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STRINGS IN FOUR DIMENSIONS*

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ABSTRACT

We review the construction and properties of four dimensional string models, using free fermions on the world-sheet. We prove that as opposed to gauge symmetries, broken space-time supersymmetry can only be restored continuously by decompactification.

INTRODUCTION

Much progress has been made recently in the study of classical string solutions [3-15]. We now know how to explicitly construct lots of consistent and phenomenologically interesting string models directly in four dimensions [7-12]. Furthermore , many calculations with them are simple , often in fact simpler than their field theory counterparts [16] ; thus there seems to be little reason for even the most pragmatic model-builder not to try and take into account the stringy constraints that guarantee a consistent unification of quantum gravity [1].

Although some 4d models can be obtained by compactification from 10 dimensions [3-5], such a geometric interpretation is not always possible because the six internal bosonic coordinates are treated in general on the same footing as their world-sheet superpartners . Futhermore, even in the case of a bonna fide compactification, the size of the internal manifold is often of the order of the

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Planck scale; the stringy nature of particles becomes then crucial and invalidates our intuition from field theories. For instance, one may obtain chirality from a non chiral 10d theory, or enhance the gauge symmetries [17], both of which would be forbidden in a traditional Kaluza Klein compactification. For these reasons it is more fruitful to abandon the language of compactification, and think of the string as moving directly in four space time dimensions with all its internal quantum numbers carried by some superconformal modular covariant model of appropriate central charge on the world-sheet [18].

The space of all such superconformal models is huge and includes such exotic possibilities as quantized Liouville modes [19], or collections of models from the minimal discrete series [11]. Nevertheless , most of what we know at present about 4d string theory can be learned even if we restrict ourselves to a much simpler class of models made out of free bosonic or fermionic fields on the world-sheet ; we may refer to these models as *Gaussian* . Different Gaussian models have the same energy-momentum tensor but may differ in the way world-sheet supersymmetry is realized and/or the choice of boundary conditions under parallel transport around the string . In this talk I will restrict myself even further to a subclass of Gaussian models which , in the fermionic language, are obtained by allowing only mutually commuting boundary conditions. Models with non-commuting boundary conditions are probably equivalent in the bosonic language to generic rational left-right asymmetric orbifolds ; the analysis of multiloop amplitudes is in this case considerably more complicated as discussed earlier by Narain [10].

The structure of this talk is as follows : in section 2 , I will briefly review the construction of consistent 4d string models using free world-sheet fermions with commuting spin-structures . In section 3 , I will show how to obtain models with space-time supersymmetry, chiral matter fields and realistic gauge groups, and discuss some of their elementary properties. Finally, in section 4 , I will examine the spontaneous breaking of symmetries; I will show that as opposed to gauge symmetries, space-time supersymmetry can only be restored continuously in a decompactification limit.

CONSTRUCTION OF 4D MODELS

The guiding principle for the construction of consistent first-quantized string theories is invariance under all (super)reparametrizations of the world-sheet. This is required for finiteness and unitarity; it also seems to imply, mysteriously enough, some of the most profound principles of modern physics such as the principle of equivalence, gauge invariance, the cancellation of anomalies and the spin-statistics connection. Let us then see how to impose reparametrization invariance on a heterotic string [2] moving in four flat space-time dimensions, with extra free world-sheet fermions carrying all its internal quantum numbers.*

Since the 2d theory is free, invariance under *infinitesimal reparametrizations* is guaranteed , provided we cancel the conformal anomaly. This fixes the total number of fermions : in the non-supersymmetric antiholomorphic side we have in addition to the space-time coordinates $\partial_{\bar{z}}X^{\mu}$ an extra 44 real fermions η^A (so that $4+\frac{44}{2}=26$). In the supersymmetric holomorphic side, on the other hand, we have the $\partial_z X^{\mu}$, their supersymmetric partners ψ^{μ} and finally an extra 18 real fermions χ^a (so that $4+\frac{4}{2}+\frac{18}{2}=26-11$) . Recall that $\frac{1}{2},1,-26$ and 11 are the contributions of a Majorana fermion ,a boson , the ghosts in the conformal gauge and their superpartners , in this order [23]. Note also that our analysis can easily be applied to type II supersymmetric strings , but these will not concern us here since their phenomenological prospects are dimmer [13,14] .

Next, we must ensure invariance under 2d *holomorphic supersymmetry transformations.* A generic candidate for the Lorentz- invariant, dimension $\frac{3}{2}$ generator of such transformations is [6,24]*

$$T_F=\psi^{\mu}\partial_z X_{\mu}+\frac{i}{3}f_{abc}\chi^a\chi^b\chi^c \tag{2.1}$$

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* This was suggested already at the dawn of dual models by Bardacki and Halpern [20]. The idea was resurrected for supersymmetric strings in ref. [6]. The modular invariance constraints were understood following the work of ref. [21,22]. These constraints were systematically analyzed in ref. [7,9,12].
* More general supercharges can be constructed if we bosonize the fermionic currents but these have not yet been completely classified.

This must obey the operator product expansion :

$$T_F(z)T_F(w) \sim \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2T_B(w)}{z-w} \tag{2.2}$$

with T_B the free energy-momentum tensor and $c = 15$ the central charge of matter fields. Using Wick's theorem, it is straightforward to check that eq. (2.2) is satisfied if and only if

$$f_{[abe}f_{cd]e} = 0 \tag{2.3a}$$

and

$$f_{acd}f_{bcd} = \frac{1}{2}\delta_{ab} \tag{2.3b}$$

where here the brackets stand for antisymmetrization in all loose indices and repeated indices are as always implicitly summed. We conclude that the coefficients f_{abc} are the appropriately normalized structure constants of a semi-simple Lie group G , since they obey the Jacobi and orthonormality conditions (2.3a,b). The dimension of G must be 18, so that it is one of only three possible groups : $SU(2)^6, SU(2) \times SU(4)$ or finally $SU(3) \times O(5)$.

The final requirement is invariance under *modular transformations* , i.e. global reparametrizations that cannot be reached continuously from the identity. This forces us to sum over different boundary conditions, or spin-structures for the fermions. Strictly speaking, a spin-structure for all fermions f_i on a world-sheet Σ is a representation of the first homotopy group $\pi_1(\Sigma)$ by orthogonal matrices : to every non-contractible loop on the surface we assign some matrix $\mathcal{A}$, so that $f_i \to \mathcal{A}_{ij}f_j$ under parallel transport around the loop. The matrix $\mathcal{A}$ should not mix left and right-movers, and should respect Lorentz- invariance and world-sheet supersymmetry. Thus, it must have the following block-diagonal form :

$$\mathcal{A} = -\delta_{\mathcal{A}} \begin{array}{c} \overbrace{\quad}^{\psi^\mu} \; \overbrace{\quad}^{\chi^a} \; \overbrace{\quad}^{\eta^A} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathcal{A}^G & 0 \\ 0 & 0 & \mathcal{A}_R \end{pmatrix} \end{array} \tag{2.4}$$

where $\delta_{\mathcal{A}}$ is a sign which, as we will soon see , plays the role of a space-time fermion parity, and $\mathcal{A}^G$ is an automorphism of the group G, i.e. :

$$f_{abc}\mathcal{A}^G_{aa'}\mathcal{A}^G_{bb'}\mathcal{A}^G_{cc'} = f_{a'b'c'} \tag{2.5}$$

This follows from the requirement that the supercharge T_F , eq. (2.1),be periodic or antiperiodic when parallel transported around a loop.

In general, since $\pi_1(\Sigma)$ is non-abelian for world-sheets Σ of genus $g \geq 2$, the matrices $\mathcal{A}$ corresponding to homotopically distinct loops need not commute. Major simplifications, however, do occur if we restrict ourselves to matrices chosen from a set of mutually commuting ones. In this case, $\pi_1(\Sigma)$ can be replaced by its abelianized version $H_1(\Sigma)$, also called the first homology group, the action of the mapping class group $Diff(\Sigma)/Diff_0(\Sigma)$ becomes that of $SL(2g,Z)$ and all fermionic determinants depend only on the period matrix of the surface and are, in fact, known explicitly in terms of Θ-functions [22].

For these purely technical reasons, we limit ourselves here to mutually commuting matrices $\mathcal{A}$, which can thus be simultaneously diagonalized in some, generally complex basis $\{f_1, f_2, ...f_K\}$ of fermions. In this basis

$$\mathcal{A} = -diag(e^{i\pi\alpha_1}, ..., e^{i\pi\alpha_K})$$

and we may denote the matrix $\mathcal{A}$ by the vector $\alpha = (\alpha_1, ..., \alpha_K)$ of phases . By convention I will only include the phase of the two real transverse , or one complex, fermions ψ^μ and will take $-1 < \alpha_i \leq 1$.

Now a particular string model is determined by a set of coefficients $C\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$; these are the weights with which a particular spin-structure $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ contributes to the one-loop amplitudes. Here α and β are as usual the boundary conditions in the space and time directions around the torus. The one- loop vacuum to vacuum amplitude for instance reads :

$$Z_{1-loop} = \mathcal{N}^{-1} \int_{Fund.dom.} \frac{d\tau d\bar{\tau}}{(Im\tau)^2} \frac{1}{(\eta\bar{\eta})^2 (Im\tau)} \sum_{spinstructs} C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \prod_i \frac{\Theta\begin{bmatrix}\alpha_i\\ \beta_i\end{bmatrix}}{\eta} \quad (2.6)$$

where τ is the modular parameter of the torus integrated over a fundamental domain in the upper complex plane , $\mathcal{N}$ is a normalization and η and Θ the well-known Dedekind and Jacobi-Riemann functions.

Modular invariance imposes the following set of necessary and sufficient [22] conditions :

$$C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = -e^{\frac{i\pi}{4}\alpha\cdot\alpha} C\begin{bmatrix}\alpha\\ \beta-\alpha+1\end{bmatrix} \quad (2.7a)$$

$$C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = e^{\frac{i\pi}{2}\alpha\cdot\beta} C\begin{bmatrix}\beta\\ -\alpha\end{bmatrix} \quad (2.7b)$$

$$C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} C\begin{bmatrix}\alpha\\ \gamma\end{bmatrix} = \delta_\alpha C\begin{bmatrix}\alpha\\ \beta+\gamma\end{bmatrix} \quad (2.7c)$$

where (2.7 a,b) come from invariance under the modular transformations of the torus ($\tau \to \tau+1$ and $\tau \to -\frac{1}{\tau}$ respectively), while (2.7 c) comes from two-loop modular invariance and the assumption of factorization of string amplitudes. In these equations , 1 stands for the vector with all entries equal to one ,the dot products are Lorentzian: left minus right-movers, and addition is always understood modulo 2 . A detailed analysis of these conditions can be found in refs. [7,9,12]. For lack of time, I will only summarize here the results that I will need in the sequel .

To start with, contributing spin-structures correspond to pairs of elements of some additive group Ξ of boundary conditions, so that :

$$|C\begin{bmatrix}\alpha\\ \beta\end{bmatrix}| = \begin{cases} 1, & \text{if } \alpha,\beta \in \Xi \\ 0, & \text{otherwise} \end{cases} \tag{2.8}$$

We shall assume Ξ is finite ,which means that all vectors $\alpha \in \Xi$ have rational components, since otherwise the normalization in (2.6) diverges and the expression is only formal. Being abelian Ξ is isomorphic to a direct sum : $Z_{n_1} \oplus Z_{n_2} \oplus ... \oplus Z_{n_\Xi}$. Furthermore it must contain the vector 1 and, since it is a group, also the vector 0 ; these correspond respectively to periodic and antiperiodic boundary conditions for all fermions .

If we were dealing with the type II supersymmetric or the bosonic string, with left-right symmetric boundary conditions, all phases in (2.7) would disappear, the absolute value in (2.8) could be dropped and this would be the end of the story. For left-right asymmetric models, however, the existence of at least one choice of phases for the coefficients $C\left[{\alpha \atop \beta}\right]$ that is consistent with eqs. (2.7) imposes extra constraints on the allowed groups Ξ . Let me describe these constraints in the simple case where fermions are allowed to be only periodic or antiperiodic. A vector α can then be interpreted as the characteristic function of a set of periodic fermions :

$$\alpha_i = \begin{cases} 1, & \text{if } f_i \text{ periodic} \\ 0, & \text{if } f_i \text{ antiperiodic} \end{cases}$$

Vector addition can be interpreted as the symmetric difference (union minus intersection) of sets . $\Xi = Z_2 \oplus Z_2 \oplus ... \oplus Z_2$ is generated by a basis $\{\beta^{(0)} = 1, \beta^{(1)}, \beta^{(2)} ...\}$ of fermion sets. Then the constraints on Ξ are that :

$$n(\beta^{(i)}) = 2n(\beta^{(i)} \cap \beta^{(j)}) = 4n(\beta^{(i)} \cap \beta^{(j)} \cap \beta^{(k)} \cap \beta^{(l)}) = 0 \quad mod8 \tag{2.9}$$

where $n(\beta)$ is the number of real left minus right-moving fermions in the set β. Note incidentally that in this case of periodic or antiperiodic fermions all coefficients $C\left[{\alpha \atop \beta}\right]$ are pure signs as follows from eq. (2.7 c) and the fact that $\alpha + \alpha = 0$.

SUSY , CHIRALITY , AND MORE

In this section I will discuss some general properties of these string theories, and will try to illustrate how easily one can obtain semi-realistic models with N=1 space-time supersymmetry , chiral families and phenomenologically acceptable gauge groups. Let us begin by noting that from the one-loop vacuum amplitude eq. (2.6) we can read off the Hilbert space of string excitations :

$$\mathcal{H} = \bigoplus_{\alpha\in\Xi} \prod_{\beta\in\Xi} \{e^{i\pi\beta\cdot F} = \delta_\alpha C^*\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\}\ \mathcal{H}_\alpha \tag{3.1}$$

I now explain this formula : $\mathcal{H}_\alpha$ is the Hilbert-space sector in which the 2d fermions have boundary conditions given by α when parallel transported around the string . This means that states in $\mathcal{H}_\alpha$ are constructed by acting on a vacuum $|0>_\alpha$ with positive frequency oscillators $f_i(n)$, where the frequency $n = \frac{1+\alpha_i}{2} + integer$. The total Hilbert space is a direct sum of sectors, one for each $\alpha \in \Xi$. F_i is the fermion-number operator that counts the fermions of type f_i . The curly brackets in (3.1) stand for a projection operator that projects out all states which do not satisfy the equality inside . The factorization condition (2.7 c) guarantees that different projectors are mutually compatible, i.e. do not kill entire sectors. The Hilbert space (3.1) is a simple generalization of the Neveu-Schwarz-Ramond , or heterotic string constructions : each time we add new sectors to a theory , we must also add new GSO-type projections [25], so that the vertex operators emitting physical states are single-valued relative to each other [26] .

The mass of a physical state in units of M_{Planck} is given by the zeroth-order Virasoro gauge conditions :

$$M^2 = \sum_{leftmovers} (frequencies) - \frac{1}{2} + \frac{\alpha_L\cdot\alpha_L}{8} \tag{3.2a}$$

$$= \sum_{rightmovers} (frequencies) - 1 + \frac{\alpha_R\cdot\alpha_R}{8} \tag{3.2b}$$

where α_L, α_R are the left- and right-parts of the vector α, and the sums run over all oscillators used to construct the state. The space-time statistics of the states in a sector $\mathcal{H}_\alpha$ depend only on whether the ψ^μ are periodic ($\delta_\alpha = -1$) or antiperiodic ($\delta_\alpha = 1$) under the boundary condition α . The reason is that in the former case the vacuum $|0>_\alpha$ must represent the Dirac algebra of zero-modes: $\{\psi^\mu(0), \psi^\nu(0)\} = \eta^{\mu\nu}$ and is therefore a space-time spinor , while in the latter it is a scalar . Furthermore oscillators cannot change the statistics since they carry at most a Lorentz-vector index .

Let us consider now some specific examples of string-models. To simplify matters, I will restrict myself to the case of only periodic or antiperiodic fermions; the operators $e^{i\pi\alpha\cdot F} \equiv (-)^\alpha$ are fermion-parities that anticommute with all the fermions in α , while commuting with the rest. I will furthermore choose the world-sheet supersymmetry group : $G = SU(2)^6$ so that the supercharge reads :

$$T_F = \psi^\mu \partial_z X^\mu + i\sum_{I=1}^{6} \chi^{I,1}\chi^{I,2}\chi^{I,3} \tag{3.3}$$

The requirement that a boundary condition leave T_F unchanged up to a sign can be checked easily by inspection .

The minimal string model has just two sectors: $\Xi = \{0,1\}$, i.e. either all fermions are periodic or they are all antiperiodic . The low-lying spectrum contains a tachyon T^A ($M^2 = -\frac{1}{2}$) in the vector representation of SO(44) :

$$\eta^A(\frac{1}{2})|0>_0 \ , \tag{3.4a}$$

a graviton, dilaton and 2-index antisymmetric tensor :

$$\psi^\nu(\frac{1}{2})\partial X^\mu(1)|0>_0 \ , \tag{3.4b}$$

gauge bosons $A^{\mu,a}$ and $A^{\mu,AB}$ of $G \times SO(44)$:

$$\chi^a(\frac{1}{2})\partial X^\mu(1)|o>_0 \quad ; \quad \psi^\mu(\frac{1}{2})\eta^A(\frac{1}{2})\eta^B(\frac{1}{2})|0>_0 \tag{3.4c}$$

and finally massless scalars $\Phi^{a,AB}$ in the (adjoint,adjoint) of $G \times SO(44)$:

$$\chi^a(\frac{1}{2})\eta^A(\frac{1}{2})\eta^B B(\frac{1}{2})|0>_0 \tag{3.4d}$$

All other states have masses of o(M_P). In particular this model contains no massless space-time fermions .

To remedy this situation, as well as get rid of the tachyons, let us change the theory by adding a new set :

$$S = \{\psi^\mu, \chi^{I,3}\} \tag{3.5}$$

to the generators of the group of sectors which becomes : $\Xi = \{0, 1, S, S+1\}$. The effect on the low-lying spectrum is twofold: firstly in the sector $\mathcal{H}_0$ the new GSO projection sets $(-)^S = \delta_0 C^*\left[{0 \atop S}\right] = -1$ where the last equality follows easily from eq. (2.7c). Thus, out of all the states (3.4) we must only keep those that have odd S-parity. This leaves the graviton and company , the six gauge bosons $A_{\mu,(I,3)}$, the gauge bosons of SO(44), and finally six scalars $\Phi^{(I,3),AB}$ in the adjoint representation of SO(44) . Secondly, massless space-time fermions now appear in the sector $\mathcal{H}_S$, namely four spin-$\frac{3}{2}$ and four spin-$\frac{1}{2}$ states :

$$\partial X^\mu(1)|0>_S \tag{3.6a}$$

as well as four spin-$\frac{1}{2}$ states in the adjoint of SO(44):

$$\eta^A(\frac{1}{2})\eta^B(\frac{1}{2})|0>_S \tag{3.6b}$$

The multiplicity of four comes from the fact that $|0>_S$ is both a Lorentz and an internal SO(6) spinor since it must represent the algebra of six zero-modes

$\chi^{I,3}(0)$. It is straightforward to check that the masssless states (3.6) together with the odd-S-parity states (3.4), form N=4 graviton and SO(44) Yang-Mills multiplets . That the theory has N=4 supersymmetry even when higher excited states and interactions are taken into account , can in fact be demonstrated by explicitly constructing the space- time supersymmetry generators [27] .

Of course N=4 theories are phenomenologically uninteresting since they don't have matter multiplets . To reduce the space-time supersymmetry , let us add one more basis element :

$$\beta^{(1)} = \{\chi^{(I=3,\ldots 6)(i=2,3)}, \eta^{A=1,\ldots 16}\} \tag{3.7}$$

to our group of sectors . Proceeding as before, we note first that the result of the $(-)^{\beta^{(1)}}$ projection is to truncate the spectrum of the SO(44) , N=4 supersymmetric theory down to the graviton, $SO(16) \times SO(28)$-Yang-Mills , and (vector,vector)- matter multiplets of N=2 . We trust the reader can , if he wants to, work out the details of this truncation , keeping in mind that the operator $(-)^{\beta^{(1)}}$ anticommutes with the four zero modes $\chi^{(I=3,..6),3}(0)$, and therefore acts on the states (3.6) as an internal SO(4) chirality . Besides satisfying the conditions (2.9), the choice of $\beta^{(1)}$ was dictated by the requirement that the new sectors contribute a massless N=2 matter multiplet, in the (spinor,1) representation of $SO(16) \times SO(28)$. These are the states:

$$|0>_{\beta_1} \quad and \quad |0>_{\beta_1+S} \tag{3.8}$$

which will give rise to chiral matter families at the next and last stage of our construction.

Indeed let us finally add the following set to our group of sectors :

$$\beta^{(2)} = \{\psi^\mu, \chi^{(I=1,..4),1}, \chi^{5,3}, \chi^{6,3}, \eta^{A=1,\ldots 10}, \eta^{A=17,\ldots 30}\} \tag{3.10}$$

The result of the $(-)^{\beta^{(2)}}$ truncation is to break the gauge group down to $SO(10) \times SO(6) \times SO(14)^2$, and to reduce space-time supersymmetry to N=1 . Furthermore, acting on the states $|0>_{\beta^{(1)}}$ the operator $(-)^{\beta^{(2)}}$ equals $(helicity) \times$

$(SO(10) chirality)$. The $\beta^{(2)}$ projection is therefore precisely a Weyl projection. To summarize, we have thus finally obtained a string model whose massless spectrum contains the states of a standard N=1 supersymmetric SO(10)- grand unified theory , with eight families, and a gauged SO(6) horizontal symmetry.

One can go on refining this model but I don't think this would at this stage be particularly illuminating . Let me instead make some general remarks. The first is that a given string tree amplitude does not depend on any details of the model, other than the external vertex operators inserted on the sphere. On the other hand, as the above construction illustrated, the massless states of many 4d models can be obtained by truncating a more symmetric theory. This makes the calculation of the effective tree Lagrangian of massless modes considerably easier [28]. For instance the N=1 theory constructed above is an exact truncation of a N=2 theory , whose Lagrangian depends on only one rather than two arbitrary functions .

The second remark concerns the graviton, dilaton and antisymmetric tensor states (3.4b), which seemed to survive all projections . This is no accident : indeed acting on these states $e^{i\pi\beta\cdot F} = \delta_\beta = C^*\left[{0 \atop \beta}\right]$, where the second equality follows easily from (2.7 b,c) . Consequently the GSO projections (3.1) are automatically satisfied, meaning that graviton and company are always in the string spectrum.

The final remark concerns space-time supersymmetry . First note that, in the case of only periodic or antiperiodic fermions, the only candidate massless spin-$\frac{3}{2}$ states are : $\partial X^\mu(1)|0>_S$, with S a set of precisely eight real left-movers* . Indeed S must contain at least 8 fermions to make the supercharge T_F periodic , and it cannot contain more since the mass , eq. (3.2), would then become non-zero . Next note that some or all components of $\partial X^\mu(1)|0>_S$ will survive the GSO projections if and only if for all $\alpha \in \Xi$ *disjoint* from S, we have

* We will encounter more general "supersymmetry generating" vectors S with non-integer components in the following section.

$$C\begin{bmatrix} S \\ \alpha \end{bmatrix} = -1 \tag{3.11}$$

Note that for α's overlapping with S , there is no restriction, since $(-)^{\alpha}$ acts like a chirality operator and cannot eliminate all components . We thus conclude that the necessary and sufficient conditions for having at least one massless gravitino is that $S \in \Xi$ and that (3.11) be satisfied. We may now prove a

Lemma : For any string model with a massless gravitino , the partition function and one-loop cosmological constant vanish .

To prove this denote for short by $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$ the contribution of a given spin-structure to the partition function , i.e. the integrand in eq. (2.6) . Then using the fact that Ξ is a group , we may write the full partition function as :

$$\sum_{\alpha,\beta\in\Xi} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \sum_{\alpha,\beta\in\Xi} \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \alpha+S \\ \beta \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta+S \end{bmatrix} + \begin{bmatrix} \alpha+S \\ \beta+S \end{bmatrix} \right\} \tag{3.12}$$

Now unless x and y are disjoint, $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is proportional to $\Theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and vanishes. The coefficients of the non-vanishing spin-structures within the curly brackets on the other hand, can be related by virtue of (3.11) and the duality and factorization conditions (2.7b,c). The result can be shown to be proportional to the Jacobi identity:

$$\Theta^4\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \Theta^4\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \Theta^4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \tag{3.13}$$

which completes the proof.

This is an example of a non-renormalization theorem [29]. Unless the theory contains anomalous U(1)'s [30], the stability of supersymmetric vacua and the vanishing of the cosmological constant presumably hold to all orders in the string-loop expansion.

SYMMETRY BREAKING

The previous section hopefully illustrated how one can construct 4d string models , which could be reasonable first approximations to our real world if we were to ignore all masses that are much much smaller than M_{Planck} . Though small, the masses of real particles are however finite, and are presumably related to the spontaneous breaking of gauge symmetries and, if it exists at all , of space-time supersymmetry. To make further progress we must therefore understand spontaneous symmetry breaking in string theory.

Gauge Symmetries

In what concerns gauge symmetries , things look good : indeed the scalar potential in most 4d string models has lots of flat directions along which the scalar vacuum expectation values can slide freely , breaking the gauge groups spontaneously at classically undetermined scales [5,31-33] . This is reminisent of no-scale models [34]. Although we do not fully understand how some of these scales will be fixed dynamically to be hierarchically smaller than M_P , the possibility that such a thing happens at least exists .

In order to be more explicit , let us consider for example the potential of the massless scalars (3.5d) in the $SU(2)^6 \times O(44)$ non-supersymmetric model of the previous section. One way of calculating this is to perturb the two- dimensional free fermionic action with the corresponding appropriately supersymmetrized scalar vertex operators at zero-momentum :

$$\begin{aligned} \mathcal{L}_{int}^{(2d)} &= \Phi^{a,AB} \int dz d\bar{z} d\theta (\chi^a + \theta f^{abc} \chi^b \chi^c) \eta^A \eta^B \\ &= \Phi^{a,AB} f^{abc} \int dz d\bar{z} (\chi^b \chi^c)(\eta^A \eta^B) \end{aligned} \tag{4.1}$$

and then calculate the β-functions [35] of the resulting generalized Thirring model. The classical string equations are :

$$\beta_{\Phi^{a,AB}} = \frac{\partial V}{\partial \Phi^{a,AB}} = 0 \tag{4.2}$$

These determine the scalar potential modulo field redefinitions , which reflect the dependence of the β -functions on the precise subtraction procedure. The two leading orders are however universal and yield [32]:

$$V(\Phi) = \frac{1}{3\sqrt{\alpha'}}\epsilon_{ijk}\sum_{I=1}^{6} Tr(\Phi^{I,i}\Phi^{I,j}\Phi^{I,k}) + \frac{1}{4}\sum_{I,J,i,j} Tr[\Phi^{I,i},\Phi^{J,j}]^2$$
$$-\frac{1}{8}\sum_{I,i,j}(Tr\Phi^{I,i}\Phi^{I,j})^2 + \frac{1}{8}\sum_{I}(\sum_{i} Tr\Phi^{I,i}\Phi^{I,i})^2 + o(\Phi^5) \tag{4.3}$$

where the traces and commutator are with respect to the suppressed SO(44) indices .

It is straightforward to check that the above potential vanishes if we give arbitrary vacuum expectation values to all $\Phi^{(I,i),AB}$ with (I,i) and (AB) chosen among some 6 and 22 , respectively , mutually commuting generators of $SU(2)^6$ and $SO(44)$. That this is true even if higher-order terms are taken into account in (4.3) , follows from the fact that the Thirring model with mutually commuting left-currents coupled to mutually commuting right- currents has exact conformal invariance . In the statistical mechanics language , these flat directions correspond to integrable marginal operators that deform continuously the spectrum of conformal weights , i.e. masses of the string states .

The scalar potential (4.3) is in some sense universal for the entire class of Gaussian string models ; the reason is that the massless scalars coming from the purely antiperiodic sector $\mathcal{H}_0$ are a subset of the $\Phi^{a,AB}$, and their potential can be obtained by appropriately truncating (4.3) . Thus for example our discussion of the flat directions can be taken over to the N= 4 , 2 and 1 supersymmetric models constructed in the previous section . Of course massless scalars may also exist in other sectors of the Hilbert space , and may give extra flat directions at appropriate multicritical points . I will refrain , however , from further discussing these points and turn now to the more crucial problem of supersymmetry breaking .

Supersymmetry

One of the main motivations for introducing supersymmetry is that it can solve the technical part of the gauge hierarchy problem , i.e. explain why radiative corrections do not drive the weak scale M_W to M_P . Of course , a completely different mechanism could be responsible for this in string theory . Otherwise following the conventional lore , we should expect space-time supersymmetry to be broken at a scale of order M_W , and hence much smaller than M_P . Furthermore this breaking is either non-perturbative [36], or else it must occur at tree level , since the non-renormalization theorems exclude the possibility that it be induced by radiative corrections*. Setting the non-perturbative effects aside, the following question arises : is it possible , as in field theory , to slightly perturb the tree-level spectrum of a 4d supersymmetric string model by introducing infinitesimal (in units of M_P) mass splittings between superpartners ? I will now show that , at least in the class of free-fermionic models considered here , this is not possible [37]: the existence of a gravitino of small mass necessarily implies the existence of a *whole tower of such states* between 0 and M_P with mass differences of the order of the supersymmetry breaking scale . Thus supersymmetry can only be restored in some sort of decompactification limit .

To prove the assertion let us start by considering the mass of a space-time fermionic string excitation . This must belong to a sector $\mathcal{H}_\alpha$ where the boundary condition α leaves the ψ^μ and the supercharge T_F periodic . Thus from (3.2 a) we have :

$$M^2 = \sum_{leftmovers} (frequencies) - \frac{1}{2} + \frac{1}{8} + \frac{\alpha^G \cdot \alpha^G}{8} \qquad (4.4)$$

where $\frac{1}{8}$ is the contribution to the mass subtraction of the periodic ψ^μ , and α^G is the phase-vector of some automorphism $\mathcal{A}^G$, as explained in section 2 .

To save time I will restrict myself here to inner automorphisms , i.e. group elements in the adjoint representation , although all my conclusions will hold

* The breaking through a one-loop induced Fayet-Iliopoulos D-term [30] does not seem phenomenologically relevant.

also for outer automorphisms . The general inner automorphism is of the form $\mathcal{A}^G = e^{i\pi\vec{\theta}\cdot\vec{H}}$ with H^l a set of mutually commuting generators in a Cartan-Weyl basis. We thus obtain :

$$\alpha^G \cdot \alpha^G = \frac{1}{2} r_G + \sum_{+ve} (\vec{\theta}\cdot\vec{\rho} - 1)^2 \tag{4.5}$$

where r_G is the rank of the group and the sum runs over all roots $\vec{\rho}$, positive with respect to $\vec{\theta}$. Now let us define , $\vec{g} = \frac{2}{c_G} \sum\limits_{+ve} \vec{\rho}$, where c_G is the Casimir in the adjoint representation. Using the Freudenthal-de Vries strange formula :

$$c_G \vec{g}\cdot\vec{g} = \frac{2}{3} d_G \tag{4.6}$$

and the fact that $\sum\limits_{+ve} \rho^i \rho^j = \frac{c_G}{2} \delta^{ij}$, we can rewrite eq. (4.5) as follows :

$$\alpha^G \cdot \alpha^G = \frac{1}{6} d_G + \frac{1}{2} c_G (\vec{\theta} - \vec{g})^2 \tag{4.7}$$

The minimum vector length is therefore $\frac{1}{6} d_G$ and is obtained for the special automorphism :*

$$\mathcal{A}_0^G = e^{i\pi\vec{g}\cdot\vec{H}} \tag{4.8}$$

which we may refer to as a superautomorphism. Plugging now eq. (4.7) back in eq. (4.4) , with $d_G = 18$, we conclude that the holomorphic part of a massless space-time spinor is necessarily the vacuum of a sector in which the fermionic boundary conditions are given by the superautomorphism (4.8). Note incidentally that we have here also proved that there are no tachyonic spinor excitations as is , of course, to be expected of a consistent string theory .

Consider now a massless gravitino . It follows easily from the above discussion that the only candidate for such a state is $\partial X^\mu(1)|0>_S$ where :

* I owe this elegant argument to Peter Goddard. A brute force but more general classification of minima, that includes outer automorphisms, was given in ref. [12].

$$S = (\underbrace{1}_{\psi^\mu} \ \underbrace{\alpha_0^G}_{\chi^a} \ \underbrace{\alpha_R = 0}_{\eta^A}) \tag{4.9}$$

In order to give a small mass to the gravitino, we must perturb slightly this boundary condition :

$$S + \delta S = (\ 1 \ \ \alpha_0^G + \delta\alpha^G \ \ \delta S_R \ ) \tag{4.10}$$

where $\delta S_i << 1$. The state

$$\partial X^\mu(1)|0>_{S+\delta S} \tag{4.11}$$

has mass

$$M^2 = \frac{1}{8}(\delta\alpha^G)^2 = \frac{1}{8}(\delta S_R)^2 \tag{4.12}$$

where the absence of a linear term is due to the fact that the vector α_0^G has minimum length , so that $\alpha_0^G \cdot \delta\alpha^G = 0$. (4.11) is actually the only candidate for a slightly massive gravitino, because at $S + \delta S$ there are no nearly periodic fermions with infinitesimal frequencies at our disposal .

Let us assume then that the Hilbert space of our model contains such a nearly massless gravitino. This means that $(S + \delta S) \in \Xi$ and, in order that (4.11) survives the GSO projections :

$$C\begin{bmatrix} S+\delta S \\ \beta \end{bmatrix} = \begin{cases} -1 & \text{if } \delta_\beta = +1 \\ \pm 1 & \text{if } \delta_\beta = -1 \end{cases} \tag{4.13}$$

This condition is due to the fact that $e^{i\pi\beta\cdot F}$ acts on (4.11) as a chirality operator if $\delta_\beta = -1$ and as the identity otherwise .

Now we will use the fact that the superautomorphism (4.8) is always an h-th root of the identity with h a small integer (for simply-laced groups h is the dual

Coxeter number). This means that $2hS = 0$, which together with the fact that Ξ is a group , leads to an entire tower of candidate low-mass gravitinos:

$$\partial X^{\mu}(1)|0>_{S+\delta S} \ ; \ \partial X^{\mu}(1)|0>_{S+(2h+1)\delta S} \ ; \ \partial X^{\mu}(1)|0>_{S+(4h+1)\delta S} \ \cdots$$

all of which satisfy level-matching due to the absence of a linear term in eq. (4.12) . Furthermore they all survive the generalized GSO projections : this can be proved by using once more the duality and factorization conditions (2.7 b,c) , to reexpress $C\left[{(2nh+1)(S+\delta S) \atop \beta}\right]$ in terms of $C\left[{S+\delta S \atop \beta}\right]$. The fact that S minimizes vector lenght is again crucial, for getting rid of the phases.

Thus, what we have concluded is that the existence of a gravitino with, for instance, a mass of $1Tev$, implies the existence of an entire tower of gravitinos with masses $(2h+1)Tev$, $(4h+1)Tev$ and so on (incidentally $h = 2, 3$ or 4 for the groups that interest us) . The physical interpretation is that supersymmetry is broken by a Sherck-Schwarz type compactification [38,31], but with the momenta in the internal dimensions related to the radii , so that the mass splittings can be made to vanish *only in the limit of decompactification* . Using different arguments, Dine and Seiberg [39], and Banks and Dixon [40] have also concluded that supersymmetry cannot be restored continuously at an analytic point of the scalar potential. As opposed to the proof given here, their arguments are not restricted to a particular class of models. However, they do not allow them to characterize the singularity uniquely as being due to decompactification. Finally I should point out that I have here taken the gravitino mass as a measure of the supersymmetry breaking scale. Thus my arguments would not apply if the mass splittings in the matter and gauge sectors were small but the gravitino mass was of order M_P.

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