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Coherent Beamsstrahlung*

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The radiation coherently emitted by a high energy bunched beam suffering an arbitrarily large disruption in a collision with an idealized undisrupted beam is calculated. The near-luminal velocity of the beam—such that the emitted radiation moves very slowly with respect to the bunch—implies that only a small part of the bunch radiates coherently and necessitates a careful treatment of the disrupted beam phase space during emission. The angular distribution and spectral density are presented. It is found that most of the radiation is at wave lengths greater than or equal to the bunch length and that the total energy lost by the beam due to coherent effects should be negligible in high energy - high luminosity linear colliders.

Energy Loss and Angular Distribution: A sufficiently dense bunch may be expected to beamsstrahlen[1] coherently. A reasonably exact criterion for and characterization of coherent Beamsstrahlung can be obtained by studying the case of a strong-beam - weak-beam collision in which a strong beam disrupts a weak beam of identical charge distribution but is itself (artificially) undisrupted. In the high energy limit the particles of the weak beam are then subjected to transverse harmonic accelerations which are focusing in both planes. The radiation field in the direction $\mathbf{n}$ at distance R at time t is[2]

$$\mathbf{E}(R\mathbf{n},t) = \frac{Ne}{R}\int d\mathbf{p}\, d\mathbf{r}\, \rho(\mathbf{r},\mathbf{p};t')\frac{(\mathbf{n}-\mathbf{v})\mathbf{n}\cdot\dot{\mathbf{v}} - (1-\mathbf{n}\cdot\mathbf{v})\dot{\mathbf{v}}}{(1-\mathbf{n}\cdot\mathbf{v})^3}$$
$$\mathbf{B} = \mathbf{n}\times\mathbf{E} \tag{1}$$

where $\rho(\mathbf{r},\mathbf{p};t')$ is the bunch's phase space distribution at the retarded time t', $\mathbf{v} = \mathbf{p}/m\gamma$, and $\dot{v}_i = -r_i/\beta_i^2$ $(i = x, y)$, $\dot{v}_z \cong -\mathbf{v}_\perp\cdot\dot{\mathbf{v}}_\perp$. The z-axis is taken as the bunch's mean direction of motion and β_i parameterizes the strength of the electromagnetic field due to the strong bunch. Note the inclusion in the form of $\dot{v}_z$ of the leading correction to the paraxial approximation—its importance will emerge below. The retarded time is a function of the phase space variables according to $t' = \mathbf{n}\cdot\mathbf{r} + t - R$, and one may envision the integration over the latter as moving through the beam with the wave front of the accumulating radiation, taking into account the evolution of the distribution as the collision progresses. The phase space coordinates may be formally regarded as particle dynamical variables at time t' which may then be subjected to a canonical transformation relating them to their initial values. The invariance of the distribution function then allows its replacement by its value prior to the collision, so that in the special case of round bunches with zero emittance

$$\mathbf{E} = \frac{Ne}{R}\int dz_0\, db_0\, d\varphi_0\, \rho(b_0,z_0)\frac{(\mathbf{n}-\mathbf{v})\mathbf{n}\cdot\dot{\mathbf{v}} - (1-\mathbf{n}\cdot\mathbf{v})\dot{\mathbf{v}}}{(1-\mathbf{n}\cdot\mathbf{v})^3} \tag{2}$$

where

$$\mathbf{v}_\perp = -\frac{\mathbf{b}_0}{\beta}\sin\left(\frac{\tilde{t}}{\beta}\right) \quad ; \quad v_z \cong 1 - \frac{1}{2}\left(\frac{1}{\gamma^2} + \mathbf{v}_\perp{}^2\right)$$
$$\dot{\mathbf{v}}_\perp = -\frac{\mathbf{b}_0}{\beta^2}\cos\left(\frac{\tilde{t}}{\beta}\right) \tag{3}$$

in which $\tilde{t} = t' - t_i$, where $t_i(z_0)$ is the time at which the beam begins to be disrupted. For conceptual simplicity we furthermore hitherto assume uniform density hard-edged bunches, although the transition to Gaussian beams is fairly trivial. Hence for an initially longitudinally independent cross section

$$\rho(b_0,z_0) = \frac{b_0\, h(\sqrt{2}\sigma_\perp - b_0)}{2\pi\sigma_\perp^2}\frac{h(\sqrt{\pi}\sigma_s - \sqrt{2}|z_0|)}{\sqrt{2\pi}\sigma_s} \tag{4}$$

where $h(x) = \{^{1,\ x>0}_{0,\ x<0}$ is the step function. Taking the origin of time to be the instant the heads of the two beams interpenetrate, we have $t_i \cong (\sqrt{2\pi}\sigma_s - 2z_0)/4$. It is evident that $\tilde{t}$ is a more appropriate 'longitudinal' variable than z_0. The relation

$$\tilde{t} + t_i - \mathbf{n}\cdot[\mathbf{b}_0 + (\sqrt{\pi/2}\sigma_s - t_i)\hat{\mathbf{k}} + \textstyle\int_0^{\tilde{t}} dt\,\mathbf{v}] = t - R$$

implies that $(1-\mathbf{n}\cdot\mathbf{v})\,d\tilde{t} \cong -(1+\mathbf{n}\cdot\hat{\mathbf{k}})\,dt_i \cong -2\,dt_i \cong dz_0$ (in which we use the fact that we expect the angle between the radiation and the beam axis to be very small). The latter eliminates all explicit z_0 dependence in (2) and exposes a crucial physical point, *viz.*: $\tilde{t}$ will vary between 0 and $\tilde{t}_{max} \cong \sqrt{\pi/2}\sigma_s$, but $\mathbf{n}\cdot\mathbf{v} \approx 1 - \mathbf{v}_\perp{}^2/2 \approx 1$ so that the range in z_0, *i.e.*, the effective bunch length over which the beam radiates coherently, is very short— $\lesssim O\big((b_0/\beta)^2\big)$ of the total bunch length. It is also clear that this obtains since, inasmuch as $|\mathbf{v}| \to 1$, the emitted wave front is nearly co-moving with the bunch and hence moves through only a very short portion of it during the collision. We define: $\mathbf{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$, and $\mathcal{V}_\perp = \mathrm{sgn}\left(\tan\tilde{t}/\beta\right)|\mathbf{v}_\perp|$, the latter period-$\pi$ function describing the betatron oscillations occurring at a fixed value of φ_0. The factor in the denominator in (2) then becomes

$$(1-\mathbf{n}\cdot\mathbf{v}) \cong \frac{1}{2}\left[\frac{1}{\gamma^2} + (\theta - \mathcal{V}_\perp)^2 + 2\theta\mathcal{V}_\perp\big(1-\cos(\varphi_0-\varphi)\big)\right] \tag{5}$$

to leading non-trivial order in θ and $\mathcal{V}_\perp$. The radiation field originating in an infinitesimal part of the beam will thus be strongly peaked (with a width $\approx 1/\gamma$) at $\theta \approx |\mathcal{V}_\perp|$, $\cos(\varphi - \varphi_0) \approx \mathrm{sgn}(\mathcal{V}_\perp)$. The numerator vector has two independent components (polarizations). The cylindrical symmetry of a round homogeneous beam dictates that the polarization lie in the $\hat{\mathbf{k}},\mathbf{n}$ plane, and indeed

$$(\hat{\mathbf{k}}\times\mathbf{n})\cdot[\,\text{numerator}\,] \propto \sin(\varphi_0-\varphi)\, f\big(\cos(\varphi_0-\varphi)\big)$$

which plainly vanishes upon integration over φ_0, *i.e.*, radiation from particles on one side of the $\hat{\mathbf{k}},\mathbf{n}$ plane cancels that from the other. This destructive interference contrasts with the situation in which the bunch radiates incoherently—where the small-scale inhomogeneities break cylindrical symmetry and each particle radiates independently, *ipso facto* proscribing interference. The non-canceling component, in the direction $\hat{\mathbf{e}}$, follows from

$$\hat{\mathbf{e}}\cdot[\,\text{numerator}\,] = -\hat{\mathbf{k}}\cdot[\,\text{numerator}\,]/\sin\theta$$
$$\cong \left[\frac{1}{2}\left(\frac{1}{\gamma^2} - \theta^2 - \mathcal{V}_\perp{}^2\right)\cos(\varphi_0-\varphi) + \theta\mathcal{V}_\perp\right]\frac{b_0}{\beta^2}\left|\cos\frac{\tilde{t}}{\beta}\right|. \tag{6}$$

All the pieces thus assembled, the energy loss per bunch per collision can now be calculated:

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$$-\Delta E \cong \frac{1}{4\pi} \int_0^{\sqrt{2\pi}\sigma_s} d(t-R) \int d\varphi\, d\theta\, \theta \left\{ Ne \int_0^{\sqrt{\pi/2}\sigma_s} \frac{d\tilde{t}}{\sqrt{2\pi}\sigma_s} \left|\cos\frac{\tilde{t}}{\beta}\right| \right.$$

$$\left. \times \int_0^{\sqrt{2}\sigma_\perp} \frac{db_0\, b_0}{2\pi\sigma_\perp^2} \frac{b_0}{\beta^2} \int_0^{2\pi} d\varphi_0 \frac{2(\gamma^{-2} - \theta^2 - \mathcal{V}_\perp{}^2)\cos\varphi_0 + 4\theta\mathcal{V}_\perp}{[\gamma^{-2} + (\theta - \mathcal{V}_\perp)^2 + 2\theta\mathcal{V}_\perp(1-\cos\varphi_0)]^2} \right\}^2$$

$$= 16\sqrt{2\pi}\sigma_s \left(\frac{Ne\sigma_\perp}{\beta^2}\frac{\beta}{\sqrt{2\pi}\sigma_s}\right)^2 \int_0 d\theta\, \theta^3 \left\{ \int_0^{\sqrt{\pi/2}\sigma_s} \frac{d\tilde{t}}{\beta}\left|\cos\frac{\tilde{t}}{\beta}\right| \right. \quad (7)$$

$$\left. \times \int_0^{\sqrt{2}\sigma_\perp} \frac{db_0\, b_0^2}{2\sqrt{2}\sigma_\perp^3}\frac{1}{\gamma^2}\frac{4\mathcal{V}_\perp}{\left[(\gamma^{-2} + (\theta-\mathcal{V}_\perp)^2)(\gamma^{-2} + (\theta+\mathcal{V}_\perp)^2)\right]^{3/2}} \right\}^2$$

to leading order in θ, $\mathcal{V}_\perp$, $1/\gamma$, and to all orders in $\mathcal{V}_\perp\gamma$. For application to large linear colliders we are mostly interested, as we shall see, in the limit $|\mathcal{V}_\perp|\gamma \to \infty$, *i.e.*, where the typical particle deflection angle $\gg$ the opening angle with respect to the particle's velocity into which most of the radiation is emitted. In this case

$$\gamma^{-2}4\mathcal{V}_\perp\left[(\gamma^{-2} + (\theta+\mathcal{V}_\perp)^2)(\gamma^{-2} + (\theta-\mathcal{V}_\perp)^2)\right]^{-3/2} \cong \theta(\theta^2 + \gamma^{-2}/4)^{-3/2}\,\mathrm{sgn}(\mathcal{V}_\perp)\,\delta(\theta - |\mathcal{V}_\perp|) \quad (8)$$

exhibiting an infinitely strong peak in the angular distribution from a single particle at its instantaneous deflection angle–leading to a sharp cutoff in the total angular distribution, and the formula for the total energy loss

$$-\Delta E \cong 16\sqrt{2\pi}\sigma_s \left(\frac{Ne\sigma_\perp}{\beta^2}\frac{\beta}{\sqrt{2\pi}\sigma_s}\frac{\beta}{\sigma_\perp}\right)^2 \int_0^{\theta_{max}} d\theta\, \theta^5 \frac{[1-(\theta/\theta_{max})^2]^2}{(\theta^2+\gamma^{-2}/4)^3}$$

$$\cong \frac{32(Ne)^2}{\sqrt{2\pi}\sigma_s}\left[\ln\left(2\sqrt{2}\gamma\frac{\sigma_\perp}{\beta}\left|\sin\frac{\sqrt{\pi/2}\sigma_s}{\beta}\right|\right) - \frac{3}{2} + O\left(\frac{\ln\gamma}{\gamma^2}\right)\right] \quad (9)$$

where $\theta_{max} \equiv \sqrt{2}\left|\sin\sqrt{\pi/2}\sigma_s/\beta\right|\sigma_\perp/\beta$, and which is valid for $\left|\sin\sqrt{\pi/2}\sigma_s/\beta\right|\sigma_\perp/\beta \gg 1/\gamma$. It is also instructive to see the opposite extreme, *i.e.*, where $\mathcal{V}_\perp\gamma \to 0$, in which from (7)

$$-\Delta E \cong 8\sqrt{2\pi}\sigma_s\left(\frac{Ne\sigma_\perp}{\beta^2}\right)^2 \times \left(\frac{\sigma_\perp}{\sqrt{2\pi}\sigma_s}\sin^2\frac{\sqrt{\pi/2}\sigma_s}{\beta}\right)^2 \int d\theta \frac{1}{\gamma^4}\frac{\theta^3}{(\theta^2+\gamma^{-2})^6}$$

$$= \frac{\sqrt{2\pi}\sigma_s}{5}\left(\frac{Ne\sigma_\perp}{\beta^2}\right)^2\gamma^4\left(\frac{\sigma_\perp}{\sqrt{2\pi}\sigma_s}\sin^2\frac{\sqrt{\pi/2}\sigma_s}{\beta}\right)^2 \quad (10)$$

valid for $\left|\sin\sqrt{\pi/2}\sigma_s/\beta\right|\sigma_\perp/\beta \ll 1/\gamma$.

Spectral Density: The equations above already incorporate the fact that the pulse of radiation accompanying a single high energy collision has an essentially square form $\propto h(t-R) \times h(\sqrt{2\pi}\sigma_s - t + R)$, implying that $\mathbf{E}(\omega) \propto 2\sin\sqrt{\pi/2}\sigma_s\omega/\omega$, independent of θ for $\theta \ll 1$, and hence that the spectral energy density

$$-dE \cong \frac{4}{\pi}\frac{\sin^2\sqrt{\pi/2}\sigma_s\omega}{\sqrt{2\pi}\sigma_s\omega^2}\,d\omega\left[-\Delta E(\mathrm{in}(7))\right]. \quad (11)$$

Thus 90.3% of the energy is found at wavelengths $\geq \sqrt{2\pi}\sigma_s$, well in accordance with most intuition but notably different from the very much shorter and less-obvious scale over which the bunch radiates coherently.

Discussion: The focusing strength of the 'strong' beam is explicitly[3] (taking $r_e = e^2/m_e$)

$$\frac{1}{\beta^2} = \frac{4r_eN}{\gamma}\frac{1}{\sqrt{2\pi}\sigma_s}\left(\frac{1}{\sqrt{2}\sigma_\perp}\right)^2 \quad (12)$$

assuming that it is identical to the weak beam. It is useful to note in particular that then

$$\frac{\sigma_\perp}{\beta} = \sqrt{\frac{2r_eN}{\sqrt{2\pi}\sigma_s\gamma}}, \quad \sqrt{D} = \left(\frac{\pi}{2}\right)^{1/4}\frac{\sigma_s}{\beta} = \sqrt{\frac{r_eN\sigma_s}{\gamma\sigma_\perp^2}} \quad (13)$$

where D, the "disruption" parameter,[3] $\approx$ (5.6×number of betatron oscillations)^2, and to recall that therefore
$|\mathcal{V}_\perp| = |\mathbf{v}_\perp| \sim O\left(\left|\sin\sqrt{(\pi/2)^{1/2}D}\right|\sigma_\perp/\beta\right) = O\left(\theta_{max}/\sqrt{2}\right)$.
Typical values can be conveniently scaled from 'nominal' SLC parameters

$\sigma_\perp/\beta \cong 1.06\cdot10^{-3}$, $\qquad \sqrt{D} \cong .91$

which assume

$\sigma_\perp = 1.3\,\mu m$, $\quad \sigma_s = 1\,mm$, $\quad N = 5\cdot10^{10}$, $\quad \gamma \cong 10^5$.

It is probably easiest to understand (9) by considering its ratio with respect to the result obtained assuming incoherence[4]

$$-\Delta E_{inco} \cong \frac{\sqrt{2\pi}\sigma_s}{6}\left(\frac{e\sigma_\perp}{\beta^2}\right)^2 N\gamma^4\left(1 + \frac{\beta}{\sqrt{2\pi}\sigma_s}\sin\frac{\sqrt{2\pi}\sigma_s}{\beta}\right). \quad (14)$$

For nominal values of the disruption parameter

$$\frac{-\Delta E}{-\Delta E_{inco}} \cong \frac{24N}{\gamma^4}\left(\frac{1}{\sqrt{(\pi/2)^{1/2}D}}\frac{\beta}{\sigma_\perp}\right)^2 \ln(\cdots). \quad (15)$$

For the SLC $\ln(\cdots) \cong 4.04$, and the ratio $\cong .042$—and evidently would most likely get smaller as the parameters are pushed in the directions desired for large linear colliders (barring the extremely unlikely possibility that the number of particles in a bunch could be increased like the 4^{th} power of the energy). (For very large D, (15) should be multiplied by 2, and of course there are comparable corrections for $\gamma^2 \gtrsim \beta^2 m_e/\sigma_\perp\hbar$ due to the suppression of high energy photons in the incoherent spectral density.) It thus seems clear that there is no pernicious coherent enhancement of the radiation loss rate. For 'sufficiently large' N the physical interpretation of (15) is plain—a value < 1 indicates the presence of destructive interference such that the actual radiation loss is smaller than its 'incoherent part' (note that (7) in principle includes the incoherent contribution—plus 'cross' terms). The E - field due to a particular particle flips sign as the particle crosses the beam axis (*cf.* (6)), engendering considerable destructive interference (from different particles) which increases with D (more axis crossings). However, the present calculation makes the fundamental approximation of treating the bunch as a continuum—an idealization that certainly breaks down if the effective coherently-radiating bunch length, which we have seen to be a very small fraction of the total, turns out in reality to contain less than one particle. The fact that even for fairly small values of the disruption parameter, such that little destructive interference can be taking place, the value of the ratio is $\lesssim 1$, indicates that within the parameter range of interest the latter situation indeed is the case and the formulae given here actually overstate the amount of coherent radiation.

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1. A. Hofmann and E. Keil, LEP-NOTE-122(1978); J.-E. Augustin, *et al.*, *Proc. Workshop on Possibilities and Limitations of Accelerators and Detectors*, Fermilab, 1978; M. Bassetti, M. Gygi-Hanney, LEP-NOTE-221(1980).
2. Gaussian units with $c = 1$ are employed.
3. R. Hollebeek, *Nucl. Instrum. and Meth.* **184** (1981) 333.
4. The form quoted is for cylindrical bunches and arbitrary disruption parameter, and was derived by the author but is very similar to previous results.[1]