula and the exponentiated result obtained using the Jackson–Scharre formula (as we have noted), and the formula of Kuraev and Fadin.[9] The results from the MC of the type described in this paper are represented by dots. They are obtained from samples of 10^4 events. The statistical error is of the size of the dot. At each energy the two cross sections correspond to two possible upper limits on the energy of the soft photon $\bar{E}_\gamma^{\mathrm{soft}}$. There is no limitation on the energy of the most energetic photon but all other ones must stay below $\bar{E}_\gamma^{\mathrm{soft}}$. One result (higher cross section) is obtained using $\bar{E}_\gamma^{\mathrm{soft}} = 2$ GeV and the other one using $\bar{E}_\gamma^{\mathrm{soft}} = 0.1$ GeV. Generally, the result of the MC comes close to the result of the naive exponentiation and it depends rather weakly on $\bar{E}_\gamma^{\mathrm{soft}}$. It is very essential, however, that this dependence is included in the calculations.

The $\overline{\beta}_0$ and $\overline{\beta}_1$ in (25) and (26) do not include the effects of renormalization group improvement. In the case at hand such improvement may be effected as follows. In $\overline{\beta}_0$, the prescription in Ref. 7 requires, here, the substitution ($\alpha(2m_{\mu,\mathrm{phys}})$ is α at the scale $2m_{\mu,\mathrm{phys}}$)

$$\alpha \to \alpha(\lambda) = \alpha(2m_{\mu,\mathrm{phys}})/[1 - 8\pi\alpha(2m_{\mu,\mathrm{phys}})b_0\ell n\lambda] \quad , \tag{31}$$

with the understanding that, in the Z° squared-coupling G^2, we write

$$\frac{G^2}{4\pi} = \frac{g_W^2(M_{Z^\circ})}{4\pi} + (M_{Z^\circ}^2/M_W^2)\alpha(\lambda) \quad , \tag{32}$$

where, here, $\lambda \simeq M_{Z^\circ}/2m_{\mu,\mathrm{phys}} \simeq 4.353621 \times 10^2$ and g_W is the SU_{2L} coupling evaluated at the scale M_{Z° so that, from Ref. 10, we may take it to be .65626. Here, $M_W \simeq 80.8$ GeV and $M_{Z^\circ} \simeq 92$ GeV. Note also that $b_0 = 11/48\pi^2$ here.

Similarly, in $\overline{\beta}_1$, the prescription in Ref. 7 requires that we leave $\alpha \simeq 1/137.03604$ in $g_{0,1,2}$ but that we make the substitutions in (31) and (32) in $d\sigma^B$. This then accounts for the renormalization group improvement of the results in (22)–(27). The improvement of $\overline{\beta}_2$ is effected in an analogous manner.

The respective renormalization group improvement effects on the cross section represented by the round dots in Fig. 1 are shown by the crosses in the figure. We see that these effects are significant if one wants high precision MC simulations.

Note added: It has recently been verified (Wim de Boer, private communication, 1987) that the total integrated cross sections associated with the Monte Carlo procedure described in this paper are in fact consistent, to three or more significant figures, with the total integrated cross sections in the second order results of Berends *et al.*[5] at $\sqrt{s} = M_{Z^\circ}$. This is an important check on the global aspects of our Monte Carlo methods. More checks of this type will be taken up elsewhere.

Acknowledgements

The authors have benefitted from their interactions with the Mark II SLC Z° Mass and Width Physics Working Group and they are grateful to Profs. J. Dorfan and G. Feldman for making these interactions possible.

References

1. F. A. Berends, R. Kleiss and S. Jadach, Comp. Phys. Commun. **29**, 185 (1983).
2. D. R. Yennie, S. C. Frautschi and H. Suura, Annals of Phys. **13**, 379 (1961).
3. J. D. Jackson and D. L. Scharre, Nucl. Instr. and Meth. **128**, 13 (1975).
4. G. Bonneau and J. Martin, Nucl. Phys. **B27**, 381 (1971).
5. F. Berends *et al.*, Phys. Lett. **185B**, 395 (1987); F. Berends and W. Beenakker, to appear.
6. S. Jadach, *Yennie–Frautschi–Suura Soft Photons in the Monte Carlo Generators*, preprint of Max–Planck–Institut München, MPI–PAE/PTh 6/87, January, 1987.
7. B. F. L. Ward, Phys. Rev. **D36**, 939 (1987); Acta Physica Polonica, 1988, to be published; and references therein.
8. F. A. Berends and R. Kleiss, Nucl. Phys. **B177**, 237 (1981).
9. E. A. Kuraev and V. S. Fadin, Sov. J. Nucl. Phys. **41**, 466 (1985).
10. W. J. Marciano, Phys. Rev. **D20**, 274 (1979); G. Altarelli *et al.*, in *Physics at LEP*, eds. J. Ellis and R. Peccei (CERN, Geneva, 1986).

Figure Caption

1. Two solid curves represent the Born and Bonneau–Martin cross sections. The dashed curve is according to Jackson and Scharre and the dotted curve is from the Kuraev–Fadin result.[a] Three types of points come from our Monte Carlo, 10^4 events, statistical error below the size of the dots. Round and square dots represent the Monte Carlo result for $\overline{\beta}_0 + \overline{\beta}_1 + \overline{\beta}_2$ and triangle points represent the $\overline{\beta}_0 + \overline{\beta}_1$ result. The most energetic photon is allowed everywhere in the phase space and the other photons are confined within a sphere $E_\gamma \leq \bar{E}_\gamma^{\rm soft}$. Two values for the $\bar{E}_\gamma^{\rm soft}$ cutoff are used: 2 GeV and 0.1 GeV. The crosses show the effect of renormalization group improvement on the round dots.

[a]The Kuraev–Fadin result is defined as follows:

$$\sigma_{KF} = \int_0^1 dx \sigma^B[s(1-x)] \left\{ \alpha A x^{\alpha A - 1}(1+\delta_R) + \alpha A(-1 + x/2) \right\},$$

$$\delta_R = \frac{3}{2}\frac{\alpha}{\pi}[\ell n(s/m_e^2) - 1] + \frac{\alpha}{\pi}\left(\frac{\pi^2}{3} - 2\right), \alpha A = \frac{2\alpha}{\pi}[\ell n(s/m_e^2) - 1] = t \quad .$$

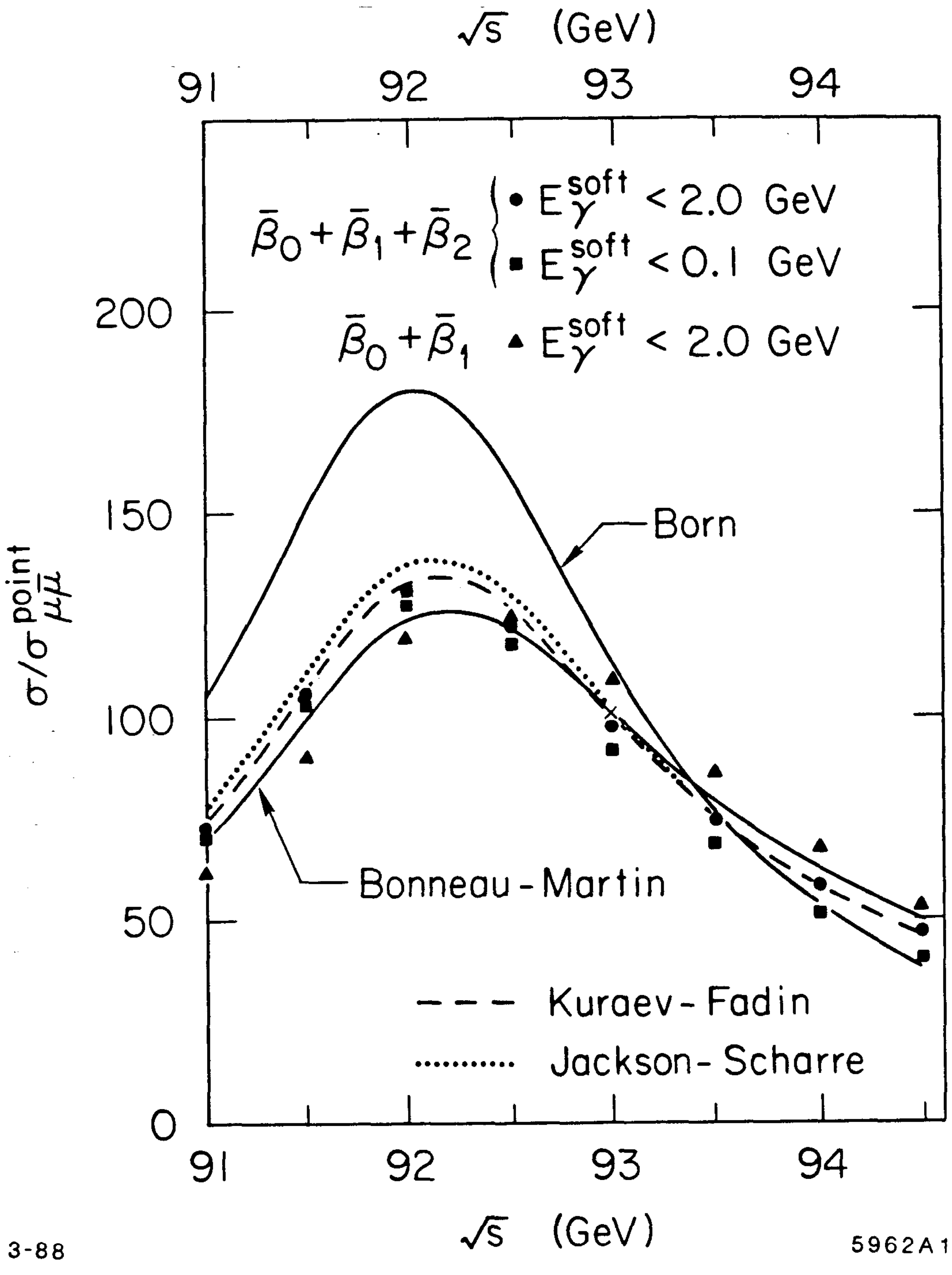
$\sqrt{s}$ (GeV)
91
92
93
94
$\bar{\beta}_0+\bar{\beta}_1+\bar{\beta}_2$
• $E_\gamma^{soft} < 2.0$ GeV
▪ $E_\gamma^{soft} < 0.1$ GeV
$\bar{\beta}_0+\bar{\beta}_1$ ▲ $E_\gamma^{soft} < 2.0$ GeV
200
150
100
50
0
$\sigma/\sigma_{\mu\bar{\mu}}^{point}$
Born
Bonneau - Martin
Kuraev - Fadin
Jackson - Scharre
3-88
5962A1

Fig. 1