SLAC – PUB – 4445
October 1987
T

RENORMALIZATION GROUP AND STRING AMPLITUDES*

IGOR KLEBANOV

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

and

LEONARD SUSSKIND†

Physics Department
Stanford University, Stanford, California, 94305

ABSTRACT

We define β-functions for open string tachyons away from the perturbative fixed point. We show that solutions of the renormalization group fixed point equations generate open string scattering amplitudes.

Submitted to *Physics Letters B*

⋆ Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.
† Work supported by NSF PHY 812280

One of the most interesting topics of recent research in string theory has been the study of the relationship between renormalization group (RG) and string dynamics. A discovery which triggered interest in this subject was made in the course of the study of string propagation in the classical backgrounds corresponding to condensates of the massless string modes [1]. To insure consistent string propagation in such backgrounds one must insist on the conformal invariance of the underlying two-dimensional theory, i.e. the β-functions corresponding to all the couplings of the two-dimensional non-linear sigma model must vanish. In all the examples considered to date it was found that the equations resulting from setting all the β-functions to zero are equivalent to variational equations for a spacetime functional, the effective action [1,2]. More precisely, if β^i is the β-function corresponding to the coupling g^i, then

$$\beta^i = G^{ij} \frac{\delta I}{\delta g^j} \tag{1}$$

to the order in the derivative expansion that could be reached by explicit calculations. Eqn. (1) indicates that the RG flow is actually a gradient flow [3]. The string effective action I which generates the flow can be used to obtain all on-shell scattering amplitudes for the massless string modes.

Recently the study of renormalization group trajectories has been extended to include massive modes [4,5], and, in some papers, all string modes [6]. It has been conjectured that in all these cases the renormalization group flow of the two-dimensional field theory describing the string propagation in background fields is a gradient flow. The functional of all couplings I which generates the RG flow could perhaps be used as the tree level action in some formulation of the string field theory [4,6]. We should also mention the weaker form of the conjecture on the relationship between the RG and strings. It states that the solutions of the RG fixed point equations are identical to the solutions of the variational equations for some spacetime effective action. This appears to be possible even if eqn. (1) is not satisfied. Therefore this conjecture is weaker than the one that postulates the existence of the gradient flow.

In this paper we present a calculation of the β-function for the tachyon background in the bosonic open string theory. The methods we use are somewhat different from the ones used in previous literature. In ref. [4,5] the β-functions were calculated as expansions in powers of the tachyon couplings which satisfy the linearized on-shell condition. The only backgrounds accessible to such calculations are the ones that can be built by assembling a collection of physical tachyons. Therefore it appears that in ref. [4,5] the β-function was only computed at the fixed point, i.e. where it must vanish. The calculation was used to construct solutions to the string equations of motion. The goal of this paper is to construct expressions for β-functions which are valid away from the RG fixed point. Such a calculation is needed to demonstrate that the RG flow exists. In principle, the expressions we construct are explicit enough to check if the RG flow is a gradient flow. In practice, however, it appears very hard to construct G_{ij} and I explicitly and thereby prove the existence of the gradient flow. Instead, we can prove rather easily that the weaker conjecture is true. We show that the solutions of the RG fixed point equations can be used to generate the open string scattering amplitudes.

As mentioned above, all the original studies of the string β-functions have been performed for massless backgrounds and in the explicitly covariant background field method. Following ref. [4] we find it simplest to include the couplings which for on-shell momenta correspond to open string tachyons. We will then compute the β-functions in powers of the background. Our procedure will be the following. First we assume the most general RG equations for some set of couplings and integrate these equations to see what kind of flow they imply. After this is done, we restrict our attention to the particular example of open strings in the tachyon background. By studying the RG flow in this model we prove the existence of non-singular β-functions away from the perturbative critical point and in fact construct explicit expressions for these β-functions.

Let us start by writing down the most general RG equations for a set of couplings g^i:

$$\beta^i = \frac{dg^i}{dt} = \lambda_i g^i + \alpha^i_{jk} g^j g^k + \gamma^i_{jkl} g^j g^k g^l + \dots \tag{2}$$

λ^i are the anomalous dimensions, and there is no summation in the first term on the RHS (we assume that the anomalous dimension matrix has been diagonalized). If we put the infrared scale at $t = 0$, let us solve for $g^i(t)$ in terms of $g^i(0)$. To the lowest order,

$$g^i(t) = \exp(\lambda_i t) g^i(0) \tag{3}$$

To study the next order, we put

$$g^i(t) = \exp(\lambda_i t) g^i(0) + a^i_{jk}(t) g^j(0) g^k(0) \tag{4}$$

Substituting (4) into (2), we find

$$\dot{a}^i_{jk} = \alpha^i_{jk} \exp(\lambda_k t) \exp(\lambda_j t) + \lambda_i a^i_{jk} \tag{5}$$

subject to the initial condition $a^i_{jk}(0) = 0$. The solution is

$$a^i_{jk}(t) = (e^{(\lambda_j + \lambda_k)t} - e^{\lambda_i t}) \frac{\alpha^i_{jk}}{\lambda_j + \lambda_k - \lambda_i} \tag{6}$$

Let us do one more order explicitly.

$$g^i(t) = e^{\lambda_i t} g^i(0) + (e^{(\lambda_j + \lambda_k)t} - e^{\lambda_i t}) \frac{\alpha^i_{jk}}{\lambda_j + \lambda_k - \lambda_i} g^j(0) g^k(0) + b^i_{jkl}(t) g^j(0) g^k(0) g^l(0) \tag{7}$$

Substituting this into (2) we find the solution

$$\begin{aligned} b^i_{jkl}(t) g^j(0) g^k(0) g^l(0) = \Bigg(& e^{\lambda_i t} \Big(\frac{2\alpha^i_{jm} \alpha^m_{kl}}{\lambda_j + \lambda_m - \lambda_i} - \gamma^i_{jkl} \Big) \frac{1}{\lambda_j + \lambda_k + \lambda_l - \lambda_i} \\ & + \Big(\frac{2\alpha^i_{jm} \alpha^m_{kl}}{\lambda_k + \lambda_l - \lambda_m} + \gamma^i_{jkl} \Big) e^{(\lambda_j + \lambda_k + \lambda_l)t} \frac{1}{\lambda_j + \lambda_k + \lambda_l - \lambda_i} \\ & - \frac{2\alpha^i_{jm} \alpha^m_{kl}}{(\lambda_j + \lambda_m - \lambda_i)(\lambda_k + \lambda_l - \lambda_m)} e^{(\lambda_j + \lambda_m)t} \Bigg) g^j(0) g^k(0) g^l(0) \end{aligned} \tag{8}$$

We have kept $g^j(0) g^k(0) g^l(0)$ to remind that the lower indices are explicitly symmetrized in all the equations. However, quite obviously, there is no symmetry between the upper index and the lower indices. This is an inherent property of the β-functions we derive. We are missing a definition of how to raise and lower indices. If such operations are defined then the gradient flow exists.

The above calculations demonstrate that the RG equations (2) imply a definite relationship between the renormalized couplings $g^i(t)$ and the bare couplings $g^i(0)$. The presence of the 'energy denominators' in the above equations makes their appearance similar to quantum mechanical amplitudes calculated in perturbation theory. We will show in the subsequent discussion that the quantum mechanical system implied by the RG equations is nothing but the first quantized string theory.

Let us now restrict ourselves to the specific example of open strings propagating in the tachyon background [4]:

$$S = \frac{1}{4\pi\alpha'}\int\limits_{y>0} dxdy\ \eta^{ab}\partial_a X_\mu \partial_b X^\mu + \int\limits_{-\infty}^{+\infty}\frac{dx}{a}\int d^{26}k\ T(k)\exp(ik\cdot X) \qquad (9)$$

26-momenta k^μ serve as the indices labeling the couplings $T(k)$. We need to determine the relationship between the renormalized and bare couplings of the two-dimensional field theory. For that purpose we introduce the explicit cut-off a into the 2-d field theory and calculate the two-dimensional effective action [7,4] using the background field method. Let us expand X^μ around some classical background $X_0^\mu(x,y)$ which satisfies the equations of motion and varies slowly compared to the cut-off scale: $X^\mu = X_0^\mu + Y^\mu$. The effective action is

$$S_{eff}(X_0) = -\log W(X_0),$$

$$\begin{aligned} W(X_0) = &\exp(-S_0(X_0))\int[DY]\exp\Big(-\frac{1}{4\pi\alpha'}\int\limits_{y>0} dxdy\ \eta^{ab}\partial_a Y_\mu \partial_b Y^\mu - \\ &\int\limits_{-\infty}^{+\infty}\frac{dx}{a}\int d^{26}k\ T(k)\exp(ik\cdot X_0)\exp(ik\cdot Y)\Big) \end{aligned} \qquad (10)$$

In calculating $W(X_0)$ all the contractions are computed in terms of the propagator on the boundary which has a built-in cut-off:

$$\langle Y(x_1)Y(x_2)\rangle = -2\log(|x_1 - x_2| + a) \tag{11}$$

where we have set $\alpha' = 1$. This appears to be a particularly convenient way of introducing cut-off into our 2-d field theory.

$W(X_0)$ can be can be conveniently calculated by expanding the exponential in powers of $T(k)$. For example, the first non-trivial term is

$$-\int_{-\infty}^{+\infty} \frac{dx}{a} \int dk\, T(k) \exp(ik{\cdot}X_0) \langle \exp(ik \cdot Y)\rangle = -\int_{-\infty}^{+\infty} dx \int dk\, a^{k^2-1} T(k) \exp(ik{\cdot}X_0) \tag{12}$$

Therefore, to this order

$$W(X_0) \sim \exp(-S_0(X_0))(1 - \int dx \int dk\, \tilde{T}(k) \exp(ik \cdot X_0)) \tag{13}$$

It follows that

$$S_{eff}(X_0) = S_0(X_0) + \int dx \int dk\, \tilde{T}(k) \exp(ik \cdot X_0) + \ldots \tag{14}$$

The ellipsis in the above equation stands for non-local terms in the effective action, i.e. the terms that cannot be written as a single integral over the x axis. In the expansion of the logarithm such terms arise naturally as the higher powers of the local terms. To find the relation between the renormalized and bare couplings we need to focus on the local terms only. We will disregard the existence of non-local terms which are present in any calculation of the effective action. Eqn. (12) states that $\tilde{T}(k)$ is the renormalized coupling given in terms of the bare coupling by

$$\tilde{T}(k) = a^{k^2-1} T(k) \tag{15}$$

to the lowest order in perturbation theory. If we identify $1/a$ with $\exp(t)$, then the cut-off is removed in the limit $t \to \infty$. Comparing (15) with (3), we conclude that $1 - k^2$ is the anomalous dimension.

Our further calculations essentially repeat the above. The second order term in the expansion of $W(X_0)$ is

$$\int_{-\infty}^{+\infty} \frac{dx_1}{a} \int_{-\infty}^{x_1} \frac{dx_2}{a} \int d^{26}k_1\ T(k_1) \int d^{26}k_2\ T(k_2) \exp(ik_1 \cdot X_0(x_1) + ik_2 \cdot X_0(x_2))$$
$$\langle \exp(ik_1 \cdot Y(x_1)) \exp(ik_2 \cdot Y(x_2)) \rangle \tag{16}$$

If X_0 varies slowly, we can expand

$$X_0(x_2) = X_0(x_1) + (x_2 - x_1)X_0'(x_1) + \dots \tag{17}$$

If we are interested in renormalization of couplings of the form $\exp(ik \cdot X_0(x))$, we can disregard the terms involving derivatives acting on X_0. Therefore, for our purposes we can replace $\exp(ik_1 \cdot X_0(x_1) + ik_2 \cdot X_0(x_2))$ by $\exp(i(k_1 + k_2) \cdot X_0(x_1))$. This makes evaluation of (16) easy. The necessary integral is

$$a^{k_1^2+k_2^2-2} \int_{-\infty}^{x_1} dx_2\ (x_1 - x_2 + a)^{2k_1 \cdot k_2} = -a^{(k_1+k_2)^2-1} \frac{1}{2k_1 \cdot k_2 + 1} \tag{18}$$

We observe that the cut-off a scales out of all the integrals. This is a convenient feature of all our calculations. We also note that the integral converges only if $2k_1 \cdot k_2 + 1 < 0$. Repeating all the arguments needed to derive (15), we find the modified relation between the bare and renormalized couplings:

$$\tilde{T}(k) = a^{k^2-1}\left(T(k) + \int d^{26}k_1 \int d^{26}k_2\ \frac{T(k_1)T(k_2)}{2k_1 \cdot k_2 + 1} \delta^{26}(k_1 + k_2 - k)\right) \tag{19}$$

In comparing this with (6) the reader may wonder about what happened to the extra term which should depend on a as $a^{-\lambda_1-\lambda_2} = a^{k_1^2+k_2^2-2}$. The answer is the following. Since the only physically relevant quantity is the ratio of the infrared and the ultraviolet cut-offs, removing the infrared cut-off is equivalent to sending

$t \to \infty$. The integral (18) converges (i.e., the infrared cut-off can be removed) only if $2k_1 \cdot k_2 + 1 < 0$, which implies $\lambda_1 + \lambda_2 < \lambda$. This means that, as $t \to \infty$, the first term in (6) is negligible compared to the second term. We define the β-functions by analytic continuation from the region of phase space where only the term $\sim \exp(\lambda_i t)$ in the expression for the renormalized couplings as a function of the bare couplings is important. Although this choice does not allow for a comparison of the subdivergences, it is particularly convenient for calculations with our cut-off propagator. Comparing with (6), we observe that

$$2k_1 \cdot k_2 + 1 = \lambda_1 + \lambda_2 - \lambda \tag{20}$$

The correct 'energy denominator' emerges from our explicit off shell calculation! We identify the off shell β-function:

$$\alpha^k_{k_1 k_2} = -\delta^{26}(k_1 + k_2 - k) \tag{21}$$

It is clear that, in general, the off shell continuation of the beta functions depends on the particular prescription for cutting off the two-dimensional field theory. Our prescription makes $\alpha^k_{k_1 k_2}$ look particularly simple. As we proceed to show, the next order contribution is not as simple but can nevertheless be calculated in closed form. The next term in the expansion of $W(X_0)$ is

$$-\int_{-\infty}^{+\infty} \frac{dx_1}{a} \int_{-\infty}^{x_1} \frac{dx_2}{a} \int_{-\infty}^{x_2} \frac{dx_3}{a} \int dk_1\, dk_2\, dk_3\; T(k_1)T(k_2)T(k_3) \exp\big(ik_1 \cdot X_0(x_1) + ik_2 \cdot X_0(x_2) + ik_3 \cdot X_0(x_3)\big) \left\langle \exp(ik_1 \cdot Y(x_1)) \exp(ik_2 \cdot Y(x_2)) \exp(ik_3 \cdot Y(x_3)) \right\rangle \tag{22}$$

The requisite integral reduces to

$$-a^{k^2-1} \int_0^{\infty} \int_0^{\infty} dx dy (x+1)^A (y+1)^C (x+y+1)^B \tag{23}$$

where we have set $2k_1 \cdot k_2 = A$, $2k_1 \cdot k_3 = B$, $2k_2 \cdot k_3 = C$. (23) can be evaluated to yield

$$\frac{-a^{k^2-1}}{(2+A+B+C)(1+B+C)}{}_3F_2(1,-1-B-C,-C;-1-A-B-C,-B-C;1)$$
$$=\frac{-a^{k^2-1}}{2+A+B+C}\left(\frac{1}{1+B+C}+\frac{1}{1+B+A}+\ldots\right) \tag{24}$$

where the ellipsis stands for the terms which have poles at $B+C=0$, $B+A=0$, etc. The above expression is understood to be completely symmetrized in the indices 1, 2, and 3 (or equivalently in A, B, and C). In comparing it with (8), let us observe the following. Once again, we have chosen the region of phase space where only the term $\sim \exp(\lambda_i t)$ is important. Comparing the overall energy denominator, we find agreement:

$$2+A+B+C=\lambda_j+\lambda_k+\lambda_l-\lambda_i \tag{25}$$

We are now ready to read off the next order term in the expansion of the β-function:

$$\int dk_1\ dk_2\ dk_3\ T(k_1)T(k_2)T(k_3)\gamma^k_{k_1k_2k_3}=\int dk_1\ dk_2\ dk_3\ T(k_1)T(k_2)T(k_3)$$
$$\frac{\delta(k_1+k_2+k_3-k)}{1+B+C}(2-{}_3F_2(1,-1-B-C,-C;-1-A-B-C,-B-C;1))$$

It is quite easy to show that the pole term $\frac{1}{1+B+C}$ cancels in the above equation. The resulting expression for γ is free of singularities in some region surrounding the locus of points specified by the equation $k_1^2=k_2^2=k_3^2=k^2=1$. We have succeeded in defining the β-function for tachyon backgrounds which do not satisfy the linearized on shell condition. However, as we can easily see by expanding the hypergeometric function ${}_3F_2$ in (24), γ has a sequence of poles at finite distances from the tachyon mass shell. This is due to the fact that the set of couplings we have taken into account is not complete. If we get far enough from the tachyon mass shell, we run into the poles due to all the other string states which have not been subtracted by our procedure.

As we mentioned before, γ^i_{jkl} is not symmetric between the upper and lower indices. If one succeeds in finding G_{ij} such that β_i has fully symmetric coefficients in the expansion in powers of the couplings (2), then a perturbative construction of the flow potential I is obvious. On the other hand, explicit construction of this metric in the space of coupling constants appears to be quite a complicated task. Therefore it is hard to prove that the RG flow is a gradient flow. Instead, we set out to prove the weaker version of equivalence between the RG and string dynamics. We will show now that solving the $\beta^i = 0$ equation perturbatively generates the correct scattering amplitudes of string theory.

$$\beta^i = \lambda_i g^i + \alpha^i_{jk} g^j g^k + \gamma^i_{jkl} g^j g^k g^l + \ldots = 0 \tag{26}$$

To the lowest order, the equation is $\lambda_i g^i = 0$. Therefore, the solution g^i_0 satisfies the linearized on-shell condition. After writing $g^i = g^i_0 + g^i_1$ and substituting into (26), we find

$$g^i_1 = -\frac{1}{\lambda_i} \alpha^i_{jk} g^j_0 g^k_0 \tag{27}$$

The diagram which summarizes the above equation is shown in fig. 1 . The presence of the on-shell couplings g_0 in (27) sets two of the three legs of the α on shell explicitly. To pick out the propagator pole corresponding to the upper index, $1/\lambda_i$, we set the leg i on shell too. The residue of the pole is the scattering amplitude for three on shell tachyons. Recalling eqn. (21) we conclude that this amplitude is 1 with our normalization. The calculation to the next order provides a non-trivial check of our procedure. We find

$$g^i_2 = \frac{1}{\lambda_i} \left(\frac{2\alpha^i_{jm} \alpha^m_{kl}}{\lambda_m} - \gamma^i_{jkl} \right) g^j_0 g^k_0 g^l_0 \tag{28}$$

where

$$\gamma^i_{jkl} g^j_0 g^k_0 g^l_0 = \left(-D^i_{jkl} + \frac{2\alpha^i_{jm} \alpha^m_{kl}}{\lambda_j + \lambda_m - \lambda_i} \right) g^j_0 g^k_0 g^l_0 \tag{29}$$

The result is shown schematically in fig. 2 . The coefficient D^i_{jkl} has been computed above in terms of the hypergeometric function:

$$D^k_{k_1k_2k_3} = \frac{\delta(k_1+k_2+k_3-k)}{1+B+C}\,{}_3F_2(1,-1-B-C,-C;-1-A-B-C,-B-C;1) \tag{30}$$

Therefore, the four-tachyon scattering amplitude is given by

$$D^i_{(jkl)} + \frac{2\alpha^i_{(jm}\alpha^m_{kl)}}{\lambda_m} - \frac{2\alpha^i_{(jm}\alpha^m_{kl)}}{\lambda_m+\lambda_j-\lambda_i} \tag{31}$$

with all the indices put on shell. Because $\lambda_i = \lambda_j = 0$ the last two terms cancel. This is exactly what had to happen for consistency of our calculations. In order to obtain the β-function from the 'off-shell amplitude' $D^i_{(jkl)}$, we subtracted the poles. On the other hand, to get back to the correct scattering amplitude, we need to add the one-particle reducible graphs back in. Adding and subtracting the poles leaves us with $D^i_{(jkl)}$ with all the indices on shell. Making use of the on shell condition for each external leg we can reduce (30) to

$$\frac{1}{1+B+C}F(-1-B-C,-C;-B-C;1) = -B(1-2k_3\cdot k, 1+2k_2\cdot k_3) \tag{32}$$

After symmetrizing this in 1, 2, and 3, we indeed recover the amplitude for scattering of four on shell tachyons with momenta k_1, k_2, k_3, and $-k$. This proves that, to the order we have considered explicitly, the solutions of the RG fixed point equations generate the scattering amplitudes of the open string theory. Extension of our calculations to the higher orders of perturbation theory does not pose any conceptual problems. The coefficients in the expansion of the β-functions in powers of renormalized couplings are free of singularities in some region surrounding the fixed point. At the fixed point they become 1PI amplitudes for on shell tachyons. In order to recover the full scattering amplitudes, we need to add the one-particle reducible graphs in which tachyons are being exchanged. Solving the RG fixed point equations effectively implements this procedure.

Let us conclude by discussing some of the problems to which the methods of this paper could be applied. Extension of our calculations to the closed string tachyon backgrounds [5] should proceed in an analogous fashion. Although we do not anticipate having closed form expressions for the off shell β-functions, it should be straightforward to prove that they have no singularities in a finite region surrounding the fixed point.

Our method can also be applied to the β-functions for the massless backgrounds. An interesting question to consider there is how the choice of the world sheet regulator affects the gauge in which the resulting spacetime equations turn out to be. Finally, we can attempt a calculation of the loop-corrected β-functions [8] along the lines of this paper. For example, adding the partition functions for the spherical and toroidal world sheets produces a modified relation between the renormalized and bare couplings. Matching it with the solutions of the loop-corrected RG equations may provide us with an unambiguous way to determine the quantum corrections to the string β-functions.

After this paper was written we received a preprint by Sathiapalan [9] which addresses off shell β-functions in a somewhat different context.

ACKNOWLEDGEMENTS

One of us (L. S.) wishes to acknowledge J.J. Atick for many discussions concerning the renormalization group. We also thank T. Banks, R. Brustein, C. Callan, W. Fischler, D. Nemeschansky, A. Sen and S. Yankielowicz for enlightening conversations. I. K. thanks A. Sen and C. Wendt for comments on the manuscript.

REFERENCES

1. C. Callan, E. Martinec, M. Perry and D. Friedan, *Nucl. Phys.* **B262** (1985) 593

   A. Sen, *Phys. Rev.* **D32** (1985) 2102, *Phys. Rev. Lett.* **55** (1985) 1846

   C. Lovelace, *Nucl. Phys.* **B273** (1986) 413

2. C. Callan, I. Klebanov and M. Perry, *Nucl. Phys.* **B272** (1986) 111

   R. Metsaev and A. Tseytlin, Lebedev Inst. preprint Print-87-0184

3. A. M. Polyakov, preprint SLAC-TRANS-0222,

   A.B. Zamolodchikov, *JETP Lett.* **43** (1986) 731

4. B. Sathiapalan, preprint UCLA/87/TEP/17

5. R. Brustein, D. Nemeschansky and S. Yankielowicz, preprint USC-87/004

6. T. Banks and E. Martinec, preprint EFI-87-12; J. Polchinski, paper in preparation

7. E. Fradkin and A. Tseytlin, *Nucl. Phys.* **B261** (1985) 1, *Phys. Lett.* **158B** (1985) 316

8. W. Fischler and L. Susskind, *Phys. Lett.* **171B** (1986) 383, *Phys. Lett.* **173B** (1986) 262

   C. Callan, C. Lovelace, C. Nappi and S. Yost, *Nucl. Phys.* **B290** (1987) 1 and preprint PUPT-1045

   W. Fischler, I. Klebanov and L. Susskind, preprint SLAC-PUB-4298

   I. Klebanov, Talk at the Boulder Workshop on Superstrings, SLAC-PUB-4429

   A. Tseytlin, Lebedev Inst. preprint, August 1987

9. B. Sathiapalan, preprint UCLA/87/TEP/35

FIGURE CAPTIONS

1. The three-tachyon amplitude.
2. The four-tachyon amplitude is the sum of a contact graph and a tachyon exchange graph.

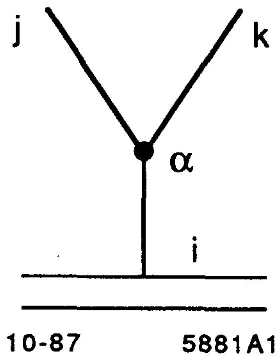

Fig. 1

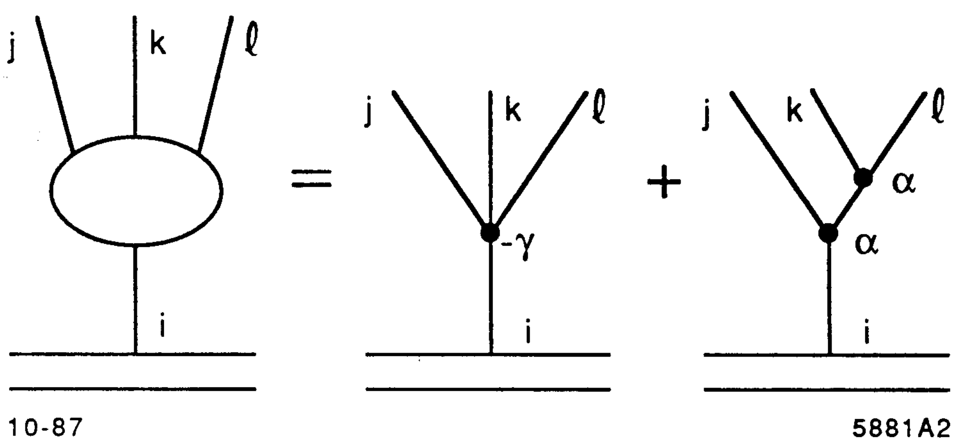

Fig. 2