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SIZE AND SHAPE OF STRINGS*

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ABSTRACT

We study numerically and analytically spatial properties of the ground state of a fundamental string in the light-cone gauge. We find that strings are smooth and have divergent average size. Their properties are very different from what is expected from particles in a conventional field theory.

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1. Introduction

String theory was originally invented to describe hadrons[1]. Ultimately this idealized mathematical theory of hadrons failed, due in part to the inability to couple strings to the external local fields, such as the electromagnetic field. The reason for this failure is the infinity of normal mode zero point fluctuations spreading the string over all space[2]. In this paper we will examine in detail the spatial properties of fundamental strings. We will also speculate on how they compare with the strings of large-N_{color} gauge theory[3]. We will be particularly interested in the following characteristics of the ground state of the fundamental string:

1. The average size of the spatial region occupied by the string;
2. The average length of the string;
3. Is the string smooth on small scales or does it exhibit rough or fractal-like behaviour?
4. How densely is space filled with string?

In order to answer these questions and to provide some intuition we have constructed a numerical method for generating 'snapshots' of the ground state of the string. For that purpose we use the exact wave function of free string in the light-cone gauge to generate a statistical ensemble of strings. In fact we find that the overwhelming majority of the ensemble have similar qualitative features. The 'snapshots' are shown in section 2 and their important features are discussed. In particular we find that both the average length of string and the average size of the region occupied by the string are infinite. The relationship between the two divergences is such that string actually packs the space densely. We also find that the string is microscopically very smooth, with no tendency to form fractal structure on small scales. The quantitative meaning of the above statements will be provided later.

In section 3 some analytic derivations are given to substantiate the qualitative findings of section 2. In section 4 we explain the physical meaning and measurability of the divergence in the size of the string. We also speculate on the possible qualitative differences between the fundamental strings and the strings of large-N_{color} QCD.

2. Snapshots of Strings

In the light-cone gauge, the transverse coordinates of the string are free fields with mode expansions

$$X^i(\sigma) = X^i_{cm} + \sum_{n>0} \left(X^i_n \cos(n\sigma) + \bar{X}^i_n \sin(n\sigma) \right) \tag{2.1}$$

The wave function for each transverse coordinate in the ground state of the string has the product form (dropping the superscript i)

$$\Psi(X(\sigma)) = \prod_n \left((\frac{\omega_n}{2\pi})^{1/2} \exp\left(-\omega_n (X_n^2 + \bar{X}_n^2)/4\right) \right) \tag{2.2}$$

with $\omega_n = n$. Squaring this gives a probability distribution for the transverse position of the string. To carry this out in practice it is necessary to truncate the mode expansion at some maximum wave number N. This is one of the ways of introducing a cut-off in the parameter space of the string. Passage to the continuum limit is achieved as $N \to \infty$.

Another cut-off procedure can be defined where string is replaced by $2N+1$ discrete mass points connected by identical springs. The normal modes are such that the positions of the mass points are given by eq.(2.1) evaluated at discrete values of the parameter $\sigma = 2\pi m/(2N+1)$, where m labels the mass points. Then the string wave function is eq. (2.2) with frequencies

$$\omega_n = \frac{2N+1}{\pi} \sin\left(\frac{\pi n}{2N+1}\right), \tag{2.3}$$

where $n = 1, \ldots, N$.

With both cut-off prescriptions a string configuration is determined by a sequence of values of X_n^i and $\bar{X}_n^i$, with $n = 1, \ldots, N$ and $i = 1, \ldots, D-2$, sampled with probability

$$P(X_n^i) = (\frac{\omega_n}{2\pi})^{1/2} \exp(-\omega_n (X_n^i)^2/2) \tag{2.4}$$

and similarly for $\bar{X}_n^i$.

For the first cut-off procedure each such configuration defines a parametrized curve in $(D-2)$-dimensional space. By necessity we show projection of string onto 2 transverse dimensions. In practice each run consists of choosing $100 \times (D-2)$ random numbers from their respective probability distributions. For each run we compute curves with $N = 10, 20, 30, 40, 50$. For convenience, let us adopt the following method: as we proceed, for example, from $N = 10$ to $N = 20$, the coefficients of the normal modes with the first 10 wave numbers are kept the same as for the $N = 10$ 'snapshot'. Similarly, we proceed from $N = 20$ to 30 to 40 to 50, always retaining the previous set of coefficients. Therefore, for each run, increasing N corresponds to observing the same string with improved resolution. The 'snapshots' generated by 2 such runs are shown in figure 1 and figure 2. For one of the runs we also show graphs of total transverse length (figure 3) and average transverse line curvature (figure 4) as a function of N. In figure 5 we show curvature as a function of length along the string for cases with $N = 10, 20$.

We find the following qualitative features:

1. Slow growth of the occupied region with N. In the next section we will show that the growth is actually logarithmic.

2. The plots of total string length vs. N appear to be linear. In the next section we will show an analytic derivation of this effect.

3. The transverse curvature averaged over the string appears to be approximately independent of the cut-off. In the next section we will show analytically that the expectation value of curvature is completely cut-off independent.

4. The growth in length with increasing N is achieved by repetition of similar smooth structures. In fact, a piece of string of given length at $N = 20$ looks similar to a piece of the same length at $N = 10$, (*cf.* Fig. 5).

5. The slow growth of the occupied volume together with the linear growth of length means that there is a strong tendency for the string to pass through the same small region many times. It is obvious that, as the cut-off is removed, the string fills space densely: there is a point on the string arbitrarily close to any point in space.

In order to elaborate on point (4) and show that no 'accidents' occur as we proceed to high values of the cut-off we have plotted the section of the string confined between $\sigma = 0$ and $4\pi/N$ for $N = 20$ and $N = 500$ (figure 6). The remarkable similarity between the two can be qualitatively regarded as a statement of conformal invariance in our approach.

All the above features, except for (3), can also be observed with the discrete regularization of the string. In figure 7 we show a typical picture at $N = 50$. It is important to note that, as more and more discrete points crowd the σ-axis, the string never becomes continuous in space. In section 3 we show that, as $N \to \infty$, the average distance in space between each pair of neighboring points approaches a constant. This, of course, is responsible for the linear growth of the total length. Thus, all the important information about the spatial properties of string can be obtained in the regularization where the string never becomes continuous in space. This fact will be essential for treatment of strings in discretized space.

3. Analytical Results

In this section we give analytic derivations of some qualitative conclusions reached in the previous section. Let us begin with the growth of the volume occupied by the string with the cut-off N. Define r to be the *rms* distance of a point on the string to its center of mass:

$$r^2 = \left\langle (\vec{X}(\sigma) - \vec{X}_{cm})^2 \right\rangle \tag{3.1}$$

Since there is no preferred point on the closed string, we can arbitrarily set $\sigma = 0$.

$$r^2 = (D-2)\left\langle \left(\sum_{n=1}^{N} X_n\right)^2 \right\rangle = (D-2)\sum_{n=1}^{N} \langle X_n^2 \rangle = (D-2)\sum_{n=1}^{N} \frac{1}{n} \tag{3.2}$$

It follows that the *rms* volume of the transverse region occupied by string $\sim (\log N)^{\frac{D-2}{2}}$.

To find the dependence of average length on the cut-off, we start with

$$\langle L \rangle = \int_0^{2\pi} \langle v \rangle \, d\sigma, \tag{3.3}$$

where

$$v = \sqrt{\left(\frac{dX^i}{d\sigma}\right)^2} \tag{3.4}$$

By translation invariance in σ

$$\langle L \rangle = 2\pi \, \langle v(\sigma = 0) \rangle \tag{3.5}$$

For each transverse direction

$$\frac{dX^i}{d\sigma}(\sigma = 0) = \sum_{n=1}^{N} n \bar{X}_n^i \tag{3.6}$$

Using the fact that each $\bar{X}_n^i$ is gaussian distributed, it is easy to show that

$dX^i/d\sigma(\sigma = 0)$ is gaussian distributed with variance

$$\Sigma^2 = \sum_1^N n = \frac{N(N+1)}{2}. \tag{3.7}$$

Therefore, $\vec{v} = d\vec{X}/d\sigma$ is distributed according to

$$P(\vec{v}) \sim \exp\left(-\frac{\vec{v}^2}{2\Sigma^2}\right) d^{D-2}\vec{v}. \tag{3.8}$$

As a result, the distribution for the length of $\vec{v}$ is

$$P(v) \sim \exp\left(-\frac{v^2}{2\Sigma^2}\right) v^{D-3}dv \tag{3.9}$$

It follows that

$$\langle v \rangle = \frac{\int_0^\infty \exp(-v^2/2\Sigma^2)v^{D-2}dv}{\int_0^\infty \exp(-v^2/2\Sigma^2)v^{D-3}dv} \sim \Sigma \sim N \tag{3.10}$$

The slope of the linear growth determined by above expression depends on dimensionality. For example, if the number of transverse dimensions is an even number $(D-2=2k)$ then (3.10) yields

$$\langle L \rangle = \frac{(2k-1)!\pi^{3/2}}{4^{k-1}\big((k-1)!\big)^2}\big(N + 1/2 + O(1/N)\big) \tag{3.11}$$

In particular, in $D = 26$ we find

$$\langle L \rangle \approx 21.54\ (N + 1/2) \tag{3.12}$$

As shown in figure 3, the data for any given run agrees well with this linear dependence. This shows that the standard deviation is small compared with the average length.

It is also interesting to study eqn. (3.11) in the limit of a large number of dimensions. Using the Stirling formula for the factorial we find that the slope of growth of transverse length with N is given by

$$\langle L \rangle / N \approx \pi\sqrt{2D-4} + O(1/\sqrt{D-2}) \tag{3.13}$$

as D becomes large. In $D = 26$ this predicts the slope of 21.76, which is very close to the exact number (3.12).

Similar analytic results can be derived in the regularization where string is replaced by a collection of mass points connected by springs. For example, the length is

$$L_d = \sum |\vec{X}_{(m+1)} - \vec{X}_{(m)}| \tag{3.14}$$

where the subscript labels the mass points. Using translation invariance,

$$\langle L_d \rangle = (2N+1) \left\langle |\vec{X}_{(2)} - \vec{X}_{(1)}| \right\rangle \tag{3.15}$$

After a few steps analogous to the ones for the continuous regularization we find

$$\langle L_d \rangle = \frac{(2k-1)!(8\pi)^{1/2}}{4^{k-1}\left((k-1)!\right)^2} (N + 1/2 + O(1/N)) \tag{3.16}$$

Note that, with the definition of length (3.14), the slope of linear growth in the discrete regularization differs slightly from (3.11) found in the continuous regularization. However, the linearity of growth and other properties important for our physical conclusions are unaffected.

Extrinsic curvature can only be investigated in the regularization where the string is kept continuous. It is conveniently expressed as

$$\kappa = \frac{|\vec{a}_\perp|}{v^2} \tag{3.17}$$

where $\vec{a}_\perp$ is the component of $\vec{a} = d^2\vec{X}/d\sigma^2$ normal to $\vec{v} = d\vec{X}/d\sigma$. Since

$$\frac{d^2X^i}{d\sigma^2}(\sigma = 0) = -\sum_{n=1}^{N} n^2 X^i_n, \tag{3.18}$$

eq. (3.6) implies that $\vec{a}$ and $\vec{v}$ are uncorrelated. Therefore,

$$\langle\kappa\rangle = \langle|\vec{a}_\perp|\rangle \left\langle\frac{1}{v^2}\right\rangle \tag{3.19}$$

From the distribution (3.9) we find

$$\left\langle\frac{1}{v^2}\right\rangle = \frac{1}{2(k-1)\Sigma^2} \tag{3.20}$$

It is important to note that this diverges in $D = 4$. Since $\vec{a}$ and $\vec{v}$ are not correlated, $\langle|\vec{a}_\perp|\rangle$ is effectively the average length of the vector (3.18) in $D-3$ dimensions. Denoting $|\vec{a}_\perp| = a$ we find the probability distribution

$$P(a) \sim \exp(-\frac{a^2}{2\tilde{\Sigma}^2})a^{D-4}da \tag{3.21}$$

with

$$\tilde{\Sigma}^2 = \sum_1^N n^3 = (\frac{1}{2}N(N+1))^2 = \Sigma^4 \tag{3.22}$$

It follows that $\langle\kappa\rangle \sim \tilde{\Sigma}/\Sigma^2$ is independent of the cut-off! After a short calculation we find

$$\langle\kappa\rangle = \frac{((k-2)!)^2 2^{2k-3}}{(2k-3)!\sqrt{2\pi}}, \tag{3.23}$$

which in $D = 26$ yields $\langle\kappa\rangle = 0.216$. This is in good agreement with figure 4 which shows the data for a sample run. In the limit of large dimensionality (3.23) reduces to

$$\kappa \approx \frac{1}{\sqrt{D-2}} \tag{3.24}$$

This confirms the intuitive expectation that increasing dimensionality makes the string smoother.

On the lower end of the range of dimensionalities the average curvature diverges in $D = 4$, (*cf.* Eq. (3.20)). We believe there is a simple intuitive reason for this which we proceed to explain. There are two kinds of singular points which can occur on a string: a kink, where the tangent vector $d\vec{X}/d\sigma$ is discontinuous, and a cusp, where it vanishes. In other words, at a cusp string turns back onto itself. From eq. (3.17) we see that at a kink the curvature has an integrable (δ-function) singularity, while at a cusp it has a non-integrable singularity. Our study of projections of strings onto 2 transverse dimensions indicate that cusps are fairly likely to occur there. We find that most of these cusps are projections of smooth configurations in higher dimensions. Therefore, we conjecture that the relatively high likelihood of cusps in $D = 4$ is responsible for the divergence in the average curvature.

4. Discussion

One may wonder whether any of the strange effects described in the previous sections are observable. In particular, the infinite *rms* radius seems very unphysical. However, it does lead to an observable effect. Consider scattering of a high-energy string from a string at rest. The interaction is mediated by string exchange. In the light-cone frame of the fast string of energy E the lifetime of the interaction is of order $\tau = 1/E$. Oscillations with frequency $> 1/\tau$ average to zero. Thus, we retain a number of modes $\sim E$. This introduces mode cut-off and gives an observable particle radius $\sim \sqrt{\log E}$. As the resolution is improved, the string 'expands'. This phenomenon leads to the well-known Regge behaviour of scattering cross-sections satisfied by the dual amplitudes.[2] Thus the effect is indeed observable and presents no obvious difficulty for scattering of strings by strings.

On the other hand, imagine that the string is being scattered by a local external field. This situation is analogous to the electromagnetic probing of hadrons. In this case the interaction is instantaneous and therefore the string must appear infinite. It is precisely for this reason that the fundamental strings cannot be consistently coupled to arbitrary external fields. That is why they are either a theory of everything or of nothing.

Let us speculate now on how the fundamental strings might differ from the large-N_{color} QCD strings. Our results indicate that the fundamental strings are smooth. This should be contrasted with the expected behaviour of QCD strings. In the limit $N_{color} \to \infty$ Migdal and Makeenko derived an exact lattice string equation[3]. If QCD had an ultraviolet fixed point then the string would be microscopically self-similar. QCD being asymptotically free is likely to make strings even rougher.

Another difference between the QCD and fundamental strings involves the spatial distribution of the longitudinal momentum p^+. For a hadron, this could be measured by interaction with external gravitational field. The result is a form factoı $F(q^2)$. For a fundamental string an analogous form factor can be obtained by observing that the distribution of p^+ is measured by the vertex operator $\partial_\alpha X^+ \partial^\alpha X^+ \exp(iq \cdot X)$, where α is the world sheet index. In the light-cone gauge $X^+ = \tau$ and the form factor reduces to

$$F(q^2) = \left\langle \int d\sigma \exp(iq \cdot X(\sigma)) \right\rangle \sim \exp(-q^2 \log N) \tag{4.1}$$

In the limit $N \to \infty$ the form factor is non-vanishing only at $q = 0$. It follows that p^+ is smeared uniformly all over space. This peculiar property applies not only to the ground state of the string but also to any finitely excited state. For any such state the change in the wave function relative to the ground state concerns only a finite number of normal modes and becomes negligible in the limit $N \to \infty$. The strange behaviour of the gravitational form factors is possibly connected with the existence of the graviton: at least for the massless spin-2 state it could be foreseen on the basis of general principles. A theorem by Weinberg and Witten [4] states that in a Lorentz invariant theory with a Lorentz invariant energy-momentum tensor the gravitational form factor of a massless spin-2 particle must satisfy $F(q^2 \neq 0) = 0$. It seems that string theory uses its infinite zero point fluctuations to allow the existence of gravitons.

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FIGURE CAPTIONS

1) (a)-(e) Projection of string onto 2 transverse dimensions with mode cut-off $N = 10,\ 20,\ 30,\ 40,\ 50$.
2) Another run, analogous to fig. 1.
3) Total transverse length *vs.* mode cut-off for one run in $D = 26$. Broken line shows the analytic result for the expectation value, Eq. (3.11).
4) Transverse extrinsic curvature averaged over the string *vs.* mode cut-off for one run in $D = 26$. Broken line shows the analytic result for the expectation value, Eq. (3.22).
5) Transverse extrinsic curvature as a function of length along the string for (a) $N = 10$ and (b) $N = 20$.
6) Section of string confined between $\sigma = 0$ and $4\pi/N$ for $N = 20$ (solid line) and $N = 500$ (dashed line).
7) A typical configuration of the ground state of 101 mass points connected by springs.

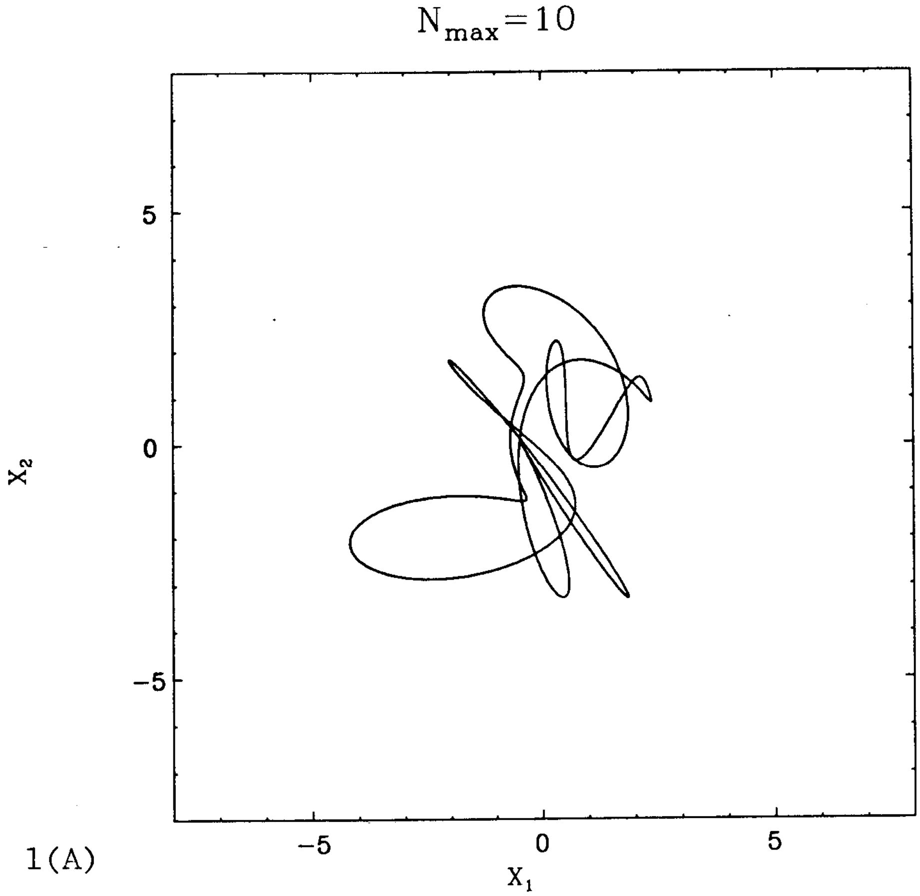


FIG. 1(A)

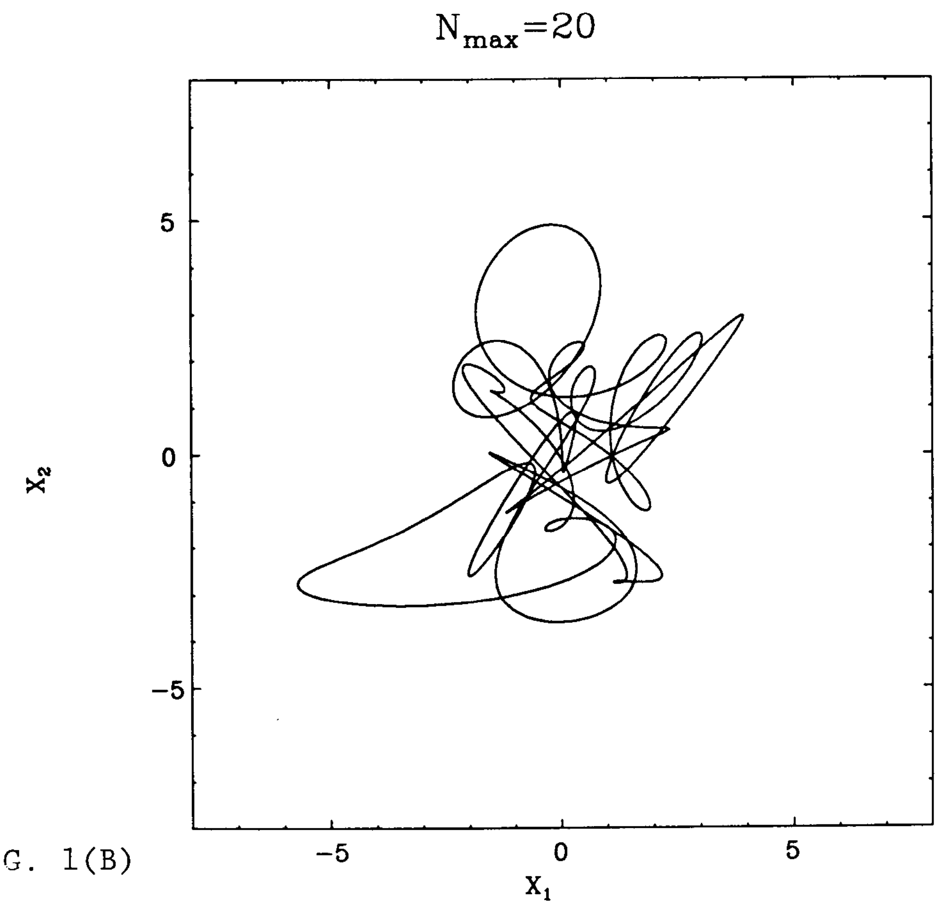


FIG. 1(B)

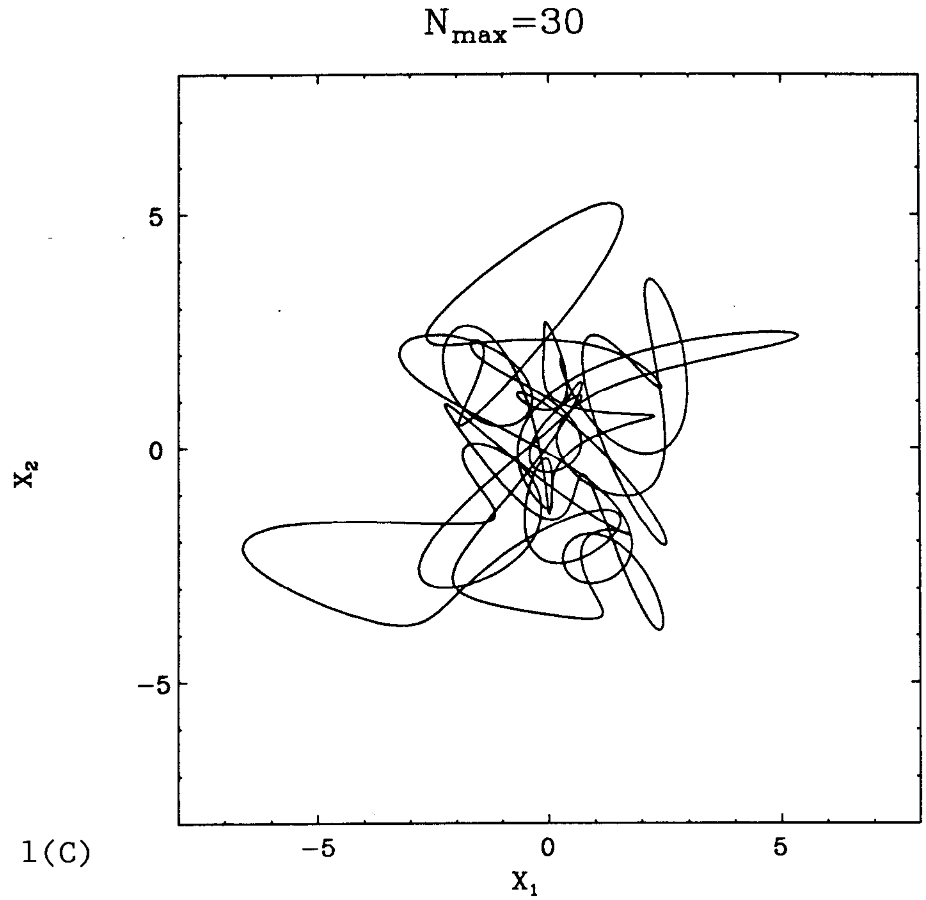


FIG. 1(C)

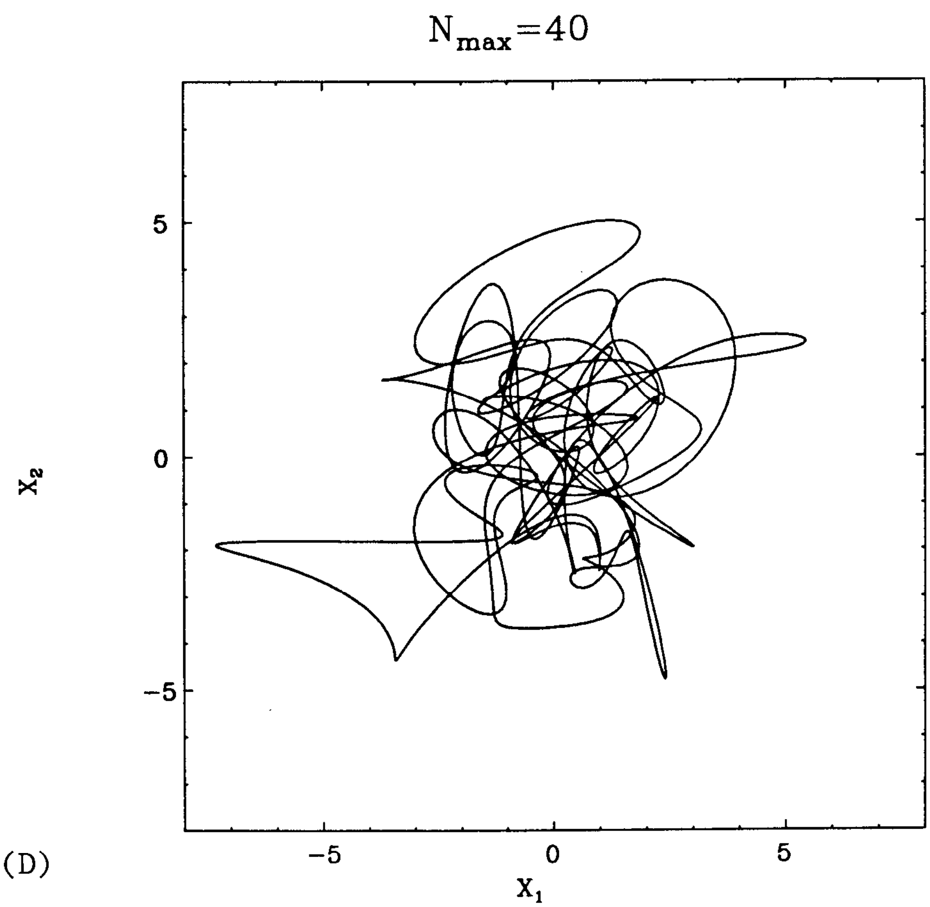


FIG. 1(D)

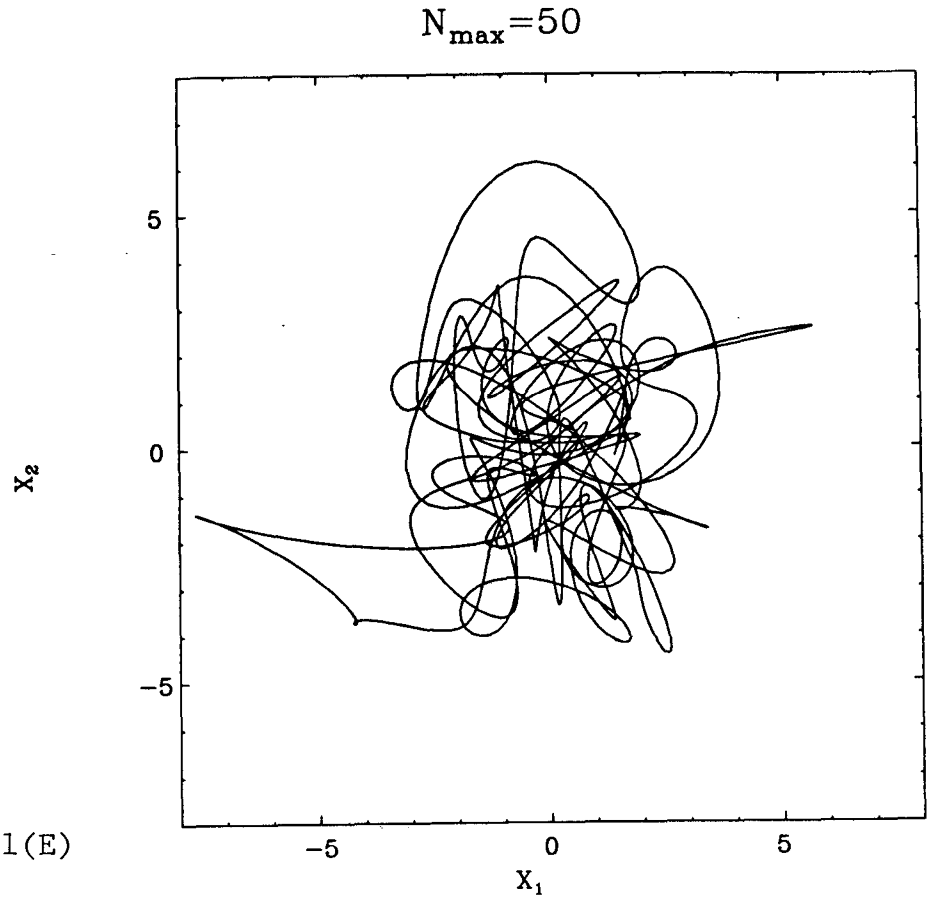


FIG. 1(E)

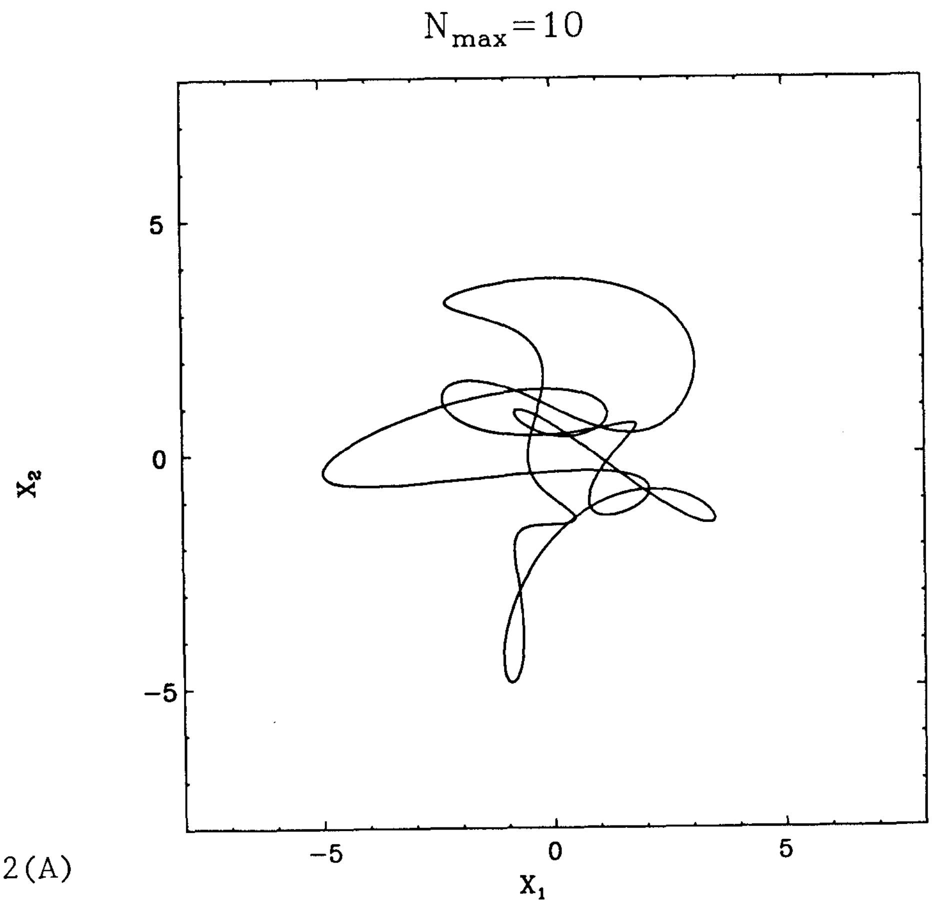


FIG. 2(A)

$N_{max}=20$

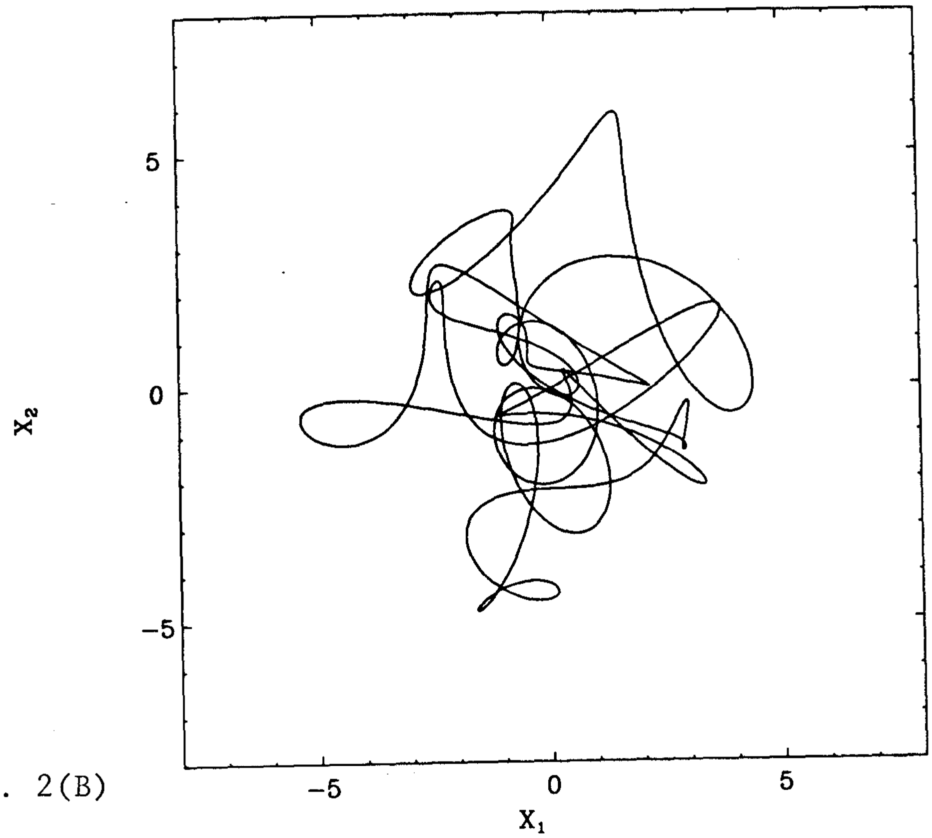


FIG. 2(B)

$N_{max}=30$

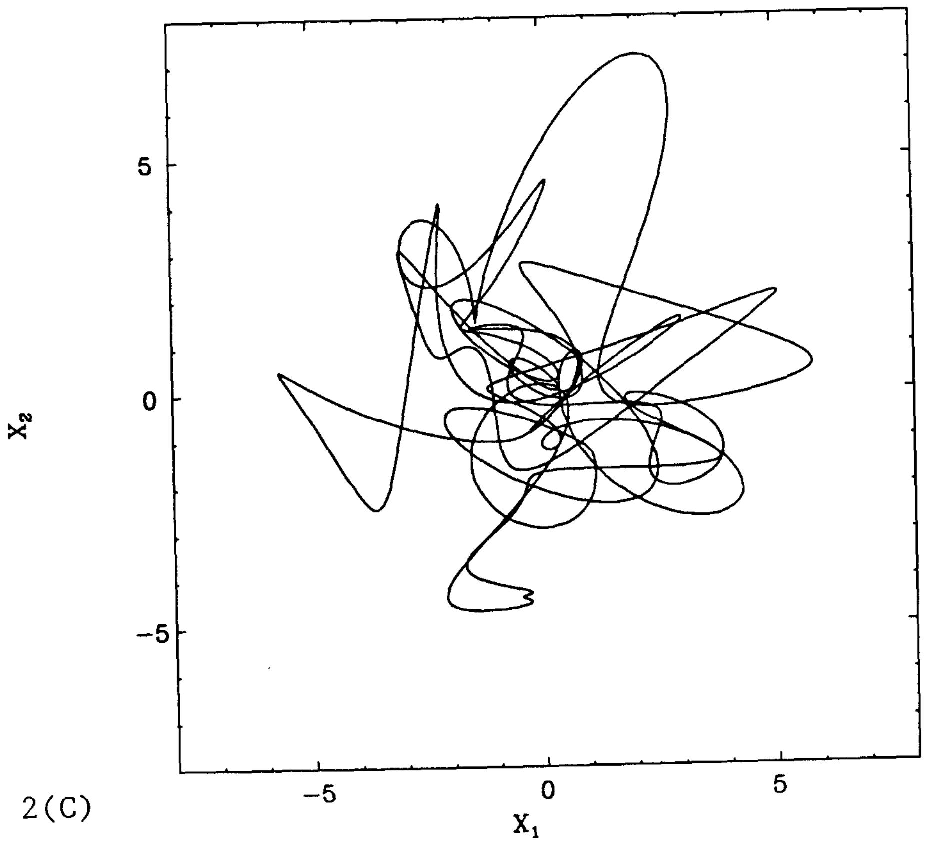


FIG. 2(C)

$N_{max}=40$

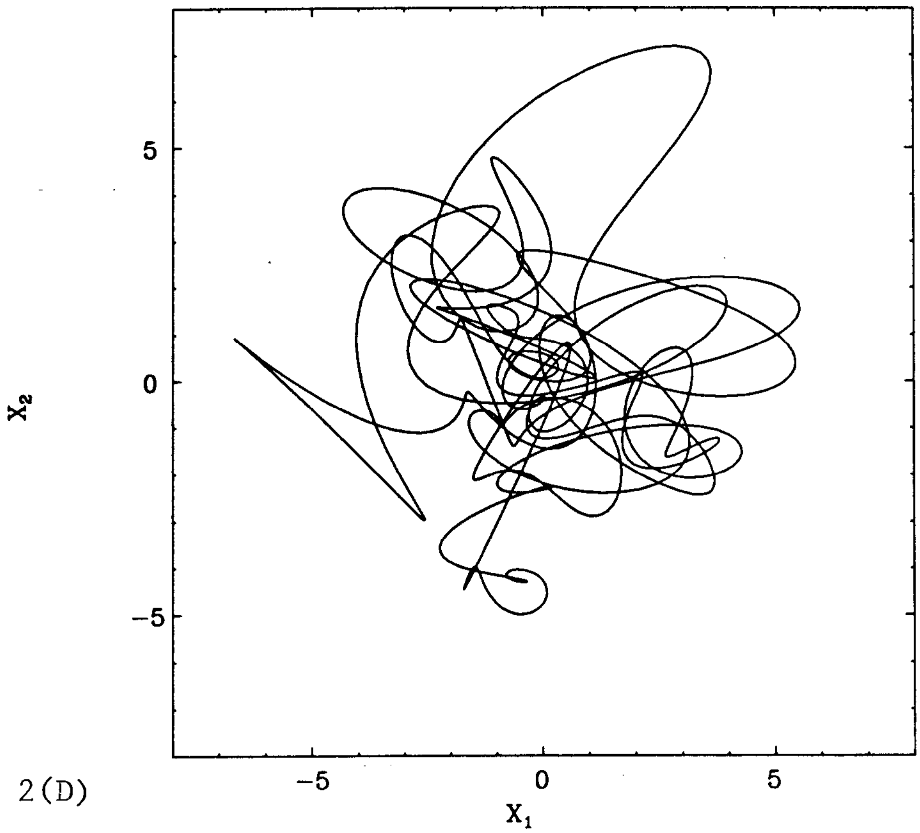


FIG. 2(D)

$N_{max}=50$

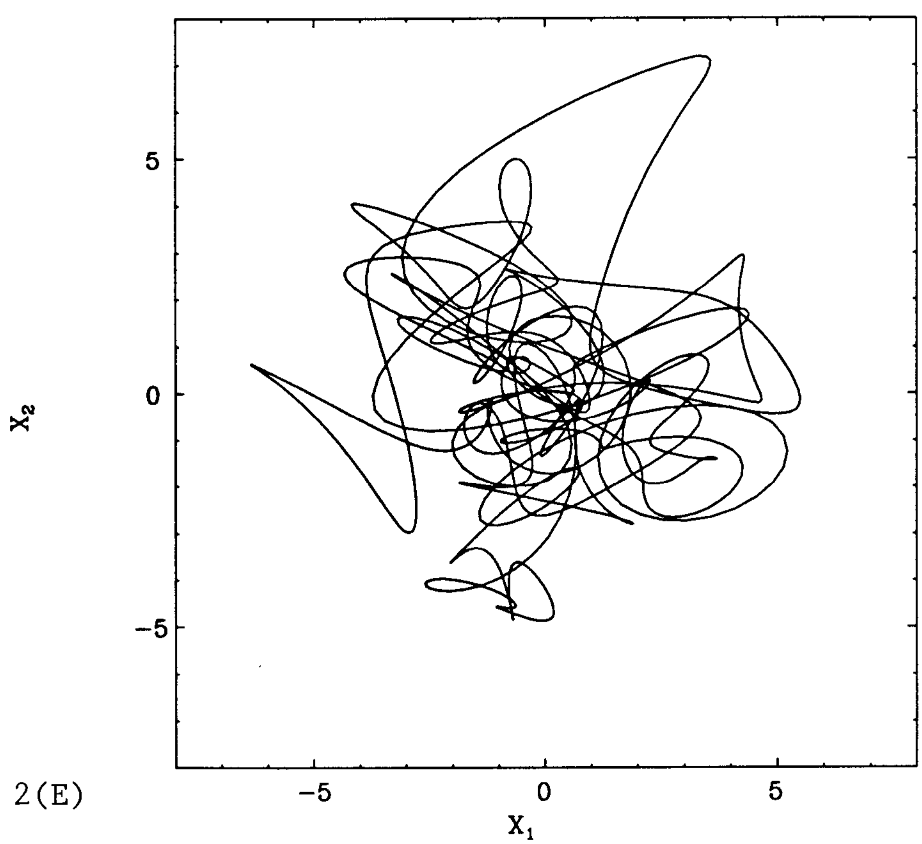


FIG. 2(E)

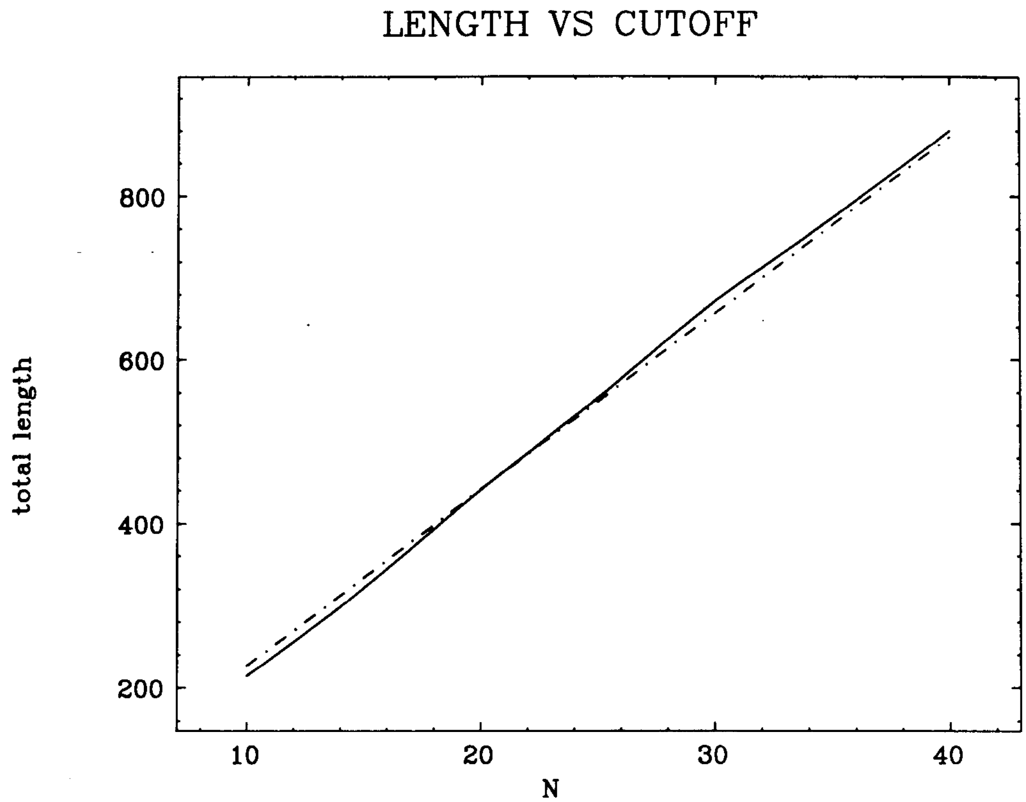


FIG. 3

FIG. 3

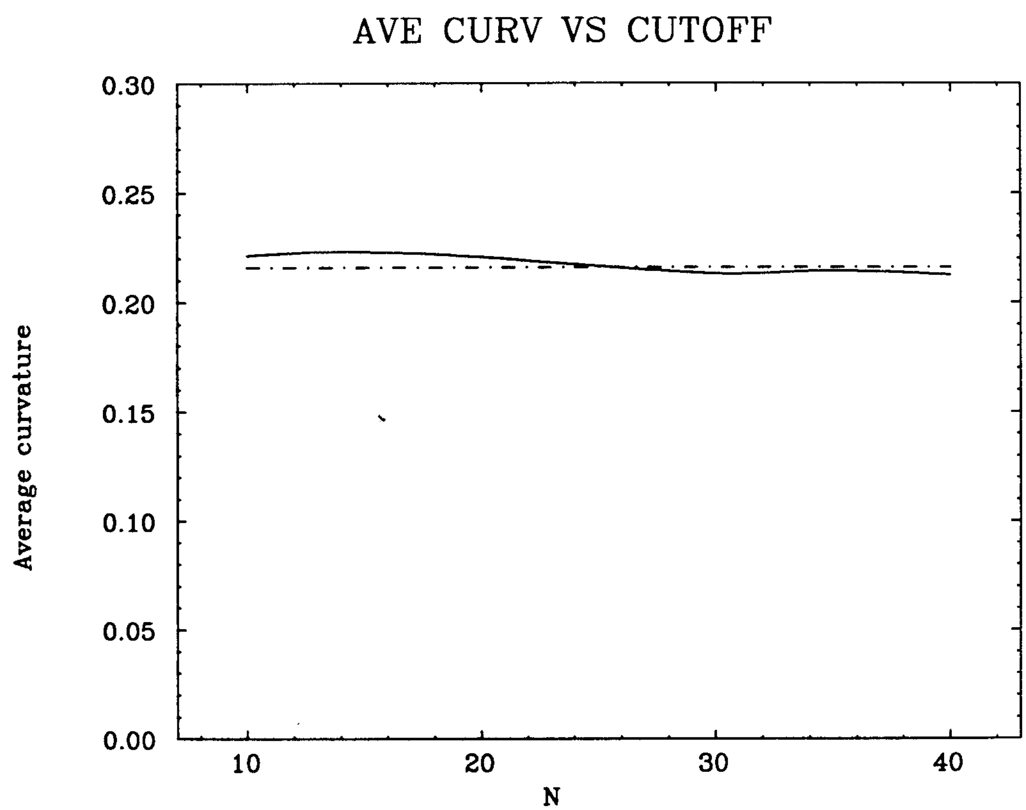


FIG. 4

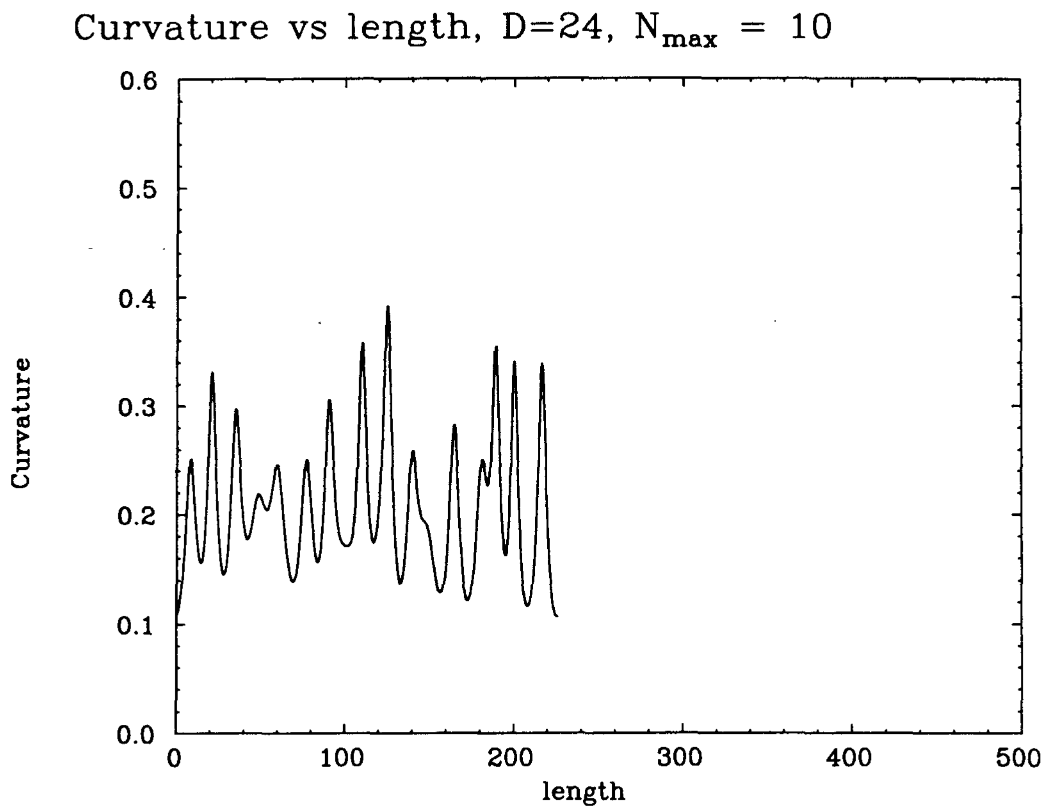
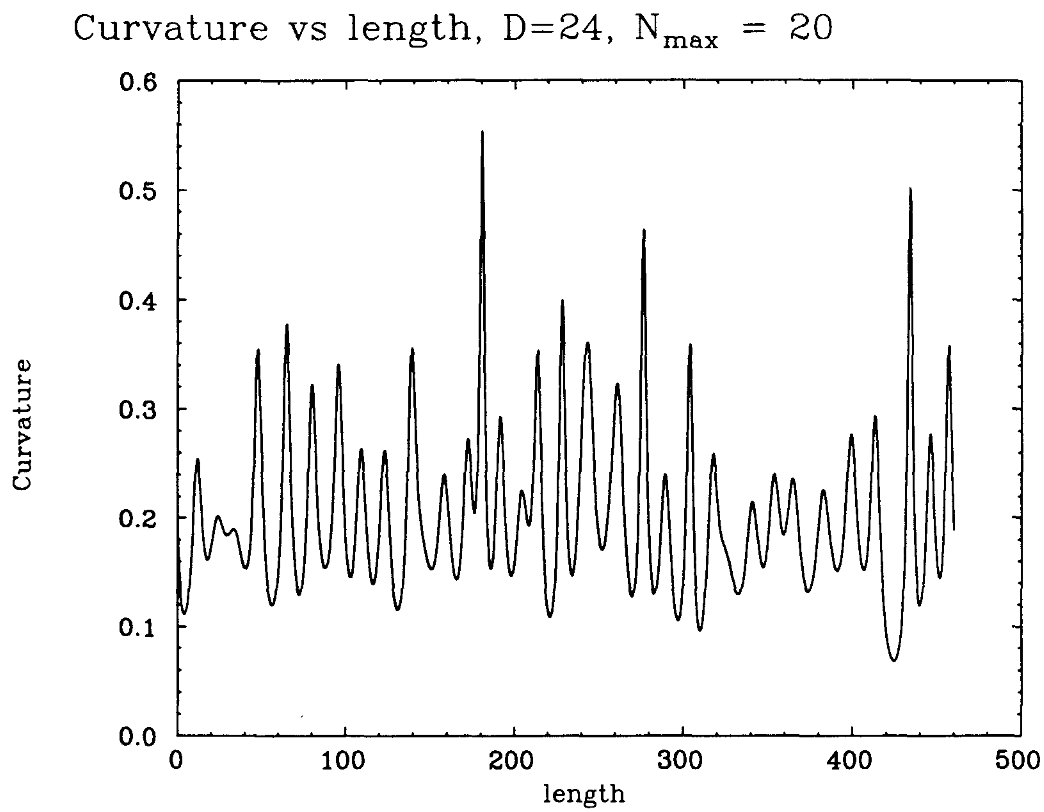

Curvature vs length, D=24, N_{max} = 10
Curvature
length
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
100
200
300
400
500

FIG. 5(A)


Curvature vs length, D=24, N_{max} = 20
Curvature
length
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500

FIG. 5(B)

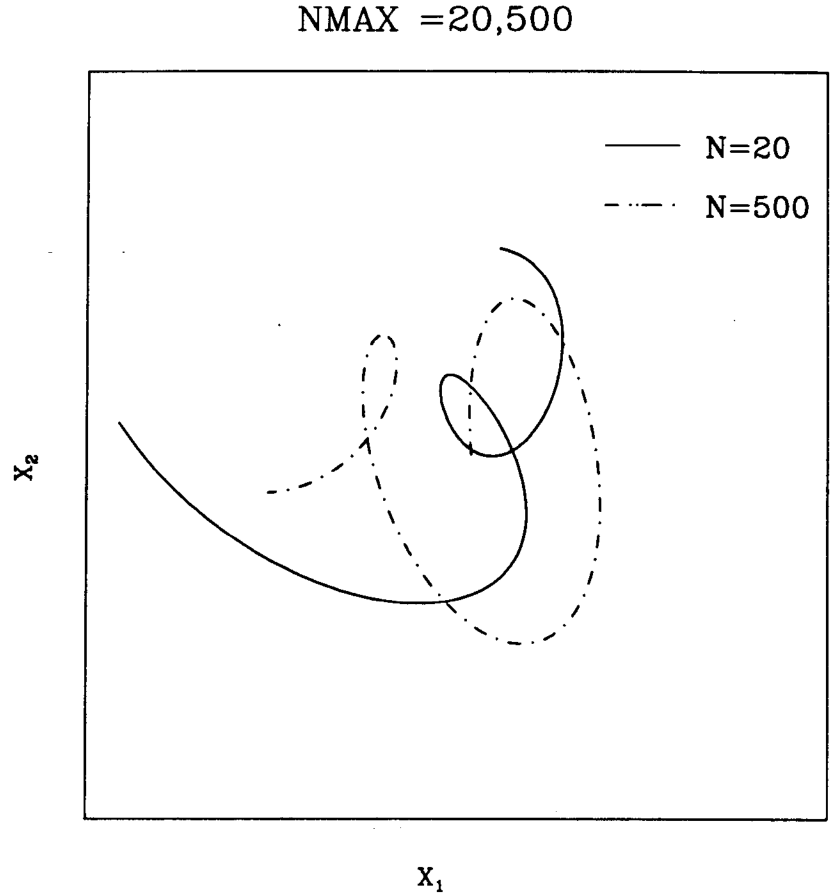

FIG. 6

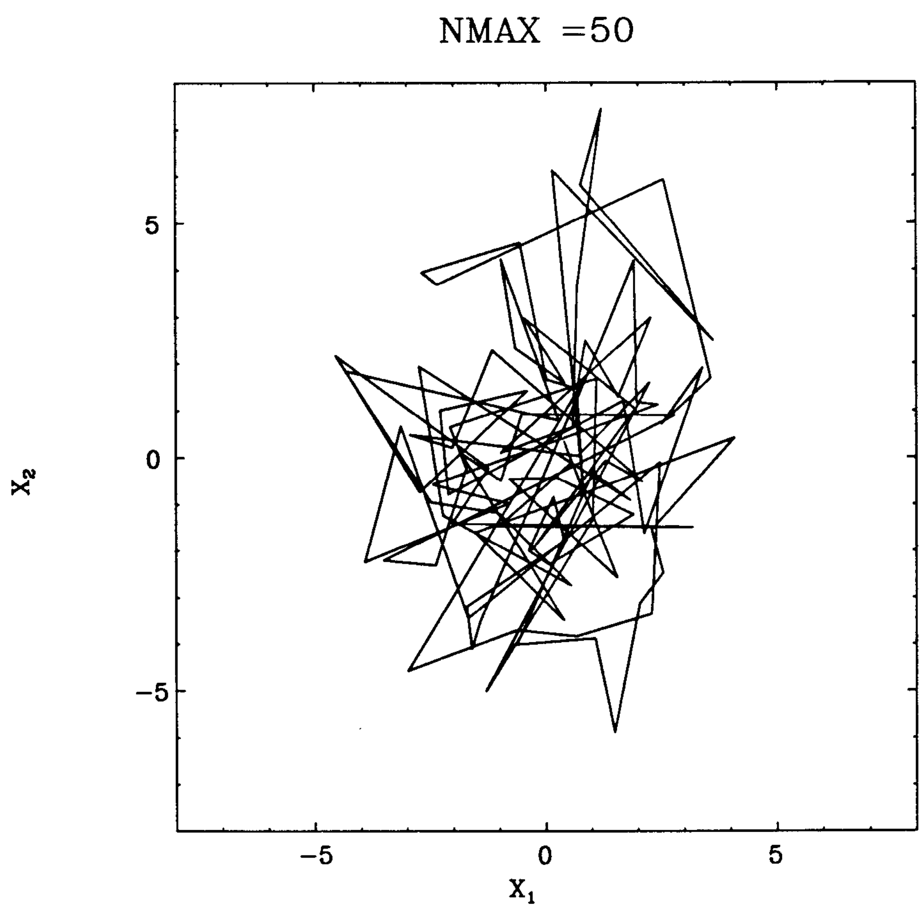

FIG. 7