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# CATOPTRIC TADPOLES*

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## ABSTRACT

Fermionic string perturbation theory is known to suffer from an ambiguity in the form of a total derivative in the moduli space. For a class of backgrounds (including $R^{10}$, orbifolds and theories with no $U(1)$ factors in gauge group) we show that these ambiguities for the partition function of heterotic string theory at any genus are proportional to massless physical tadpoles in the theory at lower genera and hence vanish in stable vacua. We also find that in $R^{10}$ the cosmological constant at a given genus is proportional to the cosmological constant at lower genera. This enables us to give an inductive argument for the vanishing of the cosmological constant in $R^{10}$ to all orders in string perturbation theory. We also address the ambiguity and finiteness of $n$-point functions. Our results indicate that in $R^{10}$ the ambiguity can be absorbed by a renormalization of the string coupling constant and the string tension. The expected sources of divergence in the $n$-point function in arbitrary tachyon-free backgrounds, besides the usual infrared divergences for $d \leq 4$, are shown to be proportional to tadpoles of physical massless fields. For type II strings in arbitrary backgrounds, we show by explicit calculation that the ambiguity vanishes at $g = 2$.

## 1. Introduction

If two mirrors are placed facing one another there is never one image, but an infinite set of receding images. In this paper we show that this effect, sometimes known as the catoptric effect, is an excellent metaphor for the cosmological constant of heterotic string theory.

In order to evaluate the cosmological constant or any string amplitude we must first make sure that it is well-defined. Recent investigation [1][2][3][4] has shown that fermionic string perturbation theory suffers from a total derivative ambiguity in the moduli space. More precisely, if we work with a choice of gauge slice for which the two dimensional metric is independent of the odd coordinates of the moduli space, and choose a delta function support for the two dimensional gravitinos, then, under a change of location of the support of the gravitino field, the measure changes by a total derivative in the moduli space[1]. This ambiguity was shown to have its origin in an intrinsic ambiguity in defining integration over the elements of the grassmann-valued coordinates of supermoduli space [2][3][4][5], and has nothing to do with any particular way one is doing string perturbation theory.

This is a somewhat disturbing state of affairs. It implies among other things that physical conclusions (like the non-renormalization theorems) are dependent on the choice of basis for the world-sheet gravitino and hence are *a priori* ambiguous.

Motivated by the need to understand how string theory copes with this problem, we examine in this paper the total derivative ambiguity in some detail. To start with we will represent moduli space as a fundamental domain of the mapping class group in teichmuller space, so we have to worry about contributions of the total derivative at boundaries of the fundamental domain. In ref. [4] it was shown that these unphysical boundaries yield mutually cancelling contributions if the gauge fixing slice satisfies so-called 'modular invariance constraints'. Therefore one criterion we shall adopt for the 'correct' choice of basis for the gravitino fields will be that it leads to a modular invariant slice in the sense of [4].

The other boundaries one needs to worry about are the physical boundaries $\Delta_0, \Delta_i$ of moduli space. There we find, as we detail in this paper, that the total derivative ambiguity in the heterotic string in general background and at any genus $g$ is proportional to physical massless tadpoles at *lower* genera. This implies that if the string quantum equations of motion are satisfied (*i.e.* all massless tadpoles vanish) order by order up to a given order $g$ in perturbation theory, then the partition function at $g+1$ is independent of choice of basis for the gravitino and in fact is free from ambiguity.

Having shown that the partition function is unambiguous we can calculate it. For strings in $R^{10}$ we give an inductive argument which shows the vanishing of the cosmological constant to all orders in perturbation theory.

A similar analysis can be done to expose the ambiguity in $n$-point scattering amplitudes in heterotic string theory. In this case we find no reason why the expression for the ambiguity should be vanishing. Surprisingly however, closer examination of that expression indicates that the ambiguity in $n$-point functions for strings in $R^{10}$ can be reabsorbed into a finite renormalization of the string tension and string coupling constant[1] and hence string amplitudes can be made well defined. Another issue we address that could render string amplitudes ill-defined is the question of finiteness. In this case we show, by ghost charge and dimension counting, that divergences in $n$-point functions at any genus $g$ in general backgrounds are related to tadpoles of massless physical states at lower genera. When the number of uncompactified dimensions $d$ is $\leq 4$ there are infrared divergences familiar from field theory. Among other things this implies that $n$-point amplitudes for heterotic string theory in $R^{10}$ are finite and well defined since the only possible tadpole in this theory is the dilaton tadpole which in turn may be related to the partition function, and hence vanishes.

[1] In fact this is also true for 0-point function, *i.e.* the cosmological constant. In that case however, the effect of the renormalization would not be seen since, as we will show, the cosmological constant is identically zero at all genera .

For the type II string the analysis for the ambiguity turns out to be technically more involved. We expect that the ambiguity is still related to massless tadpoles. We have not carried out a complete analysis. Instead we settle for an explicit analysis of the ambiguity at $g = 2$ only, and arrive at the same conclusion as that for the heterotic string.

To put the current paper in proper perspective it is perhaps worth commenting on its connection to previous work, specifically to ref.[2][4][6] where various subtleties in string perturbation theory were pointed out and discussed. In ref. [6] it was pointed out that with a standard choice of gauge fixing slice, string calculations in the type II string theory lead to physically nonsensible results. In particular it was shown that the picture-changing formalism leads to a cosmological constant at $g = 2$ which is ill-defined, and generically positive definite. In [4] we proposed a resolution of the problems in [6] by showing that there exist global obstructions to the standard gauge slice that went into those calculations and that with a choice of slice free of these obstructions a physically sensible answer is obtained (vanishing cosmological constant). The analysis in [4] did not shed light on the resolution of the integration ambiguity at arbitrary genera. In ref. [2], where the origin of the integration ambiguity was first pointed out, the criterion of $BRST$ invariance was used to determine a correct choice of basis for the world-sheet gravitino. A consistent choice was found, but it relies on special properties of genus two surfaces. In that prescription the support of the gravitino field at the boundary $\Delta_1$ of the moduli space ( where the genus two surface breaks up into two genus one surfaces $T_1$ and $T_2$) is taken at points $z_1$ and $z_2$ on $T_1$ and $T_2$. The limit $z_a \to p_a$ is then taken at the end of the calculation, where $p_1$ and $p_2$ denote the two nodes on $T_1$ and $T_2$. This prescription was used to calculate the two loop cosmological constant in the (compactified) heterotic, as well as type II superstring theories, in refs.[7][4]. What our present analysis shows is that the final answer for the partition function at arbitrary genus is independent of the location of the points $z_a$ as long as they are chosen in a modular invariant fashion and as long as the background satisfies the string quantum equations order by order. Hence the computation in refs.[7][4] can be carried out for arbitrary choice of the points $z_a$, yielding the same final answer. We should mention, however, that the node prescription (*i.e.* the prescription of taking the limit $z_a \to p_a$ ) remains a very convenient computational tool at $g = 2$.

We hasten to add that the history of this subject bids us be modest in our claims. We have not given completely rigorous *mathematical* proofs of finiteness and the vanishing of the cosmological constant. We ***have*** carefully completed the outline of the proofs initiated in [8][9] and further refined in [1] . General reasoning should always be verified with explicit calculations, and at genus two some of this has been done [10][11][12][13][4] . Among the most important of our unproven assumptions is the factorization hypothesis. The issue of the relation of the formula for the measure in the picture-changing formalism to the superMumford form on supermoduli space [14], and the related issue of contact terms in moduli space deserve further study.

The organization of this paper is as follows: In section 2 we address the question of the existence of a modular invariant slice, and prove that slices of the type we need do exist. In section 3 we give the details of the analysis necessary to see the connection between the heterotic string ambiguity in arbitrary backgrounds and physical massless tadpoles. In section 4 we give the argument for the vanishing of the cosmological constant in $R^{10}$. In the case of arbitrary backgrounds we identify possible operators whose vev at some genus would yield a nonvanishing cosmological constant at higher genera: These include some candidate $F$-terms. We do not have at this stage any argument to prove that the vev of these operators is vanishing, although analysis in low energy effective field theory leads us to believe that this is so [15]. The ambiguity in $n$-point functions in flat space is studied in section 5. There we briefly indicate how it can be absorbed into a redefinition of string coupling and tension. In this section we also briefly address the question of finiteness of $n$-point amplitudes. The ambiguity for type II string theory is studied in section 6. There we restrict our attention to $g = 2$ in arbitrary backgrounds. Section 7 contains our conclusions and some speculations. Finally some technical details have been relegated to two appendices.

## 2. Meromorphic Slices

*Global Obstructions*

In this section we fill in some gaps which were left in [4]. These considerations are technical and are only needed to assure that our starting point, eq.(3.2), is well-defined. The reader may therefore wish to proceed to section three. [2]

Moduli space is complicated and supermoduli space is supercomplicated; whenever there is a possible global obstruction it most probably will exist. In order to say anything definite about the string vacuum amplitude we will need to address three kinds of obstructions. First, it is quite possible that supermoduli space is not split, or even projected. Second, in order to enforce the $L_0 - \bar{L}_0$ projection consistently we must consistently choose a transverse coordinate $t$ to the compactification divisor of $\mathcal{M}$ in $\bar{\mathcal{M}}$ where $\mathcal{M}$ is the moduli space of curves and $\bar{\mathcal{M}}$ is its stable compactification. This is only possible if the normal bundle of $\bar{\mathcal{M}} - \mathcal{M}$ in $\bar{\mathcal{M}}$ is a trivial line bundle [3]. Third, in the picture changing formalism we will need to choose points $\{z_a\}$. There are global obstructions to various requirements one might like to impose on such choices. These last obstructions are, of course, related to the first one, since a choice of the points $\{z_a\}$ consistent with all the requirements will provide a coordinatization of the supermoduli space in which the even and the odd coordinates are explicitly split.

Although these obstructions are very interesting, we will argue now that, at least for the heterotic string, they are really irrelevant. It is possible that they are important in the type II case. The first two obstructions can be measured by cohomology groups of holomorphic vector bundles. This is obvious for the second obstruction and it follows from [16]for the first. Since $\bar{\mathcal{M}}$ is projective [17][18], by cutting out an appropriate divisor from $\bar{\mathcal{M}}$ we can make $\bar{\mathcal{M}}$ an affine variety, and using the general result that for $i > 0$,

[2] We are grateful to J.-B. Bost, P. Deligne, D. Kazhdan, C. McMullen and E. Witten for helpful discussions on the material of this section.

[3] It is not. We thank D. Kazhdan for pointing this out.

$H^i(X, E) = 0$ for a holomorphic vector bundle $E$ over an affine variety $X$ [19] we see that simply by removing a divisor we can eliminate the first two global obstructions.

Let us now consider the obstructions to the choice of points $\{z_a\}$. In [4] we used the fact that there is no holomorphic section of the universal curve over teichmuller space. Thus we are forced to use differentiable sections (which certainly exist) and we cannot take the locations of the picture changing operators to be constant if the transition functions are to be diffeomorphisms and not supersymmetry transformations. However, the integration ambiguity puts further constraints on these sections. A naive interpretation of these constraints suggests that the location of the support of the gravitino field must consist of continuous sections of the universal curve over moduli space. In fact, there is no continuous section for the universal curve over moduli space as we now explain [4]. A section of the universal curve over moduli space lifts to a section of the universal curve over teichmuller space. Consider a closed path in moduli space. The initial and final endpoints of a lift of this path will differ by a modular transformation. Therefore the initial and final points of the corresponding lift of the section of the universal curve must also differ by the action of this modular transformation. By considering closed paths shrinking to orbifold points we see that the section of the universal curve over an orbifold point must be a fixed point of the corresponding automorphism of the riemann surface. In general, an orbifold point in moduli space corresponds to a riemann surface with a nontrivial automorphism *group*. By the above reasoning, the section of the universal curve over such points must be fixed by each element of the group. Since there are riemann surfaces with automorphism groups having no common fixed point, no section of the universal curve can exist.

It is worthwhile giving a concrete example of this obstruction. Consider a genus two surface with $Z/2Z \times Z/2Z$ symmetry generated by 180 degree rotations $\alpha, \beta$ which have no common fixed point. This riemann surface ( with an appropriate marking) will be represented by a point $t_0$ in teichmuller space which is fixed by the corresponding

[4] We would like to thank C. McMullen for explaining the obstruction described below.

elements $\alpha, \beta$ of the modular group. Consider a circle in moduli space which shrinks to the corresponding orbifold point. This lifts to a set of arcs shrinking to $t_0$. (It might be helpful here to draw the picture at genus one, with semicircles shrinking to $\tau = i$.) The section of the universal curve over moduli space will lift to a section over teichmuller space, and the value of the section at the initial and final point of the arc must be related by the corresponding modular transformation. This means that the value of the section at $t_0$ must be fixed by all elements of the modular group fixing $t_0$. In our example, no point on the surface is fixed simultaneously by $\alpha$ and $\beta$ so there is no consistent choice for the section over $t_0$.

One might suspect that since the source of the trouble is the orbifold locus in moduli space simply removing this locus (which is of codimension $\geq g-2$) will remove the problem. This is not necessarily the case. Note that if we remove the orbifold point then (in the above example) one can choose a section in the neighborhood of $t_0$. However, as we encircle the point $t_0$ the section must travel through a path of finite distance (since the fixed points of $\alpha$ and $\beta$ are a finite distance apart) so that the derivatives of the section will become singular near $t_0$. Thus the string measure could have singular contributions from points like $t_0$, which would be difficult to compute. Hence, simply removing the orbifold locus does not necessarily solve the problem. Moreover, it is not clear that in more complicated examples (for example, with large and complicated nonabelian automorphism groups) the removal of the orbifold locus will allow the existence of any section at all. For these reasons we will not attempt to choose a section, even over the moduli space with the orbifold locus removed.

Fortunately we do not actually need a section of the universal curve to compute the vacuum amplitude, even when using the picture changing formalism. Although we cannot choose a section of the universal curve we need only construct sets of $2g-2$ points which, in general, vary with the teichmuller parameters, and get permuted among themselves under modular transformation. Such a choice of points is not ruled out by the considerations discussed above. Even after such a set of points is chosen, one must ensure that the gauge slice obtained this way is transverse to the gauge directions everywhere in the moduli space. Otherwise the string measure will suffer from divergences, either in the form of the arguments of two picture changing operators colliding, giving rise to a second order pole, or from spurious poles in the correlation functions involving the superghost fields (see sec.3 for more details).

While it is possible that such a set of points may be found, there is another approach for the heterotic string theory, which is in the same spirit as the resolution of the first two obstructions proposed above. Instead of choosing $z_a$'s which are continuous in the teichmuller space $\mathcal{T}$, we may try to choose them in a way that they are complex analytic in the teichmuller space outside a modular invariant analytic subvariety $\tilde{D}$ of complex codimension $\geq 1$, and satisfy the usual requirement that they get permuted among themselves under modular transformation. The integration over the moduli is now restricted to the region $\mathcal{M} - D \equiv (\mathcal{T} - \tilde{D})/G$, where $G$ is the modular group. As we shall show below, with such a choice of $z_a$'s the string integral is well defined, even though the measure may have singularities near $D$. Furthermore, we need not satisfy the criteria of the slice being transverse to the gauge directions everywhere in the moduli space, since, as we shall show, the string integral is well defined even though the measure blows up at the points where the transversality condition breaks down. In other words, we may include in $D$ the codimension $\geq 1$ subspace of the moduli space where the $\{z_a\}$ are well defined, but the transversality condition breaks down. Finally, such a choice of slice also in principle allows us to construct global sections by removing the orbifold points from the moduli space in the manner discussed above, again including them in $D$. We shall call such gauge slices *meromorphic slices.* They give a manifest splitting of supermoduli space over $\mathcal{M} - D$. Since, as we have seen, the removal of appropriate divisors from $\mathcal{M}$ makes the supermoduli space split, one would expect such meromorphic slices to exist in general, but we shall give a more concrete construction later.

Thus, from various considerations we see that in order to define the integration over moduli, we need first to remove an appropriate divisor $D$ from $\bar{\mathcal{M}}$ to eliminate global obstructions and then to define the amplitude as an integral on the complement of $D$. In the remaining sections we justify this procedure for the compactification divisor. We now explain why it makes sense for other divisors.

The heterotic string measure can develop singularities along subvarieties, but the (unphysical) singularities along divisors other than the compactification divisor can only arise from the superconformal ghost system. Thus they will be purely meromorphic. [5]We can define the integral of a measure with only meromorphic singularities along an analytic divisor $D_1$ as follows. First, we need to choose a transverse coordinate $t_1$ to $D_1$, but this cannot be done globally along $D_1$ if the normal bundle $N$ is not trivial. By cutting out a divisor $D_2 \subset D_1$ of complex codimension one in $D_1$ we can trivialize $N \to D - D_1$. Once we do that we can choose a coordinate $t$ globally on $D - D_1$ and use it to cut out a round hypertube of radius $\epsilon_1$ around $D - D_1$. By "round" we mean that the cross section of the tube, which defines a contour in the $t_1$-plane, is a circle. Using a similar procedure we can cut out a hypertube of radius $\epsilon_2$ around $D_2 - D_3$ in $D_1$. We can continue in this way cutting out hypertubes of radius $\epsilon_i$ around divisors $D_i$ of complex codimension $i$ in $\mathcal{M}$ where $i = 1, \dots n$. The measure is nonsingular on the complement of the hypertubes, by assumption, and since the terms which become singular are purely meromorphic, the phase integral about the round boundary gives zero. Thus we can unambiguously take the limit as the radius of the hypertube shrinks to zero.

As an example, suppose $m$ are the coordinates for $D$ and $t$ is transverse, and near $t = 0$

$$\mu = \frac{1}{t^2} f_0(m, \bar{m}, t, \bar{t}) + \frac{1}{t} f_1(m, \bar{m}, t, \bar{t}) + f_2(m, \bar{m}, t, \bar{t})$$

where the $f_i$ are real analytic and nonsingular in $t$. Then

$$\int dm \lim_{r \to 0} \int_{|t| > r} d^2t \; \mu$$

[5] See sec.3 for more details.

is well-defined. More generally we must specifiy the order of integration so that we first integrate over the variable transverse to the hypertube of the largest codimension. Thus we *define*:

$$\int \mu \equiv \lim_{\epsilon_1 \to 0} \cdots \lim_{\epsilon_n \to 0} \int_{\mathcal{M}-D_1} d^2 t_1 \cdots \int_{\mathcal{M}-D_1-\cdots-D_n} d^2 t_n \, \mu \tag{2.1}$$

Experience with one-loop amplitudes indicates that this is the correct procedure. In the one-loop case one must define the integral around the point $q = 0$, *e.g.* for the $O(16) \times O(16)$ string. The correct procedure is first to integrate in the $\tau_1$ direction and second to integrate in the $\tau_2$ direction. Of course, on compactification divisors the singularities need not be purely meromorphic and require much more analysis.

There is a possible difficulty with (2.1) . Although we can obtain in this way a finite well-defined answer, it might not be the correct answer, since we might have missed $\delta$-function singularities at the divisor $D$. [6]We will argue in section three that this is not the case for the heterotic string measure.

Let us now introduce another important concept, namely the topological class of a given gauge slice. Two gauge slices will be called topologically equivalent if we can continuously deform one slice to another maintaining modular invariance. To see how there can exist gauge slices which cannot be deformed to each other, let us consider two cases, one where a particular point $z_a$ travels around a non-contractible loop on the riemann surface as we travel along a closed loop in the space $\bar{\mathcal{M}} - D$, the other where the point travels along a contractible loop on the riemann surface as we travel along a closed loop in $\bar{\mathcal{M}} - D$. It is then clear that we cannot deform one configuration to the other. (The situation is more complicated since a particular point need not always define a closed loop on the riemann surface as we travel along a closed loop in the moduli space, instead the set of points $\{z_a\}$ defines a closed loop in the configuration space of the unordered set of $n$-points on the riemann surface. But the basic outcome is the same). In our analysis in sec.3 we study the change in the partition function as well as the correlation functions under infinitesimal

[6] This same point has been stressed, albeit differently, by D. Kazhdan and N. Seiberg.

changes in the locations of $z_a$ and show that the resulting change either vanishes or may be absorbed in a redefinition of the coupling constants. This, however, still leaves open the possibility that one may obtain a different result by going to a gauge slice that cannot be obtained from the ones we are working with through continuous deformations.

This remark makes it essential that we describe what characterizes the class of gauge slices that we are working with in this paper, and we shall state them now.

a)Consider a genus $g$ surface with $n$ punctures. Thus there are altogether $2g-2+n$ supermoduli on the surface [20][21]. Upon factorization into a surface $S_1$ of genus $g_1$ with $n_1+1$ punctures, and a surface of genus $g_2$ with $n_2+1$ punctures ($g=g_1+g_2$), ($n=n_1+n_2$) $2g_1-1+n_1$ of the points must lie on $S_1$, and $2g_2-1+n_2$ of the points lie on $S_2$. Furthermore, the configuration of the $2g_i-1+n_i$ points that lie on $S_i$ should be the same as the prescription that would be taken for the distribution of the points on a genus $g_i$ surface with $n_i+1$ punctures, if we had started with such an amplitude in the first place. Thus this rule relates the choice of gauge slice on a higher genus surface to those on lower genus surfaces. This prescription also tells us that the location of the points on the surface $S_1$ is completely independent of the moduli of the punctured surface $S_2$ and *vice versa* after degeneration. (Note that the counting is slightly different if $S_1$(or $S_2$) is a sphere. In this case $n_1-1$ of the points lie on $S_1$, the rest lie on $S_2$ upon degeneration).

b) The choice of gauge slice should be continuously deformable to another slice satisfying the property that at the boundary discussed in (a), of the $2g_i-n_i+1$ points on $S_i$, $n_i+1$ points coincide with the $n_i+1$ punctures, and the remaining $2g_i-2$ points are independent of the location of the punctures and coincide with the choice of points on a genus $g_i$ surface without punctures. As we shall see later, this criteria is needed to recover the amplitude with vertex operators in the zero picture from that involving the vertex operators in the $-1$ picture.

*Construction of Meromorphic Slices*

Let us now indicate how we can construct an appropriate meromorphic gauge slice, at least for the case $n = 0$ ( for $n > 0$ we can take the extra points at the location of the punctures ). Let $E \rightarrow \bar{\mathcal{M}}$ be the vector bundle of abelian differentials on the curve corresponding to a point in $\bar{\mathcal{M}}$. $E$ is a holomorphic vector bundle. (More precisely, let $\pi : \bar{\mathcal{C}} \rightarrow \bar{\mathcal{M}}$ be the universal curve. If $K$ is the relative cotangent bundle then $\pi_* K$ is a locally free sheaf of constant rank and is the sheaf of sections of a holomorphic vector bundle $E \rightarrow \bar{\mathcal{M}}$. [7]) Consider now a meromorphic section of $E$ of the form,

$$\Omega = \sum v_i(\tau)\omega_i \tag{2.2}$$

where $\omega_i$ is a basis of abelian differentials for the curve given by $\tau \in \bar{\mathcal{M}}$. Thus the $v_i(\tau)$ in (2.2) are meromorphic.

The existence of a meromorphic section follows from the basic fact that we are working with algebraically defined objects: $\bar{\mathcal{M}}$ and $\bar{\mathcal{C}}$ are projective varieties and (therefore) the bundle $E$ is algebraic, that is, its transition functions are rational functions. (Rational functions are ratios of polynomials in some projective space). These facts are far from trivial and are the result of the work of P. Deligne, F. Knudsen, D. Mumford and others. (See [18] and references therein.) From the very definition of an algebraic vector bundle [22] it is clear that there are many meromorphic sections. (To avoid various complications one may wish to remove the orbifold locus, this is an algebraic operation so these considerations apply to moduli space with the orbifold points deleted.)

A slightly more concrete variant of this argument can be given which uses teichmuller theory and some elementary theorems of algebraic geometry. For convenience we restrict ourselves here to a finite covering of $\mathcal{M}$ on which there are no orbifold singularities. Using

[7] On the divisor $\Delta_0$ the abelian differentials will develop poles. This complication is irrelevant to our computation since the boundary integrals on $\Delta_0$ can be shown to vanish on general grounds. See below.

teichmuller theory one can show that there is a positive line bundle $L \to \bar{\mathcal{M}}$ [23]. A version of the Kodaira embedding theorem states that for $n$ sufficiently large $E \otimes L^{\otimes n}$ will be the pullback of a universal bundle over a Grassmannian [24]and will hence have meromorphic sections. Thus $E$ has meromorphic sections.

The divisor of the abelian differential $\Omega$ defines a set of $2g-2$ points on the corresponding curve. We may now choose our $2g-2$ points $\{z_a\}$ to be the support of $div\Omega$. [8]In order to do this we will cut out a divisor $D \subset \bar{\mathcal{M}}$ defined by the zeroes and poles of $\Omega$ and by the curves for which two points in $div\Omega$ coincide. As we have argued before ( see also sec. 3) there is no contribution to the heterotic string measure from singularities on $D$.

We may now study the behavior of $div\Omega$ on the compactification divisor in $\bar{\mathcal{M}} - \Delta_0$. Using the standard plumbing parametrization $(m_1, m_2, p_1, p_2, t)$ of the boundary we cut out discs $U_1$, $U_2$ centered at $p_1, p_2$ from surfaces $S_1, S_2$ described by moduli $m_1, m_2$. For generic sections $\Omega$ we know that

$$\begin{aligned} \Omega(z) &\to \sum_{i=1}^{g_1} v_i\omega_i + \mathcal{O}(t) \qquad && z \in S_1 - \{z_1 |\, |z_1| < |t|^{1/2}\} \\ \Omega(z) &\to \sum_{i=g_1+1}^{g_1+g_2} v_i\omega_i + \mathcal{O}(t) \qquad && z \in S_2 - \{z_2 |\, |z_2| < |t|^{1/2}\} \end{aligned} \tag{2.3}$$

so as $t \to 0$, $2g_1 - 2$ points become the divisor of an abelian differential on $S_1$ and $2g_2 - 2$ points become the divisor of an abelian differential on $S_2$. The remaining two points of the divisor of $\Omega$ must approach the node.

We can now argue that we can choose our meromorphic section $\Omega$ so that the criterion $(a)$ on the points $\{z_a\}$ is satisfied. Once again, the key fact we use is that $\bar{\mathcal{M}}$ is algebraic. The criterion $(a)$ may be stated in local coordinates (defined by (2.2) and (2.3)) as the

[8] Since $div\Omega$ is insensitive to the overall multiplicative factor in $\Omega$, we can also use a section of the direct product bundle of $E$ and any other line bundle over the moduli space to construct a modular invariant set of points.

requirement that at $t=0$, $v_i$, $i=1,\dots g_1$ are functions only of $m_1$, while $v_i$, $i=g_1+1,\dots g$ are functions only of $m_2$. Put differently, if we have chosen

$$\Omega_{g_i} \in \Gamma[E_{g_i} \to \bar{\mathcal{M}}_{g_i}]$$

(where the subscript indicates the genus) then identifying

$$E_g|_{t=0} \cong E_{g_1} \oplus E_{g_2} \to \mathcal{C}_{g_1} \times \mathcal{C}_{g_2}$$

we require $\Omega_g|_{t=0} = \Omega_{g_1} \oplus \Omega_{g_2}$. We can construct such sections by induction. Suppose $\Omega_i$ exists for $i=1,\dots g-1$. This determines $\Omega_g$ on $\cup_{i>0}\Delta_i$, and $\Omega_g$ can be consistently defined on this set by the induction hypothesis. Recall that $\cup_{i>0}\Delta_i$ is an algebraic subvariety of $\bar{\mathcal{M}}$, and $\bar{\mathcal{M}}$ is projective. Recall also that a meromorphic function on a projective variety is rational, and hence extends to the whole projective space [25]. A simple extension of this fact shows that $\Omega$ can be expressed in terms of rational functions and hence extends to all of $\bar{\mathcal{M}}$. This concludes the inductive step.

## 3. The ambiguity in heterotic string theory

We start our discussion for the heterotic string theory. Our analysis exposing the ambiguity will be valid for a general background ( not necessarily supersymmetric) as long as the resulting string theory on that background does not have any tachyonic states in its physical spectrum. We shall show that the ambiguity in a given genus is proportional to tadpoles of physical massless fields at lower genera. Hence if all such tadpoles vanish order by order in string perturbation theory, which is needed in order for the vacuum to be an extremum (order by order) of the full quantum effective action, then the final answer for the partition function is free from any ambiguity.

*Expression for the Ambiguity*

Let us choose a basis for the two dimensional gravitino field of the form,

$$\chi_{\bar{z}}^{+}(z) = \sum_{a=1}^{2g-2} \varsigma^a \delta^{(2)}(z - z_a) \tag{3.1}$$

where $\varsigma^a$ $(a = 1, \ldots, 2g-2)$ are the odd supermoduli. In this case the partition function becomes, after integration over $\varsigma^a$,

$$W \sim \int \prod_{i=1}^{6g-6} dm_i \int D[XBC] e^{-S_0} \prod_{a=1}^{2g-2} (Y(z_a) + \partial_i \xi(z_a) D_i) \prod_{j=1}^{6g-6} (\eta_j, b) \tag{3.2}$$

where $B$ stands for the fermionic ghosts $b, \bar{b}$ and the bosonic ghost $\beta$, $C$ stands for the fermionic ghosts $c, \bar{c}$ and the bosonic ghost $\gamma$, and $X$ stands for the set of all matter fields. $m_i$ denote the moduli, $S_0$ the action in a background where the gravitino field has been set to zero, and $\eta, \xi$ are the bosonized ghost fields defined through the relation,

$$\beta = \partial \xi e^{-\phi}, \quad \gamma = \eta e^{+\phi} \tag{3.3}$$

$\partial_i$ denotes $\frac{\partial}{\partial m_i}$ (so we have $\partial_i \xi(z_a) \equiv \partial \xi(z_a) \frac{\partial z_a}{\partial m_i}$), $D_i$ denotes $\frac{\delta}{\delta(\eta_i, b)}$ , $\eta_{k\bar{z}}{}^{z}$, $\eta_{kz}{}^{\bar{z}}$ denote the beltrami differentials, and,

$$(\eta_k, b) = \int d^2 z (\eta_{k\bar{z}}{}^{z} b_{zz} + \eta_{kz}{}^{\bar{z}} b_{\bar{z}\bar{z}}). \tag{3.4}$$

Finally,

$$\begin{aligned} Y &= :e^{\phi} T_F: \equiv \{Q_B, \xi\} \\ &= c\partial\xi + e^{\phi} T_F^X - \frac{1}{4}(\partial\eta e^{2\phi} b + \partial(\eta e^{2\phi} b)) \end{aligned} \tag{3.5}$$

where $Q_B$ is the BRST charge of the system. $T_F$ denotes the fermionic component of the total stress tensor, $T_F^X$ denotes the fermionic component of the stress tensor associated with the matter field $X$. In this section we will work in the irreducible representation of the superconformal ghosts, *i.e.*, we ignore the insertion of the factor of $\xi(z_0)$ needed to absorb the $\xi$ zero mode. This is consistent as long as none of the operators involved in the manipulation contains a $\xi$ zero mode (*i.e.* $\xi$ should always appear in the combination $\partial\xi$).

For convenience we shall choose complex coordinates in the moduli space denoted by $m_i$, $m_{\bar{i}}$, and choose the $\eta_j$'s in such a way that $\eta_{iz}{}^{\bar{z}} = \eta_{\bar{i}\bar{z}}{}^{z} = 0$. Throwing away all terms which vanish by $\bar{b}$, $\bar{c}$ ghost charge conservation, we can restrict the sum on $i$ in $\partial_i\xi(z_a)D_i$ to a sum over holomorphic indices to get

$$W \sim \int_{\mathcal{M}-D} \prod_{i=1}^{3g-3} (dm_i dm_{\bar{i}}) \int D[XBC] e^{-S_0} \prod_{a=1}^{2g-2} (Y(z_a) + \partial_i\xi(z_a)D_i) \prod_{j=1}^{3g-3} (\eta_j, b)(\eta_{\bar{j}}, \bar{b}) \tag{3.6}$$

Note that we have taken the integration region over $m_i$ to be $\mathcal{M} - D$, keeping in mind the possibility of choosing the $z_a$'s in such a way that they are well-defined and vary holomorphically with $m_i$ outside a complex analytic subvariety $D$ of the moduli space $\mathcal{M}$ with complex codimension $\geq 1$. The meaning of the integration over $\mathcal{M} - D$ was discussed in the previous section.

The choice of slice which makes the $z_a$'s holomorphic function of $m_i$'s in $\mathcal{M} - D$ also has an added advantage. For certain configuration of the points $\{z_a\}$ the gravitino slice can fail to be transverse to the gauge directions. As a result, the measure will develop singularities near these points. This is manifested as either two of the $z_a$'s approaching each other, giving rise to a second order pole from the operator product of two picture changing operators, or as a spurious pole coming from the superconformal ghost correlator in (3.6)[1] [9]. However, since all the singularities come from the superconformal ghost system or from the fermionic component of the stress tensor, they must be purely meromorphic in $\{z_a\}$. This, in turn, means that these singularities are meromorphic in $m_i$, and the subspace of $\mathcal{M}$ containing these singularities is embedded complex analytically in $\mathcal{M}$. As before, we define integration over $m_i$ near these singular regions by removing a hypertube of small

[9] These spurious poles occur at the zeroes of $\prod_\delta [\delta](\sum_a \vec{z}_a - 2\vec{\Delta})$, where $\delta$ denotes the spin structure.

transverse dimension around these regions, and take the radius of the tube to zero at the end of the calculation. As we argued before, this gives a finite well-defined answer.

We shall now study the change in $W$ under an infinitesimal shift in one of the points (say $z_1$) by $\Delta z_1$. Recall that the resulting change in $W$ is a total derivative. To see this one writes $Y(z_1+\Delta z_1) - Y(z_1) = \{Q, \xi(z_1+\Delta z_1) - \xi(z_1)\}$ and deforms away the BRST contour, leaving behind a sum of residues at various poles. For example, the residues at $(\eta_k, b)$ are $(\eta_k, T)$ which are required to express the answer as a total derivative in the moduli. The net result is:

$$\Delta W \sim \int \prod_{i=1}^{3g-3} (dm_i dm_{\bar{i}}) \partial_j N^j \tag{3.7}$$

where,

$$N^j = \int D[XBC] e^{-S_0} \partial\xi(z_1) \Delta z_1 D_j \prod_{a=2}^{2g-2} (Y(z_a) + \partial_i \xi(z_a) D_i) \prod_{i'=1}^{3g-3} (\eta_{i'}, b)(\eta_{\bar{i}'}, \bar{b}) \tag{3.8}$$

For $g = 2$ this simplifies to

$$N^j = \int D[XBC] e^{-S_0} \partial\xi(z_1) \Delta z_1 Y(z_2) (-1)^{j+1} \prod_{i \neq j}^{3} (\eta_i, b) \prod_{i=1}^{3} (\eta_{\bar{i}}, \bar{b}) \tag{3.9}$$

We should make four remarks regarding (3.7)–(3.9).

1.)In general, under a modular transformation the set of points $\{z_a\}$ are permuted among themselves. Hence if we shift only one of the points, the resulting set of points may not give a modular invariant slice, which will be reflected in the fact that $N^j$ defined in (3.8) will not be a globally defined vector density in the moduli space. If $\{z_a\}$ and $\{z_a + \Delta z_a\}$ denote two modular invariant sets of points, then $\Delta W$ may be expressed as (3.7) with $N^j$ given by the sum of terms of the form (3.8). Following a procedure similar to that in ref.[4]one can show that $N^j$ defined this way is a globally defined vector density in the moduli space, and hence $\Delta W$ receives contribution only from the boundary of the moduli space (and not from the boundary of the fundamenal domain). Each individual term in $N^j$ may then be analyzed by the procedure given below.

2.)One way to understand the origin of (3.7)[2][4], is to remember that a change in the location of the support of the gravitino from $\chi = \sum_a^{2g-2} \varsigma^a \delta^{(2)}(z - z_a)$ to $\chi' = \sum_a^{2g-2} \varsigma^a \delta^{(2)}(z - z'_a)$ can be compensated by a supersymmetry transformation with an appropriate parameter $\epsilon$. However supersymmetry also shifts the metric by an even nilpotent amount. ( Recall that under supersymmetry $\delta\partial = (\chi\epsilon)\bar{\partial})$. This in turn amounts to a shift in the moduli given by $\delta m^i = (\eta_i, \chi\epsilon)$, where $\eta_i$ are the beltrami differentials). However the string amplitude $A = \int \prod_i dm^i \prod_a d\varsigma^a \mu$, just like any integral over even and odd grassmann variables, is not invariant under $m^i \rightarrow m^i + \delta m^i$ with $\delta m^i$ even nilpotent, but in fact changes by a total derivative in moduli. It is not difficult to show [2]that the total derivative computed this way exactly reproduces (3.7)with $N^{\jmath}$ as given in (3.9).

3.)In the above equations we have ignored terms which vanish by ghost charge conservation. This removes all terms of the form $\partial_{\bar{\jmath}} N^{\bar{\jmath}}$ as well as terms involving $\partial_{\bar{\imath}} z_a$ from $N^{\jmath}$. This is important to our analysis since we have removed a divisor $D$ from $\mathcal{M}$ ( see discussion in section 2), and must worry about boundary terms from $D$. (We are now defining $D$ to include the regions where the gravitino slice fails to be transverse). Although the divisor, being of real codimension two, is of measure zero, the string measure can develop singularities there. If $t$ denotes the complex coordinate transverse to $D$ (*i.e.* $t = 0$ at $D$) the relevant boundary terms will come from terms of the form $\frac{\partial}{\partial t} N^t$. In order for it to give a non-vanishing contribution (*e.g.* one which does not vanish by integration over the phase of $t$), an expansion in $t, \bar{t}$ of $N^t$ near $t = 0$ must contain inverse powers of $\bar{t}$. However, as we have remarked before, all singularities of $N^t$ near $t = 0$ are purely meromorphic and hence we never get such singularities in $N^t$.

4.) We can combine remarks (2) and (3) to argue that the definition (2.1) is a good one, and we have not missed $\delta$-function singularities on $D$. Let us change gauge slice, $z_a \rightarrow z_a + \Delta z_a$ so that $\mu + \Delta\mu = \tilde{\mu}$ is smooth at $D$. The measure $\tilde{\mu}$ will be singular at some other divisor $D'$. Alternatively, we could use $\tilde{\mu}$ in a region $U_1$ near $D$ and $\mu$ in a complementary region $U_2$, including a boundary correction on the overlap. [10]The change $\Delta\mu$ is given by (3.7) and, by the above arguments, never develops $\delta$-function singularities at $D$ or $D'$. If the correct measure were $\mu+\delta[D]$ and not $\mu$, then, since $\Delta\mu$ cannot develop a $\delta$-function singularity the string measure would be singular at $D$ for every choice of gauge. This is not the case. Alternatively, the boundary corrections on $U_1 \cap U_2$ vanish because nothing can contribute to the phase integral. From these remarks we conclude that we have not overlooked $\delta$-function singularities in the interior of moduli space. (The reader is encouraged to investigate this point more thoroughly.)

Because of remarks (3) and (4) we will henceforth ignore the divisor $D$ and concentrate on the compactification divisor of $\mathcal{M}$.

*The Ambiguity and Physical Tadpoles*

We now consider the boundary integral of (3.8) . First consider the surface term from the boundary where $S$ degenerates into two surfaces of genus $g_1$ and $g_2$ with punctures at $p_1$ and $p_2$ respectively, connected by a long cylinder $C$. Then the original $6g-6$ moduli reduce to $6g_1-6$ $(6g_2-6)$ moduli on $S_1$ $(S_2)$ (let us denote them by $m_i(\hat{m}_i)$), the location of the punctures $p_1$ and $p_2$, and the plumbing fixture variables $t$, $\bar{t}$. $(\ln|t|^{-1}$ is the length of the cylinder $C$). We shall choose the points $z_a$ according to the criteria of section two, *i.e.* in such a way that near any boundary where the genus $g$ surface breaks up into two surfaces $S_1$ and $S_2$ of genus $g_1$ and $g_2$ with punctures at $p_1$ and $p_2$ respectively, $(2g_1-1)$ points lie on the surface $S_1$, and the other $(2g_2-1)$ points lie on $S_2$. This is a natural choice, since $2g_i-1$ is the number of supermoduli of a genus $g_i$ surface with one puncture. If $g_1(g_2)=1$, then due to the translational invariance on the torus the location of the puncture $p_1(p_2)$ is not a modulus. Instead the teichmuller parameter $\tau_1(\tau_2)$ of the torus serves as a complex

[10] A good analogy to keep in mind is the magnetic monopole. One can define gauge fields on the northern and southern hemisphere, which differ by a total derivative on the overlap. This approach to handling the string measure has recently been advocated in [3] .

modulus near the boundary. Let us, for definiteness, assume that $z_1, \ldots, z_{2g_1-1}$ lie on $S_1$ in the $t \to 0$ limit, and $z_{2g_1}, \ldots, z_{2g-2}$ lie on $S_2$. Also, for convenience, denote the latter set by $\hat{z}_1, \ldots, \hat{z}_{2g_2-1}$. The possible non-vanishing contributions at the boundary come from $N^t$. The boundary integral will be infinite if, in an expansion in $t, \bar{t}$ as $t \to 0$, $N^t$ has a term of order $\bar{t}^{-1}(t\bar{t})^{-\beta}$, $\beta > 0$. If $N^t$ has such a term with $\beta = 0$ the boundary integral can be finite and nonzero.

Using the factorization hypothesis [26] we can analyze the behaviour of the correlator near the boundary $t = 0$ by inserting a complete set of states at the two ends of the cylinder $C$. Noting that $(\eta_t, \bar{b}) \sim \frac{1}{\bar{t}}\bar{b}_0$ where $\bar{b}_0$ is the $\bar{b}$ zero mode on the cylinder $C$, we may write the contribution in the $t \to 0$ limit as,

$$\begin{aligned} N^t \sim \sum_{\Phi,\Psi} t^{h_\Psi}\bar{t}^{\bar{h}_\Psi-1}\langle \partial\xi(z_1)\Delta z_1 \prod_{a=2}^{2g_1-1} (Y(z_a) + \partial_i\xi(z_a)D_i + \partial_{p_1}\xi(z_a)D_{p_1}) \\ (\eta_{p_1}, b)(\eta_{\bar{p}_1}, \bar{b}) \prod_{j=1}^{3g_1-3} (\eta_j, b)(\eta_{\bar{j}}, \bar{b})\Phi(p_1)\rangle_{S_1} \\ \langle \Phi^\dagger|\bar{b}_0|\Psi\rangle_C \langle \Psi^\dagger(p_2) \prod_{a=1}^{2g_2-1} (Y(\hat{z}_a) + \partial_{\hat{i}}\xi(\hat{z}_a)D_{\hat{i}} + \partial_{p_2}\xi(\hat{z}_a)D_{p_2}) \\ (\eta_{p_2}, b)(\eta_{\bar{p}_2}, \bar{b}) \prod_{\hat{j}=1}^{3g_2-3} (\eta_{\hat{j}}, b)(\eta_{\bar{\hat{j}}}, \bar{b})\rangle_{S_2} \end{aligned} \tag{3.10}$$

$(h_\Phi, \bar{h}_\Phi) = (h_\Psi, \bar{h}_\Psi)$ are the conformal dimensions of $\Phi, \Psi$. In the case of $g = 2$ this simplifies to

$$\begin{aligned} N^t \sim & t^{h_\Phi}\bar{t}^{\bar{h}_\Phi-1}\langle \partial\xi(z_1)\Delta z_1(\eta_1, b)(\eta_{\bar{1}}, b)\Phi(p_1)\rangle_{T_1} \\ & \langle \Phi^\dagger|\bar{b}_0|\Psi\rangle_C \langle \Psi^\dagger(p_2)Y(z_2)(\eta_2, b)(\eta_{\bar{2}}, b)\rangle_{T_2} \end{aligned} \tag{3.11}$$

where $p_1, p_2$ are the nodes at which $C$ attaches to the tori $T_1$, $T_2$.

In writing (3.10) we have assumed that in the limit $t \to 0$ the points $z_1, \ldots, z_{2g_1-1}$ that lie on the surface $S_1$ depend only on the moduli associated with the surface $S_1$, and the points $\hat{z}_1, \ldots, \hat{z}_{2g_2-1}$ that lie on $S_2$ depend only on the moduli of $S_2$. This choice exists and is natural, although it might not be necessary.[11]

We shall now show that the relevant amplitude on $S_2$ is proportional to tadpoles of physical massless fields in the theory. In order to do that we must first classify all the operators $\Psi^\dagger$ that can appear in (3.10). This is easily done by introducing the following currents:

$$\begin{aligned} J_c &= :cb: \\ J_\phi &= \partial\phi \\ J_\eta &= :\eta\xi: \\ J_{\bar{c}} &= :\bar{c}\bar{b}: \end{aligned} \tag{3.12}$$

Note that each term in $Y$ is neutral under $J_c^0 + J_\eta^0$ and carries one unit of $J_c^0 + J_\phi^0$ charge. (Here $J^0$ is the charge associated with the current $J$.) On the other hand, in order to get a non-vanishing contribution to this correlation function on a genus $g_2$ surface we must have (ignoring the $\xi$ zero mode)

$$\begin{aligned} J_\eta^{0(tot)} &= 0 \\ J_c^{0(tot)} &= -(3g_2 - 3) \\ J_\phi^{0(tot)} &= 2g_2 - 2 \\ J_{\bar{c}}^{0(tot)} &= -(3g_2 - 3) \end{aligned} \tag{3.13}$$

[11] For example, in the genus two case we have also carried out the analysis without this assumption, see for example the type II analysis in section 6.

Combining these results together, we see that we must have,

$$\begin{aligned} J_c^0(\Psi^\dagger) &= m \\ J_\phi^0(\Psi^\dagger) &= -m \\ J_\eta^0(\Psi^\dagger) &= 1-m \\ J_{\bar{c}}^0(\Psi^\dagger) &= +1 \end{aligned} \tag{3.14}$$

for some integer $m$. From this we can derive the inequality (noting that $\xi$ can appear in $\Psi^\dagger$ only through derivatives of $\xi$)

$$\begin{aligned} h_\Psi &\geq \frac{1}{2}m(m-2) && for\ m \geq 1 \\ h_\Psi &\geq \frac{1}{2}(m^2-4m+2) && for\ m \leq -1 \\ h_\Psi &\geq 1 && for\ m = 0 \end{aligned} \tag{3.15}$$

In order to get a nonvanishing contribution from $N^t$ we need $(h_\Psi, \bar{h}_\Psi) \leq (0,0)$. Thus, $m = 1,2$ are the only possibilities. Since matter fields have positive conformal dimension, we have the following possible operators

$$\begin{aligned} \Psi_1^\dagger &= \bar{c}c\partial c\partial\xi e^{-2\phi}U \\ \Psi_2^\dagger &= \bar{c}ce^{-\phi}V \end{aligned} \tag{3.16}$$

where $V$ is of dimension $(h, h+\frac{1}{2})$, $0 \leq h \leq \frac{1}{2}$, and $U$ is a dimension (0,1) operator in the theory, constructed out of the matter fields. For $g = 2$, only the operator $\Psi_1^\dagger$ can contribute since the conjugate $\Phi_2$ of $\Psi_2^\dagger$ cannot have a nonzero vev on $T_1$ by $\phi$-charge conservation. Notice that $V$ 's with $h < \frac{1}{2}$ lead to divergent boundary integrals. However, as will be shown below, in order to have a non-vanishing matrix element on $S_2$ the operator $e^{-\phi}V$ (together with a factor of $e^{ik\cdot X}$ if $h \neq \frac{1}{2}$) must correspond to vertex operator of a physical state in the $-1$ picture. For $h < \frac{1}{2}$, this corresponds to a physical tachyon. Recall that we are dealing with theories with no tachyons in their physical spectrum. Hence the only operators of the form $\Psi_2^\dagger$ that exist in a tachyon-free theory are those with $h = \frac{1}{2}$. These lead to finite boundary contributions. From now on the operator $V$ will denote an operator of dimension $(\frac{1}{2}, 1)$ from the matter sector.

We begin by examining the relevant matrix element involving the operator $\Psi_1^\dagger$. Note that this is *not* associated with the vertex operator for a massless physical field. We will now argue that in a wide class of backgrounds this matrix element vanishes. By Lorentz invariance, the operator $U$ must be constructed out of the internal fields associated with the compact dimensions. A trivial but important consequence of this is that no such term exists in $R^{10}$. Also by gauge invariance such a term cannot exist in any theory in arbitrary background *without* a $U(1)$ factor in the gauge group. Furthermore, even for an arbitrary four dimensional compactification *with* a $U(1)$ factor, the relevant matrix element in an odd spin structure $\nu$ vanishes due to the free fermion zero modes, since the picture changing operators cannot be used to soak up the free fermion zero modes in a Lorentz invariant fashion[27]. Thus we are left with the even spin structure sector. [12]

Let us define $\langle U(z)\rangle'_\nu$ as the expectation value of $U(z)$ in the matter sector for an even spin structure $\nu$. We will be able to show that $\langle U(z)\rangle'_\nu$ vanishes for $z \neq z_a$ on any riemann surface in a certain class of theories. Since the point $p_2$ is always a nonzero (at least $\mathcal{O}(t^{1/2})$) distance away from $z_a$, possible contact terms at $z = z_a$ can be ignored.

Let $\Omega$ and $\hat{\Omega}$ be the period matrices associated with two points $t$ and $\hat{t}$ in the teichmuller space, related by a modular transformation. Then,

$$\hat{\Omega} = (A\Omega + B)(C\Omega + D)^{-1} \quad \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(2g, Z). \tag{3.17}$$

[12] Throughout this paper, spin structure refers to the boundary conditions on the free right moving fermions associated with the four flat dimensions, or equivalently, the boundary condition on the fermionic component of the super-stress tensor.

We restrict ourselves to those modular transformations which keep the spin structure $\nu$ fixed. Choose a global diffeomorphism $\mathcal{D}$ on the riemann surface associated with this modular transformation, and let $\hat{z} = \mathcal{D}z$. Let us also define,

$$\left\langle\langle U(z)\rangle\right\rangle_\nu = \frac{\langle U(z)\rangle'_\nu}{\langle I\rangle'_\nu} \tag{3.18}$$

where $I$ is the identity operator. Since $U(z)$ is a dimension $(0,1)$ conformal operator it will have no singularities at any point on the riemann surface except for possible contact terms arising from contractions with $Y(z_a)$. Thus, for $z \neq z_a$ we may expand $\left\langle\langle U(z)\rangle\right\rangle_\nu$ as,

$$\left\langle\langle U(z)\rangle\right\rangle_\nu d\bar{z}|_t = \sum_{i=1}^{g} a_i(t,\bar{t})\overline{\omega_i(z,t)} \tag{3.19}$$

where $a_i$ are some coefficients to be determined. We also have,

$$\left\langle\langle U(\hat{z})\rangle\right\rangle_\nu d\bar{\hat{z}}|_{\hat{t}} = \sum_{i=1}^{g} a_i(\hat{t},\bar{\hat{t}})\overline{\omega_i(\hat{z},\hat{t})} \tag{3.20}$$

Now, modular invariance gives,

$$\left\langle\langle U(z)\rangle\right\rangle_\nu d\bar{z}|_t = \left\langle\langle U(\hat{z})\rangle\right\rangle_\nu d\bar{\hat{z}}|_{\hat{t}} \tag{3.21}$$

On the other hand we know that,

$$\omega_i(\hat{z},\hat{t}) = \omega_j(z,t)(C\Omega + D)^{-1}_{ji} \tag{3.22}$$

Combining (3.21) and (3.22) we see that,

$$a_j(t,\bar{t}) = a_i(\hat{t},\bar{\hat{t}})\overline{(C\Omega + D)^{-1}_{ji}} \tag{3.23}$$

Consider now a $PT$ transformation on the world sheet. This reverses the sense of all homology cycles, so $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} -I & 0 \\ 0 & -I \end{smallmatrix}\right)$. (The transformation can be understood as the square of the transformation which takes $\Omega \rightarrow -1/\Omega$.) This transformation leaves $\Omega$ and the spin structure $\nu$ fixed. From (3.23) we get,

$$a_j(t,\bar{t}) = -a_j(\hat{t},\bar{\hat{t}}) \tag{3.24}$$

If we now consider theories where $a_j(t,\bar{t})$ depends on $t\ \bar{t}$ only through the period matrix, (for example as in the case of four dimensional string theories constructed from free fermions [28]) then, since $\hat{\Omega} = \Omega$, eq.(3.24) may be written as $a_j(\Omega,\bar{\Omega}) = -a_j(\Omega,\bar{\Omega})$. Hence $a_j(\Omega,\bar{\Omega}) = 0$ and so $\langle\langle U(z)\rangle\rangle_\nu$ must vanish. This, in turn, shows the vanishing of $\langle U(z)\rangle'_\nu$.

In fact, the above analysis is strong enough to show the vanishing of $\langle U(z)\rangle'_\nu$ for $g = 1$ and 2 without any assumption on the dependence of the $a_j$ on the teichmuller parameters. For $g = 1$, the period matrix itself is the teichmuller parameter $\tau$, hence $a_i$ can always be taken to be a function of the period matrix. On a genus two surface, the modular transformation $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} -I & 0 \\ 0 & -I \end{smallmatrix}\right)$ corresponds to the hyperelliptic involution, which not only leaves the period matrix, but the whole teichmuller space fixed. In other words, $\hat{t} = t$. Using (3.24) we immediately conclude that $a_j(t,\bar{t}) = 0$.

For orbifolds and string models within the fermionic construction the vanishing of $\langle U\rangle_\nu$ is easy to see explicitly at arbitrary genus. Consider for instance $\frac{\langle U\rangle'_\nu}{\langle I\rangle'_\nu}$ in orbifolds. Let us take $U(\bar{z}) = \bar{\lambda}^l\lambda^l$ where $\lambda^l$ denotes a particular left moving gauge fermion and $\bar{\lambda}^l$ its complex conjugate. Then,

$$\frac{\langle U\rangle_\nu}{\langle I\rangle_\nu} = \frac{\sum_{\alpha,\beta}\omega^i\partial_i \ln\vartheta\left[\begin{smallmatrix}\vec{\alpha}^l\\ \vec{\beta}^l\end{smallmatrix}\right](0) I^\nu_{\alpha,\beta}}{\sum_{\alpha,\beta} I^\nu_{\alpha,\beta}} \tag{3.25}$$

where $(\alpha,\beta)$ contains information about the boundary condition on all the fields along all cycles except for the boundary conditions on the free right moving fermions associated with the flat directions (this information is contained in the spin structure $\nu$). This includes information about the twist structure, as well as the spin structure on the left moving gauge fermions. $I^\nu_{\alpha,\beta}$ denotes the contribution to the partition function from a sector with boundary conditions given by $\nu$ and $(\alpha,\beta)$. Finally $(\vec{\alpha}^l,\vec{\beta}^l)$ denotes the boundary conditions on the particular left moving fermion $\lambda^l$ used in constructing the $U(1)$ current $U(\bar{z})$. Modular invariance under the transformation $PT$ above requires $I^\nu_{\alpha,\beta} = I^\nu_{-\alpha,-\beta}$. However $\partial_i \ln\vartheta\left[\begin{smallmatrix}\vec{\alpha}^l\\ \vec{\beta}^l\end{smallmatrix}\right](0)$ is odd under $(\alpha,\beta)\to(-\alpha,-\beta)$. Therefore the sum receives cancelling contributions from pairs $(\alpha,\beta)$ and $(-\alpha,-\beta)$ and hence $\langle U\rangle_\nu = 0$.

We will restrict ourselves to backgrounds for which $\Psi_1^\dagger$ has zero vev henceforth.

We now consider $\Psi_2^\dagger$. The first observation to make is that $e^{-\phi}V$ is the vertex operator of a physical state of the theory in the $-1$ picture [8] .In order to prove this first note that $V$ has dimension $(\frac{1}{2},1)$ and is made of matter fields. Since matter fields have $h \geq 0$ ( actually $h = 0$ corresponds to a purely antiholomorphic operator which commutes with all the $L_n$'s and $G_n$'s) any descendent *e.g.* $[L_{-1}, \tilde{V}]$ would have conformal dimension $> 1$. (A similar analysis may be done for the left-handed conformal algebra). Thus $V$ must be a highest weight vector of the conformal algebra. Similarly, $V$ cannot be a superconformal descendent, again since the only superconformal descendent with holomorphic dimension equal to 1/2 is of the form $[G_{-1/2}, \tilde{V}]$, but $\tilde{V}$ has $h = 0$ and is therefore purely antiholomorphic, commuting with all the $L_n$'s and $G_n$'s, hence $[G_{-1/2}, \tilde{V}] = 0$. Thus the only way $V$ can avoid being a physical vertex operator is if $e^{-\phi}V$ is odd under the $Z_2$ symmetry that is being used to to make the final GSO projection (from now on we shall conform to the standard terminology of refering to this symmetry as $G$-parity [29]. However, all the other terms in the correlator are even under this $Z_2$ symmetry, hence if $e^{-\phi}V$ is odd under this symmetry, the corresponding correlator on $S_2$ will vanish. This shows that in order to get a non-vanishing contribution to (3.10), $e^{-\phi}V$ must be the vertex operator of a physical state in the $-1$ picture, and hence the relevant correlator on $S_2$ is the tadpole of a physical massless field. Thus the ambiguity is proportional to lower genus physical tadpoles.

Usually one uses vertex operators in the zero picture to calculate tadpoles, so it is worthwhile to see how we can relate the above physical tadpole in the $(-1)$ picture to that in the zero picture. Some of the techniques we use will be needed in section four. We will show that the amplitudes computed in different pictures are in fact *not* equal, and the difference is itself proportional to lower genus tadpoles, of the type identified in (3.16). We shall do this in three steps. The points $\hat{z}_a$ on $S_2$ obtained after degeneration of the original surface might depend on the location of the puncture $p_2$. By the restriction b) on the slice given in sec. 2, we know that this configuration may be continuously deformed to one where $2g_2 - 2$ of the points are independent of $p_2$, and coincide with the set of points describing the choice of gauge slice on an unpunctured genus $g_2$ surface, while the other can be taken to depend on $p_2$. The first step of our analysis is to show that the relevant matrix element on $S_2$ does not change under this continuous deformation. Thus we study the ambiguity in the one-point function

$$\begin{aligned}\langle\Psi_2^\dagger(p_2)\prod_{a=1}^{2g_2-1}(Y(\hat{z}_a)+\partial_{\hat{\imath}}\xi(\hat{z}_a)D_{\hat{\imath}}+\partial_{p_2}\xi(\hat{z}_a)D_{p_2})\\ (\eta_{p_2},b)(\eta_{\bar{p}_2},\bar{b})\prod_{\hat{\jmath}=1}^{3g_2-3}(\eta_{\hat{\jmath}},b)(\eta_{\bar{\hat{\jmath}}},\bar{b})\rangle_{S_2}\end{aligned} \tag{3.26}$$

under a change $\hat{z}_c \to \hat{z}_c + \Delta\hat{z}_c$. Since, being a physical vertex operator, $\Psi_2^\dagger$ is BRST invariant, we may conclude in the usual way that the ambiguity is

$$\begin{aligned}\sim \partial_{\imath}\Bigg[\langle\Psi_2^\dagger(p_2)(\xi(\hat{z}_c)-\xi(\hat{z}_c+\Delta\hat{z}_c))D_{\imath}\prod_{a\neq c}^{2g_2-1}(Y(\hat{z}_a)+\partial_{\hat{\imath}}\xi(\hat{z}_a)D_{\hat{\imath}}+\partial_{p_2}\xi(\hat{z}_a)D_{p_2})\\ (\eta_{p_2},b)(\eta_{\bar{p}_2},\bar{b})\prod_{\hat{\jmath}=1}^{3g_2-3}(\eta_{\hat{\jmath}},b)(\eta_{\bar{\hat{\jmath}}},\bar{b})\rangle_{S_2}\Bigg]\\ +\partial_{p_2}\Bigg[\langle\Psi^\dagger(p_2)(\xi(\hat{z}_c)-\xi(\hat{z}_c+\Delta\hat{z}_c))\prod_{a\neq c}^{2g_2-1}(Y(\hat{z}_a)+\partial_{\hat{\imath}}\xi(\hat{z}_a)D_{\hat{\imath}}+\partial_{p_2}\xi(\hat{z}_a)D_{p_2})\\ (\eta_{\bar{p}_2},\bar{b})\prod_{\hat{\jmath}=1}^{3g_2-3}(\eta_{\hat{\jmath}},b)(\eta_{\bar{\hat{\jmath}}},\bar{b})\rangle_{S_2}\Bigg]\end{aligned} \tag{3.27}$$

The total derivative with respect to $p_2$ may be dropped since there is no way to generate a pole $\sim (\bar{p}_2 - \bar{p}_2^{(0)})^{-1}$ in the correlator.[13]The first total derivative may itself be analyzed exactly as we analyzed the original ambiguity. Notice that if $\hat{z}_c$ and $p_2$ lie on different

[13] As explained below the factor of $\bar{c}(p_2)$ in $\Psi_2^\dagger$ is removed after carrying out the $z$ integral in $(\eta_{\bar{p}_2},\bar{b})$. The only dependence on $\bar{p}_2$ in $\Psi_2^\dagger$ is then through matter fields which do not develop any singularities in $\bar{p}_2$ near $T_F(z_a)$—the only other source of matter fields in (3.27).

surfaces at the boundary where $S_2$ degenerates into two surfaces, then, by criteria a) given in sec. 2, $\hat{z}_c$ at the boundary is already independent of $p_2$, and hence $\Delta\hat{z}_c$ vanishes at the boundary. On the other hand, if $\hat{z}_c$ and $p_2$ lie on the same surface when $S_2$ degenerates we can study the correlator on the surface not containing $\hat{z}_c$ and for this correlator the previous considerations apply. Hence, when $\hat{z}_c$ changes the correlator (3.26) changes by terms proportional to tadpoles of $\Psi_1^\dagger$ and $\Psi_2^\dagger$ at genus $g < g_2$. In particular we generate tadpoles of massless physical fields, again in the $(-1)$ picture, but at lower genus. In the absence of such tadpoles, (3.26) is independent of the location of $\hat{z}_c$. Thus we may take $2g_2 - 2$ of the points (say $\hat{z}_2, \ldots \hat{z}_{2g_2-2}$) to be independent of $p_2$, and take the $\hat{z}_1 \to p_2$ limit, which we shall do shortly.

The second step is a rearrangement of the terms in (3.26) which facilitates taking the $\hat{z}_1 \to p_2$ limit. The $\partial_{p_2}\hat{z}_1 \partial\xi(\hat{z}_1)\frac{\delta}{\delta(\eta_{p_2}, b)}$ term in (3.26) removes the factor of $(\eta_{p_2}, b)$ from the correlator. On the other hand, the factor of $(\eta_{p_2}, b)$ may be removed from the term involving $Y(\hat{z}_1) + \partial_{\hat{t}}\xi(\hat{z}_1)D_{\hat{t}}$ as follows. Let us note that $\eta_{p_2}$ may be expressed as $-\partial_{\bar{z}}v^z$, where $v^z$ is a vector field which is taken to be unity at $p_2$ and may be taken to vanish at all the points $\hat{z}_a$. Then $\int d^2z \eta_{p_2}{}_{\bar{z}}^{z} b_{zz} = -\int d^2z(\partial_{\bar{z}}v^z)b_{zz}$ receives contribution only from the locations where $b$ has poles, and $v^z$ is non-zero, *i.e.* $p_2$. The residue at this pole is $-\bar{c}e^{-\phi}V(p_2)$. The factor of $(\eta_{\bar{p}_2}, \bar{b})$ and $\bar{c}(p_2)$ may similarly be removed by carrying out the $z$ integral in $\int d^2z \eta_{\bar{p}_2}{}_{z}^{\bar{z}} \bar{b}_{\bar{z}\bar{z}}$. Thus the final expression for the correlator on $S_2$ may be written as,

$$\langle\{e^{-\phi(p_2)}V(p_2)(Y(\hat{z}_1) + \partial_{\hat{t}}\xi(\hat{z}_1)D_{\hat{t}}) - c(p_2)e^{-\phi(p_2)}V(p_2)(\partial_{p_2}\hat{z}_1)\partial\xi(\hat{z}_1)\}$$
$$\prod_{a=2}^{2g_2-1}(Y(\hat{z}_a) + \partial_{\hat{t}}\xi(\hat{z}_a)D_{\hat{t}})\prod_{\hat{\jmath}=1}^{3g_2-3}(\eta_{\hat{\jmath}}, b)(\eta_{\bar{\hat{\jmath}}}, \bar{b})\rangle \qquad (3.28)$$

where we have dropped the terms proportional to $\partial_{p_2}\hat{z}_a$ for $a \geq 2$, since $\hat{z}_a$ has been taken to be independent of $p_2$ for $a \geq 2$.

The third step involves taking the $\hat{z}_1 \to p_2$ limit. In this limit $\partial_{\bar{t}}\hat{z}_1 = 0$. Using (3.5) we can then write the term in the curly bracket in (3.28) as,

$$
\begin{aligned}
&\lim_{\hat{z}_1 \to p_2} [e^{-\phi(p_2)}V(p_2)Y(\hat{z}_1) - c(p_2)e^{-\phi(p_2)}V(p_2)(\partial_{p_2}\hat{z}_1)\partial\xi(\hat{z}_1)] \\
&= \{Q_B, e^{-\phi(p_2)}V(p_2)\xi(p_2)\} - [Q_B, e^{-\phi(p_2)}V(p_2)]\xi(p_2) - c(p_2)e^{-\phi(p_2)}V(p_2)\partial\xi(p_2) \\
&= \{Q_B, e^{-\phi(p_2)}V(p_2)\xi(p_2)\} - \partial(c\xi e^{-\phi}V)(p_2)
\end{aligned}
\tag{3.29}
$$

which is precisely the vertex operator in the zero picture[8]. Therefore, by induction the physical tadpole $\langle\Psi_1^\dagger\rangle$ in the $(-1)$ picture at genus $g$ is a sum of tadpoles of physical fields in the zero picture at genus $g_1 \leq g$. (Note that the zero picture is distinguished since it is in this picture that the dilaton tadpole [14]is proportional to the cosmological constant[7][4] in a given genus).

Finally we briefly consider the contribution from the boundary $\Delta_0$ where the genus $g$ surface degenerates into a genus $g-1$ surface with a long handle of length of order $\ln|t|^{-1}$ attached to it. Here $t$ is the complex coordinate transverse to the boundary (for a more complete description of the coordinate system near $\Delta_0$ see, for example, ref.[4]). The relevant contribution is the $\frac{\partial}{\partial \bar{t}}N^t$ term, and in order to give a non-vanishing contribution $N^t$ must be at least as singular as $\frac{1}{t}$ near the boundary. The part of $N^t$ that involves the conformal and the superconformal ghosts may be analyzed explicitly from the known behavior of the $\vartheta$-functions near the boundary[4]. Let $-e^{2\pi i a_1}$ denote the periodicity of the fermionic component of the stress tensor along the cycle that is being pinched, and let $e^{2\pi i b_1}$ denote the periodicity associated with the corresponding $B_1$ cycle, the cycle that intersects the pinched cycle once. (In our analysis it does not matter how we choose the $B_1$ cycle). Then for $a_1 = 0$ (Neveu-Schwarz sector) and $a_1 = \frac{1}{2}$ (Ramond sector) the

---

[14] Throughout this paper we call the expectation value of the operator $\partial X^\mu \bar{\partial} X^\mu$ the dilaton tadpole at zero momentum. There are subtleties in relating this matrix element to the tadpole of a physical dilaton field at zero momentum (they are proportional to each other, the constant of proportionality being dependent on the genus). We never need to know the precise relation between the two in our analysis.

leading behavior of the ghost correlator can be seen to be $t^{-\frac{1}{2}}\bar{t}^{-2}$ and $t^{-\frac{5}{8}}\bar{t}^{-2}$ respectively, following the analysis of ref.[4]. (These leading behaviors may be interpreted due to the propagation of the operators $c\bar{c}e^{-\phi}$ and $c\bar{c}e^{-\frac{\phi}{2}}$ respectively, an extra factor of $\bar{t}^{-1}$ comes from $\eta_{\bar{t}}$). On the other hand, the behavior of the correlator that involves the matter fields may be analyzed using the factorization hypothesis, inserting a complete set of states at the two ends of the long cylinder $C$. The net result may be expressed as,

$$\frac{1}{(\ln|t|^{-1})^{\frac{d}{2}}}\sum_i A_i t^{h_i}\bar{t}^{\bar{h}_i} \tag{3.30}$$

where $d$ is the dimension of the uncompactified space-time, and $(h_i, \bar{h}_i)$ may be interpreted as the conformal dimension of the operator inserted at the two ends of the cylinder $C$. The $(\ln|t|^{-1})^{-\frac{d}{2}}$ factor comes from integration over the $d$ dimensional momenta. In the Ramond sector the lowest lying state has $h_i = \frac{5}{8}$ due to the relation $L_0 = G_0^2 + \frac{5}{8}$. Combining this with the ghost contribution, and keeping only those terms that survive after integration over the phase of $t$, we see that the leading $t$ behavior of $N^t$ is $\frac{1}{(\ln|t|^{-1})^{\frac{d}{2}}}\frac{1}{t}$. This is softer than $\frac{1}{t}$, and hence does not give any boundary contribution at any genus. The total contribution from the Neveu Schwarz sector, on the other hand, may be written as,

$$N^t \sim \frac{1}{(\ln|t|^{-1})^{\frac{d}{2}}}\sum_i A_i t^{h_i-\frac{1}{2}}\bar{t}^{\bar{h}_i-2} \tag{3.31}$$

Thus we see that potentially divergent boundary contribution may come from operators $V_i$ of dimension $(h, h+\frac{1}{2})$, for $0 \leq h < \frac{1}{2}$. So far the spin structure $b_1$ has never entered our calculation, but now we shall sum over the spin structure $b_1$. The two contributions are equal in magnitude, the relative phase between them is $+(-)$ if the operator $e^{-\phi}V_i$ is even (odd) under $G$-parity. Thus only operators that are even under $G$-parity can lead to a divergent contribution. However, as has already been pointed out, such operators correspond to the existence of a physical tachyon in the Neveu-Schwarz sector. Thus in physically sensible theories, which do not have a physical tachyon in the tree level spectrum, there is no boundary contribution from $\Delta_0$.

Finally we would like to add that using the same method, the leading contribution to the partition function near the boundary $\Delta_0$ may be shown to be of the form

$$\sim \int dt d\bar{t} \frac{1}{(\ln|t|^{-1})^{\frac{d}{2}}} \frac{1}{t\bar{t}} \tag{3.32}$$

where the extra factor of $t^{-1}$ compared to $N^t$ comes from $\eta_t$. The integral over $t$ is finite showing that there is no divergent contribution at $\Delta_0$ for $d > 2$.

## 4. The cosmological constant at arbitrary genus in heterotic string theory in $R^{10}$

In the previous section it was shown that the ambiguities of the heterotic string vacuum amplitude are proportional to physical tadpoles on strictly lower genus surfaces. In particular, when these tadpoles vanish we can define an unambiguous amplitude at genus $g$, and it makes sense to try to prove things about it. It was also shown that for heterotic strings in $R^{10}$–the case to which we restrict ourselves in this section–only one tadpole can arise; the dilaton tadpole in the zero picture. As has been shown in refs.[7][4], this tadpole is itself proportional to the cosmological constant. This observation suggests the possibility of proving that the cosmological constant vanishes in $R^{10}$ using an inductive argument. The vacuum amplitude is known to be zero at $g = 1$ by explicit computation. We assume that the amplitude is zero order by order up to genus $g - 1$ and hence is well-defined at genus $g$. In this section we provide the inductive step showing that the amplitude at genus $g$ is then indeed zero. Before we are able to complete this inductive programme we also have to make sure that the expression for the cosmological constant is well defined, in the sense that it receives no divergent contributions. We shall check this as we carry out the argument for the vanishing of the cosmological constant.

By standard manipulation[1][7][4], dropping terms which vanish by ghost charge conservation, the partition function may be expressed as

$$W \sim \int \prod_{i=1}^{6g-6} dm^i \sum_j \partial_j M^j \tag{4.1}$$

where,

$$M^{\bar{\jmath}} = 0, \tag{4.2}$$

and,

$$M^{\jmath} = \sum_{l} \int_{\Sigma - \{z_a\}} d^2y \oint_{r_l} \frac{dx}{2\pi i} \langle \xi(z_1)\xi(\tilde{z}_1) J_\alpha(x) V^\alpha(y)$$
$$D_\jmath \prod_{a=2}^{2g-2} (Y(z_a) + \partial_\imath \xi(z_a) D_\imath) \prod_{\imath=1}^{3g-3} (\eta_\imath, b)(\eta_{\bar{\imath}}, \bar{b})\rangle \tag{4.3}$$

where,

$$J_\alpha = e^{-\frac{\phi}{2}} S_\alpha \tag{4.4}$$

are the space-time supersymmetry currents in the $-\frac{1}{2}$ picture[8] and,

$$V^\alpha(y) = \bar{\partial} X^\mu (\gamma^\mu)^{\alpha\beta} \{ e^{\frac{\phi}{2}} \partial X^\nu (\gamma^\nu)_{\beta\gamma} S^\gamma + \frac{1}{2} e^{\frac{3\phi}{2}} \eta b S_\beta \} \tag{4.5}$$

is the dilatino vertex operator in the $\frac{1}{2}$ picture. $z_1, \cdots z_{2g-2}$ are the points where the picture changing operators were originally introduced in the partition function. [15]In writing down eq.(4.3) we have used the reducible representation of the $\xi$, $\eta$ algebra, *i.e.* all factors of $\xi$, including those which soak up the $\xi$ zero mode, are now displayed explicitly in the correlator. In carrying out the integration over $y$, we cut out small holes of radius $\epsilon$ around the points $z_a$, restrict the region of integration over $y$ to be outside the holes, and take the $\epsilon \to 0$ limit at the end of the calculation[4]. $\tilde{z}_1$ is an arbitrary point on the surface $\Sigma - \{z_a\}$, the expression for $M^\jmath$ may be shown to be independent of $\tilde{z}_1$ point by point in

[15] Note that in writing down (4.3) we have isolated a point $z_1$ from the rest. Since in general a modular transformation permutes the points $z_a$, we loose manifest modular invariance. As a result, $M^\jmath$ may not be a globally defined vector density in the moduli space, although $\partial_\jmath M^\jmath$ is still a globally defined scalar density. We may rectify this state of affairs by symmetrizing the right hand side of eq.(4.3) with respect to all the $z_a$'s. Our analysis below then may be applied to each term in $M^\jmath$ separately.

the uncompactified moduli space. Finally, $r_l = r_l(y, \{z_a\})$ are the locations of the zeros in $x$ of the function,[16]

$$f(x) = \prod_{\delta} \vartheta[\delta](\frac{1}{2}\vec{y} - \frac{1}{2}\vec{x} + \sum_{a=1}^{2g-2} \vec{z}_a - 2\vec{\Delta}) \tag{4.6}$$

Here $\delta$ runs over all the spin structures, $\vec{z} = \int_{P_0}^{z} \vec{\omega}$, where $P_0$ is some arbitrary base point, $\omega_i$ are the holomorphic abelian differentials, and $\vec{\Delta}$ is the vector of riemann constants[30]. If we compute the correlator (4.3), then as a function of $x$ it has two sets of spurious poles $\{r_l\}$ and $\{r_l'\}$, besides the poles dictated by the operator product expansion. $r_l'$ are given by the zeros of (4.6) with $z_1$ replaced by $\tilde{z}_1$. These two sets of poles may be interpreted as due to the $\xi$ zero mode being absorbed by $\xi(\tilde{z}_1)$ and $\xi(z_1)$ respectively. Thus, for example, if we replace $\xi(\tilde{z}_1)$ by $\partial\xi(\tilde{z}_1)$ in (4.3), the set of poles $\{r_l\}$ disappears from the correlator, whereas replacing $\xi(z_1)$ by $\partial\xi(z_1)$ removes the set of poles $\{r_l'\}$.

We will need to know how the $r_l$ behave at the boundary of moduli space. From (4.6) one learns that on a genus $g$ surface there are $g2^{2g-2}$ such poles[7]. From the same formula one also learns that on a boundary $\Delta_{g_1}$, $g_1 2^{2g_1-2}$ of them lie on the surface of genus $g_1$, each repeated $2^{2g_2}$ times for $2^{2g_2}$ different spin structures on the surface of genus $g_2$, and *vice versa*.

The total derivative terms might have nonzero integrals on the boundaries of the moduli space where the genus $g$ surface breaks up into two lower genus surfaces $(\Delta_i)$ or a genus $g-1$ surface with an infinitely long handle attached to it $(\Delta_0)$. Following analysis similar to the one given at the end of sec. 3 (see also ref. [4]), it can be shown that there is no contribution from $\Delta_0$. Thus we need only consider the boundary $\Delta_i$ for $i > 0$.

[16] As we have pointed out before, in order for the gauge slice to be transverse, the expression $\prod_\delta \vartheta[\delta](\sum_a \vec{z}_a - 2\vec{\Delta})$ is never zero on $\mathcal{M} - D$. This only guarantees that the spurious poles $r_l$ never coincide with $y$. However there is no way to avoid the existence of those poles. The origin of the $r_l$'s can be understood as due to the occurence of accidental zero modes in the $(\beta, \gamma)$ system [7].

Introducing the usual variables, the relevant term near the boundary $\Delta_{g_1}$ is $M^t$, and is given by,

$$M^t \sim \sum_l \int d^2y \oint_{r_l} dx \quad t^{h_\Psi} \bar{t}^{\bar{h}_\Psi - 1} \langle \xi(z_1) \prod_q \mathcal{O}_q \prod_{a=2}^{2g_1-1} (Y(z_a) + \partial_i \xi(z_a) D_i + \partial_{p_1} \xi(z_a) D_{p_1})$$
$$(\eta_{p_1}, b)(\eta_{\bar{p}_1}, \bar{b}) \prod_{j=1}^{3g_1-3} (\eta_j, b)(\eta_{\bar{j}}, \bar{b}) \Phi(p_1) \rangle_{S_1}$$
$$\langle \Phi^\dagger | \bar{b}_0 | \Psi \rangle_C \langle \Psi^\dagger(p_2) \xi(\tilde{z}_1) \prod_{q'} \hat{\mathcal{O}}_{q'} \prod_{a=1}^{2g_2-1} (Y(\hat{z}_a) + \partial_{\hat{i}} \xi(\hat{z}_a) D_{\hat{i}} + \partial_{p_2} \xi(\hat{z}_a) D_{p_2})$$
$$(\eta_{p_2}, b)(\eta_{\bar{p}_2}, \bar{b}) \prod_{\hat{j}=1}^{3g_2-3} (\eta_{\hat{j}}, b)(\eta_{\bar{\hat{j}}}, \bar{b}) \rangle_{S_2} \tag{4.7}$$

In writing down (4.7) we have used the independence of the correlator of $\tilde{z}_1$ to choose $\tilde{z}_1$ on a surface opposite to the one where $z_1$ lies. The set of operators $\{\mathcal{O}, \hat{\mathcal{O}}\}$ contains $J_\alpha(x)$ and $V^\alpha(y)$. There are thus four possibilities,

I) $x, y \in S_1$;

II)$x \in S_2, y \in S_1$;

III) $x \in S_1, y \in S_2$;

IV) $x, y \in S_2$.

We will list the possible operators $\Psi^\dagger$ that can appear in (4.7), remembering that we need $M^t \sim \bar{t}^{-1}$ in order to get a non-vanishing boundary contribution, and hence $\Psi^\dagger$ must have dimension (0,0). Since we also would like to check finiteness we will also check if any operator with dim $< (0,0)$ could propagate. This is an important consistency check, since such operators would lead to divergent boundary contributions. The analysis proceeds as in the previous section using the ghost charge conservation given in (3.14), which needs to be modified slightly when $x$ and $y$ lie on different surfaces. Another difference from the previous treatment is that since we are using the reducible representation of the $\xi, \eta$ algebra, all factors of $\xi$, including the ones that absorb the $\xi$ zero mode, must be present explicitly in the correlator. We will restrict the analysis to $R^{10}$ since, although we could classify all possible operators that could appear in arbitrary backgrounds, we have not been able to evaluate all the relevant matrix elements on those backgrounds.

It will be convenient to recall that only those operators which carry even $J^0_\phi + J^0_\psi$ charge can appear. (Here $J^0_\psi$ is the charge associated with the world-sheet fermion $\psi$). This is a conserved charge on the riemann surface modulo 2, and all the other operators *e.g.* the $Y$'s carry even $J^0_\phi + J^0_\psi$ charge. Thus, for example, only operators of the form $e^{(2m-\frac{1}{2})\phi} S_\beta$ and $e^{(2m+\frac{1}{2})\phi} S^\beta$ will appear, operators of the form $e^{(2m+\frac{1}{2})\phi} S_\beta$ and $e^{(2m-\frac{1}{2})\phi} S^\beta$ cannot appear since they carry odd $J^0_\phi + J^0_\psi$ charge (here $m$ is an integer). Similarily only operators of the form $e^{2m\phi}$ and $e^{(2m+1)\phi}\psi$ could appear but not those of the form $e^{2m\phi}\psi$ or $e^{(2m+1)\phi}$. Finally, since we have a manifestly Lorentz as well as gauge invariant formalism we will often use Lorentz and gauge invariance to drop correlators. We now consider each of the four configurations in turn.

I) $x, y \in S_1$: In this case the operator $\Phi(p_1)$ must contain an explicit factor of $\xi(p_1)$, since in the absence of such operators the $\xi$ zero mode on $S_1$ must be absorbed by the factor of $\xi(z_1)$, and as a result, the correlator will not have any pole at $r_l$. Thus $\Psi^\dagger$ cannot contain any $\xi$ zero mode. Therefore the analysis of section three, which used the irreducible algebra may be applied, and the possible candidates for $\Psi^\dagger$ are precisely those listed in (3.16). The relevant correlators on $S_2$ may be shown to vanish using the induction hypothesis and the analysis used for the ambiguity. (Actually, in this case there is no Lorentz invariant U(1) current, so the contribution from the first term in (3.16) vanishes by Lorentz invariance). There is no operator that conserves all the ghost charges and $J^0_\phi + J^0_\psi$ with dim $(h, \bar{h}) < (0,0)$ and hence there are *no* potential singular contributions to this configuration.

II) $y \in S_1, x \in S_2$: The possible operators are,

$$
\begin{aligned}
a) \quad & \Psi^\dagger(p_2) = \bar{c} c e^{-\frac{1}{2}\phi} S_\beta \bar{\partial} X^\mu(p_2) \\
b) \quad & \Psi^\dagger(p_2) = \bar{c} c \partial c e^{-\frac{3}{2}\phi} \xi S^\beta \bar{\partial} X^\mu(p_2)
\end{aligned}
\tag{4.8}
$$

Before we analyse the resulting matrix elements of the above operators, let us emphasize that again no operators with potential divergent contributions exist for this configuration. First consider (a). This operator is allowed by Lorentz invariance, the matrix element on $S_2$ being proportional to $(\gamma^\mu)_{\alpha\beta}$. In this case $\xi(\tilde{z}_1)$ is the only operator which can soak up the $\xi$ zero mode, as a result the only set of unphysical poles as a function of $x \in S_2$ is $\{r_l\}$. This follows since no other term in the correlator on $S_2$ contains a factor of $\xi$ without derivatives. We may then deform the $x$ contour away from the poles $r_l$ and try to shrink it to a point. In this process we pick up residues at $p_2$, the residue being the operator:

$$\bar{c}ce^{-\phi}(\gamma^\nu)_{\alpha\beta}\psi^\nu\bar{\partial}X^\mu(p_2).$$

By Lorentz invariance the expectation value of $\bar{c}ce^{-\phi}\psi^\nu\bar{\partial}X^\mu$ on $S_2$ must be proportional to that of $\delta^{\mu\nu}c\bar{c}e^{-\phi}\psi^\rho\bar{\partial}X^\rho$. But this is nothing but the dilaton tadpole in the $-1$ picture on $S_2$, and hence vanishes. On the other hand, the relevant correlator involving the operator b) on $S_2$ vanishes by Lorentz invariance.

In passing we note that in a general compactified theory, there will be an operator $\Psi^\dagger(p_2) = \bar{c}c\partial ce^{-\frac{3}{2}\phi}\xi S^\beta V$, where $S^\beta$ is the four dimensional spin operator, and $V$ is some dimension $(\frac{3}{8}, 1)$ operator. The conjugate of this operator $\bar{c}ce^{-\frac{1}{2}\phi}S_\beta V^\dagger$ is the vertex operator of a physical fermion in the $(-\frac{1}{2})$ picture. Taking this to be a gaugino vertex gives the non-vanishing contribution to the cosmological constant at two-loop order[7] due to Fayet-Iliopoulos $D$-term at one loop. This is the only contribution to the partition function at two loops. However, in general, taking this operator to be the vertex operator associated with a fermion field belonging to a chiral scalar supermultiplet gives possible contribution to the partition function from the $F$-terms. We do not have any argument at this stage why the $F$-term contribution to the cosmological constant should vanish in general, although arguments based on low energy effective field theory indicates that this might be true [15].

III)$y \in S_2, x \in S_1$: In this case, again the operator $\Phi(p_1)$ on $S_1$ must contain a factor of $\xi(p_1)$, otherwise we would not have any poles at $r_l$ as a function of $x$.Thus $\Psi^\dagger$ is free from any $\xi$ zero mode. The possible candidates for $\Psi^\dagger$ are,

$$
\begin{aligned}
a) \quad & \Psi^\dagger(p_2) = \bar{c}ce^{-\frac{3}{2}\phi}S^\beta\bar{\partial}X^\mu(p_2) \\
b) \quad & \Psi^\dagger(p_2) = \bar{c}c\partial ce^{-\frac{5}{2}\phi}S_\beta\partial\xi\bar{\partial}X^\mu(p_2)
\end{aligned}
\tag{4.9}
$$

These are the only operators that are allowed by all the relevant charges. Since they are dimension $(0,0)$ they lead to finite contributions. Consider operator (a). For this term we use the fact that the final answer is independent of the choice of the points $z_a$, $\hat{z}_a$. As before we take them to satisfy the constraint that at any boundary, all but one (say $\hat{z}_1$ ) of the points $\hat{z}_a$ on $S_2$ are taken to be independent of $p_2$ (with a similar constraint on $S_1$). $\hat{z}_1$, on the other hand satisfies the requirement that on any boundary of $S_2$ it always lies on the same surface as $p_2$. Since $\Psi^\dagger$ is BRST invariant we may use manipulations similar to the one before (3.28) first to remove the factors of $(\eta_{p_2}, b)$ and $(\eta_{\bar{p}_2}, \bar{b})$ from the correlator, and then to take the limit $\hat{z}_1 \to p_2$. In this limit, $\partial_i\hat{z}_1 = 0$ for all the moduli of the surface. On the other hand, the $\partial_{p_2}\xi(\hat{z}_1)D_{p_2}$ term can easily be seen to cancel the $c\partial\xi$ term in $Y(\hat{z}_1)$. Thus we are left with two terms in $Y(\hat{z}_1)$ proportional to $e^\phi$ and $e^{2\phi}$ respectively. Each of these terms can be seen to give vanishing contribution near $e^{-\frac{3}{2}\phi}S^\beta\bar{\partial}X^\mu(p_2)$. Hence the net contribution from this term is zero. Contribution from the operator (b) vanishes by Lorentz invariance.

IV) $x, y \in S_2$: There are four possible operators. They are,

$$
\begin{aligned}
a) \quad & \Psi^\dagger(p_2) = \bar{c}ce^{-\phi}\psi^\mu\bar{\partial}X^\mu(p_2) \\
b) \quad & \Psi^\dagger(p_2) = \bar{c}c\partial ce^{-2\phi}\xi\partial X^\mu\bar{\partial}X^\mu(p_2) \\
c) \quad & \Psi^\dagger(p_2) = \bar{c}c\partial c\partial^2 ce^{-3\phi}\xi\partial\xi\psi^\mu\bar{\partial}X^\mu(p_2) \\
d) \quad & \Psi^\dagger(p_2) = \bar{c}c\partial ce^{-2\phi}\xi
\end{aligned}
\tag{4.10}
$$

Notice that operators $a, b, c$ could lead to finite contributions, while the operator in $d$ if it has a nonvanishing matrix element would lead to a divergent boundary contribution.

Let us analyse the matrix elements corresponding to all these operators carefully. Consider (a). Since $\xi(\tilde{z}_1)$ is the only factor of $\xi$ on $S_2$, the only unphysical poles are at $r_l$. Thus we may deform the $x$ contour away from $r_l$. The residue at $p_2$ is the operator $\bar{c}ce^{-\frac{3}{2}\phi}S^\beta\bar{\partial}X^\mu$ which goes away after taking the $\hat{z}_1 \to p_2$ limit as in case III. (a) above. The other residue is at $y$, the effect of which is to convert the operator at $y$ to the dilaton vertex operator $\partial X^\mu(y)\bar{\partial}X^\mu(y)$ in the zero picture.

As we have already mentioned, the rule of integration over $y$ is to cut out small holes around the points $z_a$ and do the integration over the rest of the surface. In this case, the same prescription must be adopted for the point $p_2$, since at any finite $t$, the surface $S_2$ is really a genus $g_2$ surface with a small hole around $p_2$ cut out. With this prescription one can show, using the manipulations given in [4], that the the result of inserting a factor of $\int \partial X^\mu(y)\bar{\partial}X^\mu(y)$ in a correlator is just an overall multiplicative factor. Thus we may remove this factor from the correlator. The resulting expression is a dilaton tadpole on $S_2$ in the $-1$ picture and hence vanishes.

Consider now case (b). In this case, the relevant operator $\Phi(p_1)$ on $S_1$ is $\bar{c}c\partial X^\mu\bar{\partial}X^\mu(p_1)$. This looks very much like the dilaton tadpole in the zero picture, but one needs to be somewhat careful. First of all, note that there is one $Y$ ($Y(z_1)$) missing from $S_1$. If this is the one which could depend on $p_1$, then we are done, since the rest of the $z_a$'s are independent of $p_1$ and the result is precisely a dilaton tadpole [17]– and hence vanishes. If, on the other hand, some other $z_a$ (say $z_2$) is the one which depends on $p_1$, then we do not have a standard dilaton tadpole. In this case we can interchange the points $z_1$ and $z_2$ using the picture changing operation, and get a standard dilaton tadpole. During this process we also generate terms which are total derivatives in the moduli space, but it has been shown in appendix A that all these extra total derivative terms have zero boundary contribution. Thus the contribution from this term vanishes.

[17] In general, one can start from a $z_1$ which satisfies this property near a particular boundary, but it is not possible to choose a $z_1$ which satisfies this property near every boundary.

Consider case (c). The conjugate of this operator, *i.e.* $\Phi(p_1)$, does not have any factor of $c$. If we now express $\eta_{p_1}{}^z_{\bar z}$ as $-\partial_{\bar z}v^z$, where $v^z$ is a vector field which is unity at $p_1$, and vanishes at all the points $z_a$, we see that $(\eta_{p_1}, b) = -\int d^2z\partial_{\bar z}v^z b_{zz} = 0$ by integration by parts, since $b_{zz}$ is meromorphic, and does not have a pole at $p_1$.

Finally consider case ($d$). In this case the matrix element of this operator can be shown to be proportional to lower genus cosmological constants. The proof follows exactly the same analysis as the one we used for case ($b$). (Everywhere in that argument replace $\partial X\bar\partial X$ by $I$). Hence if the lower genera cosmological constants vanish, no divergent contributions are encountered at higher genus.

This concludes our argument for the vanishing of the cosmological constant in the heterotic string theory in flat space-time to all orders in perturbation theory.

## 5. The ambiguity in $n$-point functions

In this section we comment on the ambiguity in $n$-point functions in the heterotic string. Our results here are not complete, but we will isolate some of the relevant issues. This analysis will also indicate how the proof of the nonrenormalization theorems for massless two- and three- point functions might proceed. We also briefly discuss the issue of finiteness of the $n$-point function.

We begin with the formula for the $n$-point function

$$\int\prod_{i=1}^{6g-6} dm_i \int\prod_{k=1}^{n} d^2x_k\langle\prod_{k=1}^{n} c\bar cV_{-1}(x_k)\prod_{a=1}^{2g-2+n}\left(Y(z_a)+\partial_i\xi(z_a)D_i+\partial_{x_k}\xi(z_a)D_{x_k}\right) \prod_{j=1}^{6g-6}(\eta_j,b)\prod_{k=1}^{n}(\eta_{x_k},b)(\eta_{\bar x_k},\bar b)\rangle \tag{5.1}$$

Here punctures are regarded as moduli, with associated beltrami differentials $\eta_{x_k}$. We have also assumed that the beltrami differentials for punctures are independent of the supermoduli as is the case for ordinary beltrami differentials. $V_{-1}(x_k)$ are the vertex operators for the massless bosonic external states in the $-1$ picture. (Many of our statements will generalize to other amplitudes, but for simplicity we restrict attention to this case.) A genus $g$ surface with $n$ punctures has $2g-2+n$ supermoduli[20][21] , and we have chosen their basis to be $\delta^{(2)}(z-z_a)$ $(a=1,\cdots 2g-2+n)$. The locations of $\{z_a\}$ are chosen in a way discussed in sec.2; if the genus $g$ surface breaks up into a surface $S_1$ of genus $g_1$ with $n_1+1$ punctures, and a surface $S_2$ of genus $g_2$ with $n_2+1$ punctures $(g=g_1+g_2,\ n=n_1+n_2)$, then $2g_i-1+n_i$ of the $z_a$'s lie on $S_i$ $(i=1,2)$. The counting is slightly different if $g_1(g_2)=0$, in this case $n_1-1$ of the $z_a$'s lie on $S_1(S_2)$, the rest lie on $S_2(S_1)$. Note that we may recover the usual zero picture vertex operators by taking the limit $z_k\to x_k$ $(k=1,\ldots n)$ and the rest of the $z_a$'s to be independent of $x_k$, once we prove that the final answer is independent of the locations of the $z_a$'s.

*Contact terms*

In the presence of supermoduli the usual formulae for the vertex operators must be corrected. These corrections cancel the effects of contact terms [31] from collisions of vertex and picture-changing operators. To see how this works let us consider the graviton vertex in the zero picture. In a graviton background the action is

$$S=\int G_{\mu\nu}(\partial X^\mu\bar\partial X^\nu+\psi^\mu(\bar D\psi)^\nu+\bar\chi\psi^\mu\partial X^\nu) \tag{5.2}$$

so the graviton vertex operator is

$$V_0=\epsilon_{\mu\nu}\big[(\partial X^\mu+i\psi^\mu k\cdot\psi)(\bar\partial X^\nu+\bar\chi\psi^\nu)+\psi^\mu\bar\partial\psi^\nu\big]e^{ik\cdot X} \tag{5.3}$$

Notice that we have not used the equations of motion and that the vertex is corrected by the gravitino. Since the vertex is corrected the integral over the supermoduli contains a term where the picture-changing operator is replaced by

$$\epsilon_{\mu\nu}(\partial X^\mu+i\psi^\mu k\cdot\psi)\psi^\nu(z_a) \tag{5.4}$$

However, from

$$\langle\bar{\partial}X(z)\partial X(w)\rangle = \frac{1}{2}\delta(z-w) + \text{ non-singular}$$
$$\langle\bar{\partial}\psi(z)\psi(w)\rangle = \frac{1}{2}\delta(z-w) \tag{5.5}$$

we learn that the contact terms in

$$\int V_0|_{\bar{\chi}=0}\cdot Y(z_a)$$

precisely cancel the effect of this correction.

One can also see this in the $-1$ picture. In the presence of the gravitino field, every $\bar{\partial}X^\mu$ in a vertex operator $V$ is replaced by $\bar{\partial}X^\mu + \frac{1}{2}\bar{\chi}\psi^\mu$[18]. If we use $\bar{\chi}$ in the vertex operator to integrate over the super modulus $\varsigma^a$ we get a factor of $V(z_a)$ with $\bar{\partial}X^\mu$ in $V$ replaced by $\frac{1}{2}\psi^\mu$. On the other hand if we use the $\varsigma^a$ integration to bring down a factor of $-\psi^\mu\partial X^\mu(z_a)$ from $e^{-S}$ ( which is ultimately included in $Y$) then the contraction of $\bar{\partial}X^\mu$ in $V$ with $\partial X^\mu(z_a)$ gives a factor of $-V(z_a)$ again with $\bar{\partial}X^\mu(z_a)$ in $V$ replaced by $\frac{1}{2}\psi^\mu$. Thus these two factors cancel each other.

Thus, we can forget about both the contact terms and the corrections to the vertex operator if we restrict the $x_k$ integration in (5.1) to lie outside circles of radius $\epsilon$ around $z_a$, and then take the $\epsilon \to 0$ limit.

*Ambiguity in $R^{10}$*

Let us now study the ambiguity in the scattering amplitude. If we move $z_1$, then, since the vertex operators $c\bar{c}V$ are BRST invariant we see that the ambiguity is a total derivative

$$\partial_I\langle\prod_{k=1}^{n} c\bar{c}V_{-1}(x_k)\partial\xi(z_1)\Delta z_1 D_I \prod_{a\neq 1}^{2g-2+n}(Y(z_a)+\partial_i\xi(z_a)D_i+\partial_{x_k}\xi(z_a)D_{x_k})$$
$$\prod_{j=1}^{6g-6}(\eta_j,b)\prod_{k=1}^{n}(\eta_{x_k},b)(\eta_{\bar{x}_k},\bar{b})\rangle \tag{5.6}$$

[18] With our normalization convention $\bar{\partial}X^\mu + \frac{1}{2}\bar{\chi}\psi^\mu$ and $\bar{\partial}\psi^\mu - \frac{1}{2}\bar{\chi}\partial X^\mu$ give covariantized forms of $\bar{\partial}X^\mu$, $\bar{\partial}\psi^\mu$ respectively.

where the index $I$ runs over $x_k$ and $\imath$.

Since we have defined the $x_k$ integration by cutting out small holes around the points of insertion of the $Y$'s, we first need to worry about whether the total derivative terms may receive any contribution from these unphysical boundaries after integration over the moduli. In order to get a non-vanishing contribution, the integrand must diverge as $(\bar{x}_k - \bar{z}_a)^{-1}$ near these points. However, by examining the relevant correlators one can verify that there are no such singularities ( see comment in footnote 14). Thus the only contribution comes from the physical boundaries of the moduli space of punctured riemann surfaces. Again, as before, one can show that there is no contribution from $\Delta_0$, hence the only contribution comes from the region where the original punctured surface breaks up into two punctured surfaces $S_1$ and $S_2$. Furthermore, if each of these surfaces carries more than one external vertex operator, then the corresponding boundary term as $t \to 0$ has a factor of $(t\bar{t})^{(\sum_{i\in S_1} k_i)^2}$ and may be made to vanish by analytic continuation in momenta. (Note that we might need to make different analytic continuation at different boundaries). Thus the only relevant contribution comes from the configuration where either all $n$ or $n-1$ of the vertex operators are on one surface. Let us call this surface $S_1$.

Consider first the case with all $n$ vertex operators on $S_1$. If $\partial\xi(z_1)$ lands on $S_1$ then the previous ambiguity analysis applies, and only the one-point function $\Psi^\dagger = c\bar{c}e^{-\phi}\psi\cdot\bar{\partial}X$ on $S_2$ contributes. This is the dilaton tadpole at genus $(\leq g)$, and hence vanishes. The equality sign comes since $S_1$ can be a sphere, which corresponds to the boundary where all the $x_k$'s come together. This is in fact a codimension one boundary if all the $x_k'$s approach one another with a single scale. Next we suppose that $\partial\xi(z_1)$ lands on $S_2$. The amplitude on $S_2$ is, again by the analysis leading to (3.16),

$$\begin{aligned}\langle c\partial c\bar{c}e^{-\phi}\psi\cdot\bar{\partial}X(p_2)\partial\xi(z_1)\prod_{a=1}^{2g_2-2}(Y(\hat{z}_a)+\partial_{\hat{\imath}}\xi(\hat{z}_a)D_{\hat{\imath}}+\partial_{p_2}\xi(\hat{z}_a)D_{p_2})\\ (\eta_{p_2},b)(\eta_{\bar{p}_2},\bar{b})\prod_{\hat{\jmath}=1}^{3g_2-3}(\eta_{\hat{\jmath}},b)(\eta_{\bar{\hat{\jmath}}},\bar{b})\rangle_{S_2}\end{aligned} \tag{5.7}$$

Unfortunately there seems to be no obvious reason why this is zero or even a total derivative. We will now show that even if this correlator does not integrate to zero, all need not be lost. The reason is that the correlator on $S_1$ is the $n$-point function at genus $g_1$ with a zero-momentum dilaton (in the $-1$ picture) inserted. From soft-"dilaton" "theorems"[19]we expect that this will be related to the $n$ point function $\mathcal{A}_n^{(g_1)}$ by

$$(2T\frac{\partial}{\partial T} + 10\lambda\frac{\partial}{\partial \lambda})\mathcal{A}_n^{(g_1)} \tag{5.8}$$

where $T$ is the string tension and $\lambda$ is the string loop expansion parameter. We have sketched a proof of the theorem in appendix $B$. Thus if we denote by $C^{(g_2)}$ the amplitude given in (5.7) we may express the ambiguity as,

$$\sum_{g_1} C^{(g-g_1)}(2T\frac{\partial}{\partial T} + 10\lambda\frac{\partial}{\partial \lambda})\mathcal{A}_n^{(g_1)} \tag{5.9}$$

In carrying out the analysis further, we shall assume (as stated in criterion (a) in sec. 2), that the following criterion is satisfied by the choice of points $z_a$ on the genus $g$ surface (The same criterion applies to the set of points $z_a + \Delta z_a$ corresponding to the different choice of slice): At the boundary where the genus $g$ surface with $n$ punctures degenerates into a genus $g_1$ surface $S_1$ with $n_1$ punctures and a genus $g_2$ surface $S_2$ with $n_2$ punctures, the locations of $z_a$ on $S_1(S_2)$ coincide with the choice of the points $z_a$ on a genus $g_1(g_2)$ surface we would make if we were trying to compute an $n_1 + 1(n_2 + 1)$ point amplitude on such a surface. Thus the locations of $z_a$ on $S_1(S_2)$ is a universal function of $g_1(g_2)$ and the moduli of the surface $S_1(S_2)$ and does not care about either the genus $g_2(g_1)$ or the moduli of the other surface $S_2(S_1)$. As a result, if we define $\mathcal{A}_n = \sum_g \mathcal{A}_n^{(g)}$, we may write the double sum as a product of two independent sums as follows,

$$\Delta\mathcal{A}_n = (\sum_{g_1} C^{(g_1)})(2T\frac{\partial}{\partial T} + 10\lambda\frac{\partial}{\partial \lambda})\sum_{g_2} \mathcal{A}_n^{(g_2)} \equiv (2CT\frac{\partial}{\partial T} + 10C\lambda\frac{\partial}{\partial \lambda})\mathcal{A}_n \tag{5.10}$$

[19] What we are calling here soft-"dilaton" "theorem" is what we discuss in appendix B. Again this may be a misnomer but it is in accord with our terminology in this paper of calling $\partial X\bar{\partial} X$ the dilaton (see footnote 15). Soft-dilaton theorems in bosonic strings have been discussed in [32].

As a result, the ambiguity may be absorbed into a redefinition of the string tension $T$ and the loop counting parameter $\lambda$ of the form,

$$\begin{aligned} T &\to T(1+2C) \\ \lambda &\to \lambda(1+10C) \end{aligned} \tag{5.11}$$

Let us now consider the case with $(n-1)$ vertex operators on $S_1$. We shall only consider the case when the isolated vertex operator is that of a massless particle. [20]The vertex operators that can propagate must have the same Lorentz quantum numbers. The ghost charge analysis of section three still applies, so if, *e.g.* we have a graviton vertex operator $\partial X^\mu \bar\partial X^\nu$ at $p=0$, then the operator that propagates in the tube must be $\bar c c e^{-\phi}\psi^\mu \bar\partial X^\nu$. Thus there are two cases we must consider. If $\partial\xi(z_1)$ lands on $S_1$ we obtain a two-point function of massless physical states on $S_2$ with one of the vertex operators in the $(-1)$ picture. This vanishes due to gauge invariance.[21]If $\partial\xi(z_1)$ lands on $S_2$ we obtain the amplitude $\mathcal{A}_n^{g_1}$, but with one of the vertex operators in the $(-1)$ picture. Again, this should be a sum of terms $\mathcal{A}_n^{g_1}$ with $g_1 < g$. In this case the multiplicative factor depends on which vertex operator is on $S_2$, and the resulting contribution may be absorbed into a multiplicative renormalization of the vertex operators.

From the above analysis it is also apparent how to give an inductive argument for the vanishing of massless two and three point functions as in the previous section.

*Remarks on Finiteness in Arbitrary Backgrounds*[22]

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[20] For massive particles $e^{ik\cdot X}$ has negative conformal dimension, and hence may be combined with operators of high conformal dimension to give possible candidates for $\Psi^\dagger$ of conformal dimension (0,0). This causes additional complication, and we do not yet have a complete solution of this problem.

[21] Similar contribution for massive particles will not vanish, but precisely cancel against the counterterms needed to make the vertex operators $BRST$ invariant[33].

[22] The question of multiloop finiteness in bosonic string theory in $R^{10}$ has been addressed in [34][35]

Before we conclude this section, we should mention that starting from the expression (5.1) for the $n$-point function it is easy to see that the divergence in the amplitude from the boundary of the moduli space comes solely from the physical massless tadpoles in the theory (and possibly from two point functions if there is a mass renormalization due to string loop effects). The analysis of divergence may again be done by using the factorization hypothesis. The possible operators of conformal dimension (0,0) or less, which can give rise to singularities of the form $t^{-\beta}\bar{t}^{-\beta}$ (with a factor of $t^{-1}\bar{t}^{-1}$ coming from $(\eta_t, b)(\eta_{\bar{t}}, \bar{b})$) near a boundary $\Delta_i$ $(i \geq 1)$ of the moduli space are again severely restricted by ghost charge conservation. Ghost charge conservation allows any operator of the form $\bar{c}ce^{-\phi}\tilde{V}$, where $\tilde{V}$ is any operator of dimension $(h, h+\frac{1}{2})$. For $0 \leq h \leq \frac{1}{2}$ these can lead to divergent boundary integrals. As before $G$-parity conservation forces the operator $e^{-\phi}\tilde{V}$ to be even under this symmetry in order to have a non-vanishing matrix element. However as mentioned earlier for $h < \frac{1}{2}$ these operators would correspond to physical tachyonic states which do not exist for the tachyon free backgrounds we are considering. So we only need to worry about operators with $h = \frac{1}{2}$. As we have argued, these correspond to massless physical states.

The operator $\Psi_1^\dagger$ listed in (3.16) does not contribute, since its conjugate state on the cylinder is $c_{-1}|0\rangle$, which is annihilated by $(\eta_t, b) = t^{-1}b_0$, $b_0$ being the $b$ zero mode on the cylinder[4]. Thus if we now consider the boundary where all the vertices are on one surface, the relevant matrix element on the other surface is given by the tadpole of a physical massless field, and vanishes in the absence of such tadpoles. Note that in this formalism the divergence due to the unphysical tachyon exchange[36], which corresponds to the exchange of the operator $\bar{c}c$ in our language, never appears, since it has the wrong ghost charge. Contribution from the boundaries where all but one of the vertex operators are on one surface may similarly be related to two point functions, and vanish if there is no mass renormalization in the theory. (In the presence of mass renormalization these divergences indicate that the external vertex operator is not the right one and must be modified [37][38][33] by changing its momenta so as to keep it on-shell. Also there may be extra intermediate operators contributing for external states which are massive at the string tree level, hence we restrict ourselves to massless external states.) Possible divergences from boundaries where each surface contains more than one vertex operator are eliminated by analytic continuation in the external momenta.

Let us now consider the contribution from $\Delta_0$. (The following considerations were motivated by the recent work of V. Kaplunovsky [39].) First note that if all the vertex operators lie on the genus $(g-1)$ surface sufficiently far away from the nodes $p_1$ and $p_2$, the analysis given at the end of sec.3 may be used to show that there is no divergence from this boundary. On the other hand, the situation is trickier when some of the vertex operators approach $p_1$ or $p_2$, or, equivalently, when the vertex operators lie on the neck. We shall now show that such configurations give rise to the familiar infrared divergences for $d \leq 4$. In order to carry out the analysis, it is most convenient to convert the $-1$ picture vertex operators to zero picture ones by taking the limit where the locations of some of the $Y$'s approach the locations of the vertex operators. Let us take $m$ of the $n$ vertices to lie on the neck. The operator analysis may be carried out in the same way, with the result that the leading contribution comes from the insertion of physical massless vertex operators at the nodes $p_1$ and $p_2$. But the resulting contribution to the matrix element from the neck now must include the effect of the insertion of the external legs.[23] Let $(x_1 \ln(|t|^{-1}), \theta_1)$, $((x_1 + x_2) \ln(|t|^{-1}), \theta_2) \ldots ((x_1 + x_2 + \ldots x_m) \ln(|t|^{-1}), \theta_m)$ be the coordinates of the external vertex operators in the cylindrical coordinate system with $(0 \leq x_i \leq 1)$, $\sum x_i \leq 1$, $0 \leq \theta_i \leq 2\pi$. Let $k_1, \ldots k_m$ be the momenta carried by the external vertices, and $k$ be the momentum carried by the operator inserted on the genus $(g-1)$ surface at $p_1$. Then the resulting matrix element on the cylinder may be obtained

[23] For $\Delta_i$ such an insertion will only change the matrix element on the neck but still leads to a massless tadpole on one of the surfaces and hence would disappear if the tapole matrix element vanishes.

by dividing the whole cylinder into time segments of length $x_1$, $x_2 \ldots$ carrying states with $L_0 = \bar{L}_0$ eigenvalues $\frac{1}{2}k^2$, $\frac{1}{2}(k+k_1)^2 \ldots$. Thus the net contribution is of order,

$$exp\{\ln|t|(k^2x_1 + (k+k_1)^2x_2 + \ldots + (k+\ldots+k_{m-1})^2x_m \\ +(k+\ldots+k_m)^2(1-x_1-\ldots-x_m))\} \tag{5.12}$$

The integration measure over the locations of the vertex operators is given by

$$(\ln(|t|^{-1}))^m \prod_i (dx_i d\theta_i) \qquad ,$$

whereas the beltrami differentials give the usual factors of $t^{-1}\bar{t}^{-1}$. Combining everything, the $t$ integral may be written as,

$$\int \frac{dt}{t}\frac{d\bar{t}}{\bar{t}}(\ln(|t|^{-1}))^m \\ exp\{\ln|t|(k^2x_1 + (k+k_1)^2x_2 + \ldots + (k+\ldots+k_{m-1})^2x_m + (k+\ldots+k_m)^2x_{m+1})\} \\ \sim \frac{1}{(k^2x_1 + (k+k_1)^2x_2 + \ldots + (k+\ldots k_m)^2x_{m+1})^{m+1}} \tag{5.13}$$

where we have defined $x_{m+1} = 1 - \sum x_i$. Since the $\theta_i$ integrals are trivial, the final answer may be written as,

$$\int d^dk \int dx_1 \ldots \int dx_{m+1}\delta(1-x_1-\ldots-x_{m+1}) \\ \frac{1}{(k^2x_1 + (k+k_1)^2x_2 + \ldots + (k+\ldots k_m)^2x_{m+1})^{m+1}} \tag{5.14}$$

which may be easily recognized as the Feynman integral representation of the product of $m+1$ massless propagators. As is well known, for $d \leq 4$, integration over $k$ in general leads to infrared and collinear divergences. As we can see, string theory is not free from such difficulties.

## 6. The ambiguity in type II strings

The ambiguity in this case is technically more complicated, but the same techniques should show whether the ambiguity is related to physical tadpoles or not. We have not done that. Here we shall only present the details of the calculation which show in all backgrounds that the ambiguity at genus 2 vanishes.

For the superstring the formula analogous to (3.2) for the partition function at $g = 2$ is,

$$W \sim \int (\prod_{i=1}^{6} dm_i) \int D[XBC] e^{-S_0} \prod_{a=1}^{4} (\check{Y}(z_a) + \partial_i \check{\xi}(z_a) D_i) \prod_{k=1}^{6} (\eta_k, b) \tag{6.1}$$

where $z_a$ are the basis for the gravitino fields defined as,

$$\begin{aligned} \chi_{\bar{z}}^{+} &= \sum_{a=1}^{2} \varsigma^a \delta^{(2)}(z - z_a) \\ \chi_{\bar{z}}^{-} &= \sum_{a=3}^{4} \varsigma^a \delta^{(2)}(z - z_a) \end{aligned} \tag{6.2}$$

and $\check{Y}(z_a)(\check{\xi}(z_a))$ stands for $Y(z_a)(\xi(z_a))$ for $a = 1, 2$, and $\bar{Y}(z_a)(\bar{\xi}(z_a))$ for $a = 3, 4$. Using the same trick we may compute the change in $W$ under a shift $\Delta z_1$ in $z_1$:

$$\Delta W \sim \int \prod_{i=1}^{6} dm_i \sum_{j=1}^{6} \frac{\partial}{\partial m_j} N^j \tag{6.3}$$

where,

$$N^j = \int D[XBC] e^{-S_0} \partial \xi(z_1) \Delta z_1 D_j \prod_{a=2}^{4} (\check{Y}(z_a) + \partial_i \check{\xi}(z_a) D_i) \prod_{k=1}^{6} (\eta_k, b) \tag{6.4}$$

Again $N^j$ receives contribution from the boundary $\Delta_1$. We calculate the boundary contribution taking $z_1$ and $z_3$ to lie on the torus $T_1$ and $z_2$ and $z_4$ to lie on the torus $T_2$ in this limit. In order to get non-vanishing boundary contribution, $N^t$ must be of order $\bar{t}^{-1}$ and/or $N^{\bar{t}}$ must be of order $t^{-1}$. Using various ghost charge conservation, we may write,

$$\begin{aligned} N^t \sim & \langle \partial \xi(z_1) \Delta z_1 Y(z_2) \{ \bar{Y}(z_3) \bar{Y}(z_4) + \bar{Y}(z_3) \bar{\partial} \bar{\xi}(z_4) \partial_{\bar{t}} \bar{z}_4 D_{\bar{t}} \\ & + \bar{Y}(z_4) \bar{\partial} \bar{\xi}(z_3) \partial_{\bar{t}} \bar{z}_3 D_{\bar{t}} \} \prod_{k \neq t} (\eta_k, b) \rangle \end{aligned} \tag{6.5}$$

and,

$$N^{\bar{t}} \sim \langle \partial\xi(z_1)\Delta z_1 Y(z_2)\{\bar{Y}(z_3)\bar{\partial}\bar{\xi}(z_4)\partial_\imath \bar{z}_4 D_\imath + \bar{Y}(z_4)\bar{\partial}\bar{\xi}(z_3)\partial_\imath \bar{z}_3 D_\imath\} \prod_{k\neq\bar{t}} (\eta_k, b)\rangle \tag{6.6}$$

where we have chosen complex coordinates $m_\imath$, $m_{\bar{\imath}}$ in the moduli space, and have chosen the beltrami differentials in such a way that $\eta_{\imath z}{}^{\bar{z}} = \eta_{\bar{\imath}\bar{z}}{}^{z} = 0$.

We shall now analyze each term in $N^t$ and $N^{\bar{t}}$ using the factorization hypothesis. Let us start with $N^{\bar{t}}$. The relevant terms are,

$$N_1^{\bar{t}} = t^{h_\Phi}\bar{t}^{\bar{h}_\Phi}\langle \partial\xi(z_1)\Delta z_1 \bar{Y}(z_3)(\eta_1, b)(\eta_{\bar{1}}, b)\Phi(p_1)\rangle_{T_1} \langle \Phi^\dagger|\Psi\rangle_C \langle \Psi^\dagger(p_2) Y(z_2)\bar{\partial}\bar{\xi}(z_4)(\eta_2, b)(\eta_{\bar{2}}, b)\rangle_{T_2}\partial_t \bar{z}_4 \tag{6.7}$$

We need $\Phi$ to have a dimension (-1,0) in order to get an order $t^{-1}$ contribution. But conservation of ghost charge on $T_1$ forces $\Phi$ to contain holomorphic ghost fields $c\eta$, which has $h_\Phi = 0$. Thus there is no net boundary contribution from this term.

$$N_2^{\bar{t}} = t^{h_\Phi - 1}\bar{t}^{\bar{h}_\Phi}\langle \partial\xi(z_1)\Delta z_1 \bar{Y}(z_3)(\eta_1, b)(\eta_{\bar{1}}, b)\Phi(p_1)\rangle_{T_1} \langle \Phi^\dagger|b_0|\Psi\rangle_C \langle \Psi^\dagger(p_2) Y(z_2)\bar{\partial}\bar{\xi}(z_4)(\eta_{\bar{2}}, b)\rangle_{T_2}\partial_{\tau_2}\bar{z}_4 \tag{6.8}$$

We now need $(h_\Phi, \bar{h}_\Phi) = (0,0)$. Ghost charge conservation forces this operator to be $\Psi^\dagger = c\partial\xi e^{-2\phi}\bar{c}\bar{\eta}$. The relevant holomorphic ghost correlator on the torus $T_2$ for spin structure $\nu$ is then,

$$\langle c(p_2)\partial\xi(p_2)e^{-2\phi(p_2)}Y(z_2)\rangle_\nu \propto \Big( \langle c(p_2)\partial b(z_2)\rangle \langle \partial\xi(p_2)e^{-2\phi(p_2)}e^{2\phi(z_2)}\eta(z_2)\rangle_\nu + \langle c(p_2)b(z_2)\rangle \Xi \langle \partial\xi(p_2)e^{-2\phi(p_2)}e^{2\phi(z_2')}\eta(z_2)\rangle_\nu \Big) \tag{6.9}$$

where,

$$\Xi \equiv \lim_{z_2'\to z_2}\left(\frac{\partial}{\partial z_2'} + 2\frac{\partial}{\partial z_2}\right) \tag{6.10}$$

was introduced in ref.[4]. The first term in (6.9) vanishes due to the absence of a $b$ zero mode on the torus. The second term is proportional to,

$$\Xi \frac{\vartheta_\nu(2z_2' - 2z_2)}{\vartheta_\nu^2(2z_2' - z_2 - p_2)} \frac{(\vartheta_1(z_2' - p_2))^4}{(\vartheta_1(z_2 - p_2))^2} = 0 \tag{6.11}$$

This can be seen to vanish in an even spin structure $\nu$ by explicit computation. (The contribution from the odd spin structure vanishes due to the free fermion zero modes). Thus there is no net contribution from this term. [24]

$$\begin{aligned} N_3^{\bar{t}} =& t^{h_\Phi - 1}\bar{t}^{\bar{h}_\Phi}\langle \partial\xi(z_1)\Delta z_1 \bar{Y}(z_3)(\eta_{\bar{1}}, b)\Phi(p_1)\rangle_{T_1} \\ & \langle \Phi^\dagger | b_0 | \Psi\rangle_C \langle \Psi^\dagger(p_2) Y(z_2) \bar{\partial}\bar{\xi}(z_4)(\eta_2, b)(\eta_{\bar{2}}, b)\rangle_{T_2} \partial_{\tau_1}\bar{z}_4 \end{aligned} \tag{6.12}$$

We need $(h_\Phi, \bar{h}_\Phi) = (0,0)$. Ghost charge conservation, and the requirement of absorbing the $b, c$ zero modes on the torus $T_1$ constrains the holomorphic part of $\Phi(p_1)$ to contain $: bc : \eta$, which has $h_\Phi = 2$. Hence there is no contribution from this term.

$$\begin{aligned} N_4^{\bar{t}} =& t^{h_\Phi}\bar{t}^{\bar{h}_\Phi}\langle \partial\xi(z_1)\Delta z_1 \bar{\partial}\bar{\xi}(z_3)(\eta_1, b)(\eta_{\bar{1}}, b)\Phi(p_1)\rangle_{T_1} \\ & \langle \Phi^\dagger | \Psi\rangle_C \langle \Psi^\dagger(p_2) Y(z_2) \bar{Y}(z_4)(\eta_2, b)(\eta_{\bar{2}}, b)\rangle_{T_2} \partial_t\bar{z}_3 \end{aligned} \tag{6.13}$$

We need $(h_\Phi, \bar{h}_\Phi) = (-1,0)$. But ghost charge conservation on $T_1$ gives $\Phi \sim: c\eta :$ which has $h_\Phi = 0$. Thus this term cannot give rise to any boundary contribution.

$$\begin{aligned} N_5^{\bar{t}} =& t^{h_\Phi - 1}\bar{t}^{\bar{h}_\Phi}\langle \partial\xi(z_1)\Delta z_1 \bar{\partial}\bar{\xi}(z_3)(\eta_1, b)(\eta_{\bar{1}}, b)\Phi(p_1)\rangle_{T_1} \\ & \langle \Phi^\dagger | b_0 | \Psi\rangle_C \langle \Psi^\dagger(p_2) Y(z_2) \bar{Y}(z_4)(\eta_{\bar{2}}, b)\rangle_{T_2} \partial_{\tau_2}\bar{z}_3 \end{aligned} \tag{6.14}$$

---

[24] Since the identity (6.11) plays a crucial role in our analysis, we shall give an intuitive understanding of this identity. Note that $c\partial\xi e^{-2\phi(p_2)}$ is the BRST invariant inverse picture changing operator [40]. The shift in (6.9) under a shift in the point $z_2$ may then be shown to vanish by expressing $Y$ as $\{Q_B, \xi\}$ and deforming the BRST contour. This shows that (6.9) is independent of the location of $z_2$. (6.9) may then be shown to vanish by taking the $z_2 \to p_2$ limit, and studying the operator product expansion. A generalization of this argument might be useful in the study of the higher genus partition function.

We need $(h_\Phi, \bar{h}_\Phi) = (0,0)$. A possible operator is $\Psi^\dagger = c\partial\xi e^{-2\phi}\bar{c}\bar{\partial}\bar{c}\bar{\partial}\bar{\xi}e^{-2\bar{\phi}}$. The relevant correlator on $T_2$ in the holomorphic sector is $\langle c(p_2)\partial\xi(p_2)e^{-2\phi(p_2)}Y(z_2)\rangle$. This is precisely what appears in the left hand side of (6.9) and was shown to vanish.

$$N_6^{\bar{t}} = t^{h_\Phi - 1}\bar{t}^{\bar{h}_\Phi}\langle\partial\xi(z_1)\Delta z_1\bar{\partial}\bar{\xi}(z_3)(\eta_{\bar{1}}, b)\Phi(p_1)\rangle_{T_1}$$
$$\langle\Phi^\dagger|b_0|\Psi\rangle_C\langle\Psi^\dagger(p_2)Y(z_2)\bar{Y}(z_4)(\eta_2, b)(\eta_{\bar{2}}, b)\rangle_{T_2}\partial_{\tau_1}\bar{z}_3 \qquad (6.15)$$

We need $(h_\Phi, \bar{h}_\Phi) = (0,0)$. But ghost charge conservation, and the requirement of absorbing $b$, $c$ zero modes on $T_1$ forces $\Phi$ to contain $:bc:\eta$, which has $h_\Phi = 2$. Hence the contribution vanishes.

Let us now turn to $N^t$. This contains the following terms,

$$N_1^t = t^{h_\Phi}\bar{t}^{\bar{h}_\Phi - 1}\langle\partial\xi(z_1)\Delta z_1\bar{Y}(z_3)(\eta_1, b)\Phi(p_1)\rangle_{T_1}$$
$$\langle\Phi^\dagger|\bar{b}_0|\Psi\rangle_C\langle\Psi^\dagger(p_2)Y(z_2)\bar{\partial}\bar{\xi}(z_4)(\eta_2, b)(\eta_{\bar{2}}, b)\rangle_{T_2}\partial_{\bar{\tau}_1}\bar{z}_4 \qquad (6.16)$$

We need $(h_\Phi, \bar{h}_\Phi) = (0,0)$. Conservation of ghost charge on $T_1$ gives $\Phi = c\eta\bar{c}\bar{\partial}\bar{\xi}e^{-2\bar{\phi}}$. The relevant antiholomorphic ghost correlator on $T_1$ vanishes by eqs.(6.9)-(6.11).

$$N_2^t = t^{h_\Phi}\bar{t}^{\bar{h}_\Phi - 1}\langle\partial\xi(z_1)\Delta z_1\bar{Y}(z_3)(\eta_1, b)(\eta_{\bar{1}}, b)\Phi(p_1)\rangle_{T_1}$$
$$\langle\Phi^\dagger|\bar{b}_0|\Psi\rangle_C\langle\Psi^\dagger(p_2)Y(z_2)\bar{\partial}\bar{\xi}(z_4)(\eta_2, b)\rangle_{T_2}\partial_{\bar{\tau}_2}\bar{z}_4 \qquad (6.17)$$

We need $(h_\Phi, \bar{h}_\Phi) = (0,0)$. Conservation of ghost charge, and the requirement of absorbing the $\bar{b}$, $\bar{c}$ zero modes on $T_2$ forces $\Psi^\dagger$ to contain $:\bar{b}\bar{c}:\bar{\eta}$ which has $\bar{h}_\Phi = 2$. Hence there is no boundary contribution from this term.

$$N_3^t = t^{h_\Phi}\bar{t}^{\bar{h}_\Phi}\langle\partial\xi(z_1)\Delta z_1\bar{Y}(z_3)(\eta_1, b)(\eta_{\bar{1}}, b)\Phi(p_1)\rangle_{T_1}$$
$$\langle\Phi^\dagger|\Psi\rangle_C\langle\Psi^\dagger(p_2)Y(z_2)\bar{\partial}\bar{\xi}(z_4)(\eta_2, b)(\eta_{\bar{2}}, b)\rangle_{T_2}\partial_{\bar{t}}\bar{z}_4 \qquad (6.18)$$

We now need $(h_\Phi, \bar{h}_\Phi) = (0,-1)$. Ghost charge conservation on $T_2$ forces the antiholomorphic part of $\Psi^\dagger$ to contain $\bar{c}\bar{\eta}$, which has $\bar{h}_\Phi = 0$. Hence the contribution from this term vanishes.

$$
\begin{aligned}
N_4^t =& t^{h_\Phi}\bar{t}^{\bar{h}_\Phi - 1}\langle \partial\xi(z_1)\Delta z_1 \bar{\partial}\bar{\xi}(z_3)(\eta_1, b)\Phi(p_1)\rangle_{T_1} \\
& \langle \Phi^\dagger|\bar{b}_0|\Psi\rangle_C \langle \Psi^\dagger(p_2)Y(z_2)\bar{Y}(z_4)(\eta_2,b)(\eta_{\bar{2}},b)\rangle_{T_2}\partial_{\bar{\tau}_1}\bar{z}_3
\end{aligned}
\tag{6.19}
$$

We need $(h_\Phi, \bar{h}_\Phi) = (0,0)$. Ghost charge conservation and the requirement of absorbing the $\bar{b}$, $\bar{c}$ zero modes on $T_1$ forces $\Phi$ to contain $:\bar{b}\bar{c}:\bar{\eta}$ which has $\bar{h}_\Phi = 2$. Hence there is no contribution.

$$
\begin{aligned}
N_5^t =& t^{h_\Phi}\bar{t}^{\bar{h}_\Phi - 1}\langle \partial\xi(z_1)\Delta z_1 \bar{\partial}\bar{\xi}(z_3)(\eta_1, b)(\eta_{\bar{1}},b)\Phi(p_1)\rangle_{T_1} \\
& \langle \Phi^\dagger|\bar{b}_0|\Psi\rangle_C \langle \Psi^\dagger(p_2)Y(z_2)\bar{Y}(z_4)(\eta_2,b)\rangle_{T_2}\partial_{\bar{\tau}_2}\bar{z}_3
\end{aligned}
\tag{6.20}
$$

We need $(h_\Phi, \bar{h}_\Phi) = (0,0)$. Ghost charge conservation on $T_2$ gives $\Psi^\dagger = c\partial c e^{-2\phi}\partial\xi\bar{c}e^{-2\bar{\phi}}\bar{\partial}\bar{\xi}$. The resulting antiholomorphic ghost correlator on $T_2$ vanishes by eqs.(6.9)-(6.11).

$$
\begin{aligned}
N_6^t =& t^{h_\Phi}\bar{t}^{\bar{h}_\Phi}\langle \partial\xi(z_1)\Delta z_1 \bar{\partial}\bar{\xi}(z_3)(\eta_1, b)(\eta_{\bar{1}},b)\Phi(p_1)\rangle_{T_1} \\
& \langle \Phi^\dagger|\Psi\rangle_C \langle \Psi^\dagger(p_2)Y(z_2)\bar{Y}(z_4)(\eta_2,b)(\eta_{\bar{2}},b)\rangle_{T_2}\partial_{\bar{t}}\bar{z}_3
\end{aligned}
\tag{6.21}
$$

We need $(h_\Phi, \bar{h}_\Phi) = (0,-1)$. Ghost charge conservation on $T_1$ forces $\Phi$ to contain $\bar{c}\bar{\eta}$ which has $\bar{h}_\Phi = 0$. Hence there is no contribution.

$$
\begin{aligned}
N_7^t =& t^{h_\Phi}\bar{t}^{\bar{h}_\Phi - 1}\langle \partial\xi(z_1)\Delta z_1 \bar{Y}(z_3)(\eta_1, b)(\eta_{\bar{1}},b)\Phi(p_1)\rangle_{T_1} \\
& \langle \Phi^\dagger|\bar{b}_0|\Psi\rangle_C \langle \Psi^\dagger(p_2)Y(z_2)\bar{Y}(z_4)(\eta_2,b)(\eta_{\bar{2}},b)\rangle_{T_2}
\end{aligned}
\tag{6.22}
$$

We need $(h_\Phi, \bar{h}_\Phi) = (0,0)$. If we pick up the part of $\bar{Y}(\bar{z}_3)$ proportional to $\bar{c}\bar{\partial}\bar{\xi}$, ghost charge conservation on $T_1$ forces $\Phi$ to contain $\bar{\eta}$, and hence $\bar{h}_\Phi = 1$. On the other hand, if we pick up the part of $\bar{Y}(z_3)$ proportional to $e^{2\bar{\phi}}$, then the part of $\bar{Y}(z_4)$ which contribute is $\bar{c}\bar{\partial}\bar{\xi}$, and the same argument forces $\Psi^\dagger$ to contain $\bar{\eta}$, giving $\bar{h}_\Phi = \bar{h}_{\Psi^\dagger} = 1$. Thus the only part of $\bar{Y}(z_3)$ and $\bar{Y}(z_4)$ which may contribute is the $e^{\bar{\phi}}\bar{T}_F^X$ term. Consequently $\Phi$ must have the form $\bar{c}c\eta e^{-\bar{\phi}}u$, where $u$ is a dimension $(0, \frac{1}{2})$ operator in the matter sector.

The relevant correlator on $T_1$ is now,

$$\langle \partial\xi(z_1) e^{\bar{\phi}(z_3)} \bar{T}_F^X(z_3)(\eta_1, b)(\eta_{\bar{1}}, b) e^{-\bar{\phi}(p_1)} c(p_1)\eta(p_1)\bar{c}(p_1)u(p_1)\rangle \tag{6.23}$$

Let us concentrate on an even spin structure $\nu$, since the contribution from an odd spin structure vanishes due to the free fermion zero modes. By studying the operator product of $e^{\bar{\phi}}\bar{T}_F^X$ with $e^{-\bar{\phi}}u$ one can see that there is no singularity at $z_3 = p$. Thus the only singularity in (6.23) as a function of $\bar{z}_3$ is at the zero of $\vartheta_\nu(z_3 - p_1)^*$, since $\langle e^{\phi(z_3)} e^{-\phi(p_1)}\rangle \sim \frac{\vartheta_1(z_3 - p_1)}{\vartheta_\nu(z_3 - p_1)}$. Furthermore, $e^{\bar{\phi}}\bar{T}_F^X$ must be periodic in all spin structures. Since the only periodic function on the torus with a single pole is a constant, we conclude that (6.23) is independent of $\bar{z}_3$. Hence we may take $z_3 \to p_1$ limit in (6.23) and get,

$$\langle \partial\xi(z_1)(\eta_1, b)(\eta_{\bar{1}}, b) c(p_1)\eta(p_1)\bar{c}(p_1)U(p_1)\rangle \tag{6.24}$$

where,

$$U(p_1) = \lim_{z \to p_1} e^{\bar{\phi}(z)} \bar{T}_F^X(z) e^{-\bar{\phi}(p_1)} u(p_1) \tag{6.25}$$

Eq.(6.24) now has exactly the same form as the heterotic string theory (see eq.(3.11)) and may be shown to vanish in the same way. [25]This concludes our analysis for the type II string at genus two.

At higher genus there are many more terms in the type II analysis. (Note that the similar ambiguity analysis in the heterotic case at $g = 2$ would only involve three terms, compared to thirteen in the type II case.) There will be analogs of the terms appearing in (6.8) which must be treated carefully. We believe that again in this case an iterative scheme may be developed relating the ambiguity to lower genus tadpoles, but we have not done this.

---

[25] In this case, in computing $\langle U(z)\rangle'_\nu$ we fix the spin structure on the fermionic component of the holomorphic stress tensor to be $\nu$, and sum over all spin structures for the antiholomorphic stress tensor.

## 7. Conclusion

There are three results in this paper. First, we have shown that within a class of gauge slices, the heterotic string partition function in arbitrary background remains invariant under continuous deformations of the gauge slice, provided the background satisfies the quantum equations of motion order by order in string perturbation theory. Second we have shown, working within this special class of gauges, that the partition function for the heterotic string in flat space-time vanishes to all orders in string perturbation theory. Third, for the $n$-point function we have seen that the ambiguity can be absorbed in a redefinition of the string coupling constant and string tension. The last result is somewhat surprising and a deeper understanding of it and its implications is clearly needed.

We have seen in our analysis that there is a very intricate relation between higher genus amplitudes and those at lower genera. Unless the equations of motion are satisfied *order by order* in string perturbation theory we cannot even define vacuum amplitudes and $n$-point functions unambiguously. This is yet another indication of the difficulties that arise in attempts to use the perturbative formulation of string theory to give a background independent formulation of strings. We expect, however, that this difficulty will be resolved by using a Fischler-Susskind mechanism [41],*i.e.*, if the string equations of motion are not satisfied order by order in perturbation theory, but rather as a result of cancellation between different orders, the corresponding ambiguity will (presumably) also cancel between different orders. This is quite plausible in view of the fact that the ambiguity comes from an obstruction to deforming the $BRST$ contour, and the Fischler-Susskind mechanism precisely implements cancellation of $BRST$ anomaly between different orders in perturbation theory. We leave the explicit demonstration of this phenomenon to future work.

It is worth pointing out that we have used an inductive argument to relate the cosmological constant at higher loops to that at one loop. As is well-known, all closed oriented string theories, both bosonic and fermionic, have vanishing zero-loop cosmological constant. On the other hand, nonsupersymmetric strings almost always have nonvanishing one-loop cosmological constant. Although the proof relating the $g$-loop cosmological constant to that at one-loop makes essential use of the spacetime supersymmetry current, it is tempting to speculate that there might be some kind of theorem with the flavor of the Adler-Bardeen theorem for anomalies: Under appropriate circumstances the vanishing of the cosmological constant would be guaranteed by the vanishing of the one-loop cosmological constant. Specifically, the catoptric effect–that the measure is a total derivative with residues which are themselves total derivatives of the same kind–might occur for certain nonsupersymmetric backgrounds. The realization of such a possibility would be interesting since there exists a mechanism for the vanishing of the one, two, and three-loop cosmological constant of a nonsupersymmetric theory, but this mechanism seems to have no natural extension beyond three loops [42]. A combination of these ideas with the catoptric phenomenon might yield consistent nonsupersymmetric strings to all orders of perturbation theory.


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## Appendix A.

In this appendix we discuss the term that appeared in the analysis of case IV.(b) (same analysis also applies to term in IV.(d)) in the text in the calculation of the heterotic string partition function, and show how it vanishes. In this case, the relevant correlator on $S_1$ is,

$$\begin{aligned}&\langle(\bar{c}c\partial X^\mu\bar{\partial}X^\mu)(p_1)\xi(z_1)(Y(z_2)+\partial_\iota\xi(z_2)D_\iota+\partial_{p_1}\xi(z_2)D_{p_1})\\&\prod_{a=3}^{2g_1-1}(Y(z_a)+\partial_\iota\xi(z_a)D_\iota)(\eta_{p_1},b)(\eta_{\bar{p}_1},\bar{b})\prod_\iota(\eta_\iota,b)(\eta_{\bar{\iota}},\bar{b})\rangle\end{aligned}\tag{A.1}$$

where, for definiteness, we have taken $z_2$ to be the one which depends on $p_1$, all the other $z_a$'s are independent of $p_1$. Using the standard trick of removing the factor of $(\eta_{p_1}, b)$ from the correlator, this expression may be written as,

$$\langle\Big\{ -\partial X^\mu \bar\partial X^\mu(p_1)(Y(z_2)+\partial_\imath\xi(z_2)D_\imath) + c(p_1)\partial X^\mu\bar\partial X^\mu(p_1)\partial_{p_1}\xi(z_2)\Big\}\xi(z_1) \prod_{a=3}^{2g_1-1}(Y(z_a)+\partial_\imath\xi(z_a)D_\imath)\prod_\imath(\eta_\imath,b)(\eta_{\bar\imath},\bar b)\rangle \tag{A.2}$$

We may now use the expression (3.5) for $Y(z_2)$, and deform the BRST contour away from $z_2$ around $z_1$. In this process we pick up total derivative terms due to non-vanishing commutator of $Q_B$ with $(\eta_\imath, b)$ etc. The final answer is,

$$\begin{aligned} &-\langle\partial X^\mu\bar\partial X^\mu(p_1)\xi(z_2)(Y(z_1)+\partial_\imath\xi(z_1)D_\imath)\prod_{a=3}^{2g_1-1}(Y(z_a)+\partial_\imath\xi(z_a)D_\imath)\prod_\imath(\eta_\imath,b)(\eta_{\bar\imath},\bar b)\rangle \\ &+\frac{\partial}{\partial m^\imath}\tilde N^\imath \\ &+\frac{\partial}{\partial p_1}\langle((c\partial X^\mu+\gamma\psi^\mu)\bar\partial X^\mu)(p_1)\xi(z_1)\xi(z_2) \prod_{a=3}^{2g_1-1}(Y(z_a)+\partial_\imath\xi(z_a)D_\imath)\prod_\imath(\eta_\imath,b)(\eta_{\bar\imath},\bar b)\rangle \end{aligned} \tag{A.3}$$

where,

$$\tilde N^\jmath \sim \langle\partial X^\mu\bar\partial X^\mu(p_1)\xi(z_2)\xi(z_1)D_\jmath\prod_{a=3}^{2g_1-1}(Y(z_a)+\partial_\imath\xi(z_a)D_\imath)\prod_\imath(\eta_\imath,b)(\eta_{\bar\imath},\bar b)\rangle \tag{A.4}$$

Consider first the total derivative with respect to $p_1$. This term can receive a boundary contribution from regions where there are poles of the form $(\bar p_1 - \bar p_1^{(0)})^{-1}$. However, no such poles exist. There are no contact terms since the singular points have been removed. The first term in (A.3) is the standard dilaton tadpole, and hence vanishes. The second term might become singular on the boundary. Let us consider the contribution from the boundary where $S$ breaks up into two surfaces $\tilde S_1$ and $\tilde S_2$ of genus $\tilde g_1$ and $\tilde g_2$, with nodes at $\tilde{p}_1$ and $\tilde{p}_2$ respectively. Let us define $\tilde{S}_1$ to be the surface containing $p_1$, then by our previous requirement $z_2$ must also belong to the same surface. If $z_1 \in \tilde{S}_1$, then there are $2\tilde{g}_2 - 1$ insertions of $Y$ on $\tilde{S}_2$, and the relevant operator at $\tilde{p}_2$ after factorization is $\bar{c}c\xi e^{-\phi}\psi^\mu\bar{\partial}X^\mu$. This gives a dilaton tadpole on $\tilde{S}_2$, and hence vanishes. On the other hand, if $z_1 \in \tilde{S}_2$, then there are $2\tilde{g}_1 - 1$ insertions of $Y$ on $\tilde{S}_1$, and the relevant operators inserted on $\tilde{S}_1$ at $\tilde{p}_1$ are the ones given in eqs.(4.10)(a-d).

Each of these terms may be handled in a way similar to case IV in the text. For example, for the operator $\Psi^\dagger(\tilde{p}_1) = \bar{c}ce^{-\phi}\psi^\mu\bar{\partial}X^\mu(\tilde{p}_1)$, the only zero mode of $\xi$ on $\tilde{S}_1$ comes from $\xi(z_2)$, and hence there is no dependence of the correlator on $z_2$ (and hence no intrinsic dependence on $p_1$ through $z_2$). This, in turn, means that the only dependence of the correlator on $p_1$ is through $\partial X^\mu(p_1)\bar{\partial}X^\mu(p_1)$. Hence $\int d^2p_1\partial X^\mu(p_1)\bar{\partial}X^\mu(p_1)$ may be removed from the correlator at the cost of an overall multiplicative factor. The resulting correlator on $\tilde{S}_1$ is just the dilaton tadpole. The case where $\Psi^\dagger(\tilde{p}_1)$ is given by (4.10)(b) (d) may be shown to vanish by applying the result of the text and this appendix on $\tilde{S}_2$. Finally, the case where $\Psi^\dagger(p_1)$ is given by (4.10)(c), there is no $c(\tilde{p}_2)$ insertion on $\tilde{S}_2$, and the result vanishes by the $z$ integral in $\int \eta_{\tilde{p}_2}{}^z_{\bar{z}} b_{zz} d^2z$.

### Appendix B. Soft Dilaton Theorem

In this appendix we sketch a proof of the soft dilaton theorem used in the text. Let us normalize the fields $X^\mu$ such that

$$\partial X^\mu(y)\partial X^\nu(z) \sim \frac{\delta^{\mu\nu}}{(y-z)^2} \quad \text{as} \quad y \to z \tag{B.1}$$

With this normalization,

$$\langle\langle\bar{\partial}X^\mu(y)\partial X^\nu(z)\rangle\rangle = \pi\delta^{\mu\nu}\overline{\omega_i(y)}(Im\Omega)^{-1}_{ij}\omega_j(z) \tag{B.2}$$

where in this section $\langle\langle\ \rangle\rangle$ denotes the correlator normalized by the partition function. We use the convention $d^2y \equiv \frac{i}{2\pi}dy \wedge d\bar{y}$.

The result of inserting $\int d^2y \bar{\partial} X^\mu(y) \partial X^\mu(y)$ in a correlator may be divided into two parts using Wick's theorem–one where $\bar{\partial} X^\mu(y)$ contracts with $\partial X^\mu(y)$, and the other where both $\bar{\partial} X^\mu(y)$ and $\partial X^\mu(y)$ contracts with the other fields in the correlator. The part involving the contraction of $\bar{\partial} X^\mu(y)$ with $\partial X^\mu(y)$ may be written as,

$$\pi d \int d^2y \overline{\omega_i(y)} (Im\Omega)^{-1}_{ij} \omega_j(y) \mathcal{A}^{(g)}_n = gd\mathcal{A}^{(g)}_n \tag{B.3}$$

where $d$ is the dimension of space-time and $g$ is the genus of the surface.

Let us now turn to the terms other than the one involving the self contraction of $\bar{\partial} X^\mu(y)$ with $\partial X^\mu(y)$. These terms may be written as, [26]

$$\int d^2y \partial_y \langle X^\mu(y) \bar{\partial} X^\mu(y) \prod_i V_i(x_i) \ldots \rangle \tag{B.4}$$

where ... denote the insertion of the picture changing operators and the beltrami differentials. In order to get a non-vanishing boundary contribution from (B.4) we need a pole in $y$ of the form $(\bar{y} - \bar{y}_0)^{-1}$. This can happen near an operator $\bar{\partial} X^\nu(y_0)$ or $e^{ik\cdot X(y_0)}$. Now,

$$\begin{aligned} X^\mu(y) \bar{\partial} X^\mu(y) \bar{\partial} X^\nu(y_0) &\sim X^\nu(y) \frac{1}{(\bar{y} - \bar{y}_0)^2} - \bar{\partial} X^\nu(y) \frac{1}{(\bar{y} - \bar{y}_0)} \\ &\sim X^\nu(y_0) \frac{1}{(\bar{y} - \bar{y}_0)^2} + \text{non} - \text{singular terms} \end{aligned} \tag{B.5}$$

Thus there is no single pole near an operator $\bar{\partial} X^\nu(y_0)$ (or any of its derivatives) and hence (B.4) does not receive any boundary contribution from operators of this form. On the other hand, [27]

$$X^\mu(y) \bar{\partial} X^\mu(y) e^{ik\cdot X(y_0)} \sim \frac{ik \cdot X(y_0)}{(\bar{y} - \bar{y}_0)} e^{ik\cdot X(y_0)} + \text{non} - \text{singular terms} \tag{B.6}$$

[26] Note that the rule of integration over $y$ (which was originally the location of the puncture) is to cut out small holes around the locations of the vertex operators and the picture changing operators, and to restrict the region of integration to be outside these holes. This can be traced to the fact for any finite $t$ none of the vertex operators or the picture changing operators can actually coincide with the puncture, they are always a distance $|t|^{\frac{1}{2}}$ away from the puncture. As a result we do not have to worry about delta-function contact terms.

[27] The operator $ik \cdot X(y_0) e^{ik\cdot X(y_0)}$ may be defined as $k^\mu \frac{\partial}{\partial k^\mu} e^{ik\cdot X(y_0)}$.

Thus (B.4) may be written as,

$$\begin{aligned}\sum_i \oint_{|y-x_i|=\epsilon} (-\frac{d\bar{y}}{2\pi i})\langle\frac{ik_i \cdot X(x_i)}{(\bar{y}-\bar{x}_i)} \prod_j V_j(x_j)\ldots\rangle \\ = -\sum_i \langle ik_i \cdot X(x_i) \prod_j V_j(x_j)\ldots\rangle \\ = -\sum_i k_i^\mu \frac{\partial}{\partial k_i^\mu} \mathcal{A}_n^{(g)}\end{aligned} \tag{B.7}$$

Let $\lambda$ be the string loop expansion parameter and $T$ the string tension. Then from dimensional analysis we get, [28]

$$\mathcal{A}_n^{(g)} = T^{\frac{d}{2}} \lambda^{g-1} f(k_i/\sqrt{T}) \tag{B.8}$$

where $f$ is some function of the momenta. Then from (B.3) and (B.7) we see that,

$$\int d^2y \langle \partial X^\mu(y) \bar{\partial} X^\mu(y) \prod_i V_i(x_i)\ldots\rangle = (d\lambda\frac{\partial}{\partial\lambda} + 2T\frac{\partial}{\partial T})\mathcal{A}_n^{(g)} \tag{B.9}$$

This proves the soft dilaton theorem.

[28] Here we are using dimensionless vertex operators, thus $\mathcal{A}_n^{(g)}$ has the same dimension as the cosmological constant in $d$ dimensions.

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