SLAC-PUB-4444
October 1987
(N)

TIME, ENERGY, AND THE LIMITS OF MEASUREMENT*

M. H. Partovi
Department of Physics, California State University,
Sacramento, California 95819

R. Blankenbecler
Stanford Linear Accelerator Center, Stanford, California 94305

ABSTRACT

We present a discussion of our microstatistical formalism for multitime quantum measurements. We show that this formalism is capable of dealing with time in quantum mechanics in a rigorous way, and enables one to precisely state and derive time-energy uncertainty relations. Another application to the problem of the quantum limit of accuracy of position measurements in the context of gravitational wave detection is briefly discussed.

1 *INTRODUCTION*

The general problem of treating incomplete information occurs at the fundamental level of quantum measurements in an unavoidable manner. The measurement of the state of a microscopic system in general requires a determination of the $N^2 - 1$ elements of its density matrix, N being the dimensionality of the Hilbert space of the states of the system. Now in general N is infinite, implying that an exhaustive measurement is in principle impossible. Stated simply, measurements performed on the most irreducible systems in nature are necessarily incomplete; see our previous contribution to these proceedings for further discussion and for quantitative examples.

The maximum entropy principle (MEP) provides a natural solution to the above problem. Indeed, given a properly formulated measure of entropy, the solution is formally identical to that of the standard problem, well known in statistical mechanics (Jaynes, 1957), of maximizing entropy subject to a set of constraints. Precisely such an entropy for quantum measurements was proposed by Deutsch (1983) and developed by one of us (Partovi, 1983). Proposing a *maximum uncertainty principle* (MUP) as the quantum version of MEP, we developed the statistical mechanics of microscopic systems (Blankenbecler and Partovi, 1985, and these proceedings). It will be convenient for the following development to summarize this formalism here.

The state of a quantum system is in general specified by a density matrix, $\widehat{\rho}$, which is a self-adjoint operator whose eigenvalues are a discrete set of probabilities; i.e., they lie between zero and one and add up to unity. A measurement of

Published in Maximum-Entropy and Bayesian Methods in Science and Engineering, Volume #1, pgs. 249-255

*Work supported by the Department of Energy, contract DE-AC03-76SF00515

the state of the system in general entails the measurement of a number of observables of that system (e.g., energy, position, spin), say $\widehat{A}^\nu$, by producing a large number of copies of the system under identical conditions, and subjecting a sufficiently large fraction of these to interaction with the measuring devices D^ν. In general each D^ν breaks up the range of possible values of A^ν (i.e., the spectrum of A^ν) into a number of bins, α_i^ν, and measures the frequencies $\mathcal{P}_i^\nu$ with which the values of the observable A^ν are found to lie in the bin α_i^ν. For each bin α_i^ν one can introduce a projection operator $\widehat{\pi}_i^\nu$, so that in symbols one has

$$\mathcal{P}_i^\nu = \mathrm{tr}\ \widehat{\rho}\widehat{\pi}_i^\nu \ . \tag{1.1}$$

We are now in a position to state the generic microstatistical problem and its solution: Given that measurements have yielded a set of $\mathcal{P}_i^\nu$ and that no other information is known about the system, how is $\widehat{\rho}$ to be determined? The answer is: maximize the ensemble entropy $-tr\,\widehat{\rho}\,\ell n\,\widehat{\rho}$ subject to the constraints expressed in (1.1). The solution is

$$\widehat{\rho} = Z^{-1} \exp\left[-\sum_{\nu,i} \lambda_i^\nu \widehat{\pi}_i^\nu\right] \ , \tag{1.2}$$

where

$$tr\,\widehat{\rho} = 1 \ , \quad \mathcal{P}_i^\nu = -(\partial/\partial\lambda_i^\nu)\,\ell n\, Z \ . \tag{1.3}$$

Notice the similarity to the analogous expressions of equilibrium statistical mechanics, as well as the very important difference that no constraint on the energy of the system such as would appear in, e.g., the canonical ensemble occurs here. Notice also the fact that in all of the above we have assumed the various measurements to be simultaneous in the sense that each copy of the quantum system is submitted to the measuring device at precisely the same relative time subsequent to its preparation

In summary, then, Eq. (1.2) and (1.3) specify the density matrix of a quantum system subsequent to a single-time measurement according to the maximum entropy/uncertainty principle. Next, we shall tern to a discussion of the meaning of time and subsequently to the treatment of multitime measurements.

2 *TIME*

Time has always played a rather elusive role in quantum mechanics. The reason is simply that time is a parameter, and not a dynamical variable, for any system that obeys Hamiltonian dynamics. In quantum mechanics, for example, the change in the mean value of any dynamical observable $\widehat{A}$ in the state $\widehat{\rho}$ of a system with Hamiltonian $\widehat{H}$ is proportional to $i\,tr\,\widehat{\rho}[\widehat{H},\widehat{A}]$, for sufficiently small changes dA, with the constant of proportionality independent of $\widehat{A}$. Hence,

for a pair of observables $\widehat{A}$ and $\widehat{B}$, one has $dA/dB = (tr\,\widehat{\rho}[\widehat{H},\widehat{A}])/(tr\,\widehat{\rho}[\widehat{H},\widehat{B}])$ for the ratio of their respective rates of change. Clearly, it is natural as well as convenient to introduce a standard for this sort of comparison by parametrizing the evolution of the system in the usual way: $dA = dt \cdot i\,tr\,\widehat{\rho}[\widehat{H},\widehat{A}]$. The choice of this parameter is in principle an arbitrary matter, although any choice other than our present one (or a linear function of it) would appear odd and unnatural to us (e.g., a healthy person's heart beat would slow down indefinitely if we choose $t' = \exp[t/t_0]$ as the new parameter).

Now it would be extremely convenient if we had an observable $\widehat{C}$ for which dC/dt had a constant value *independent* of the state of the system. Indeed if there were such a $\widehat{C}$, we would use it (or a linear function of it) as the standard chronometric variable and could thereby deal with time as simply another dynamical variable of the system. Unfortunately, such a state-independent universal $\widehat{C}$ does not exist, so that time has to be dealt with as a parameter characterizing the evolution of a dynamical system as described above.

Having clarified the meaning of t, we will now proceed to find a precise measure of the accuracy with which it can be measured. Suppose we wish to measure the time of an event using a system in a state $\widehat{\rho}$ as the clock and the value of a dynamical observable $\widehat{A}$ as the chronometric variable. Now the expectation value $A = tr\,\widehat{\rho}\,\widehat{A}$ is a function of time, giving us the required mapping $A(t)$, or (assuming invertibility) its inverse $t(A)$, for determining the value of t when a measured value of A is obtained. In general, the measured values of A will have a distribution, $\mathcal{P}(A)\,dA$, so that one will have a corresponding distribution in the corresponding values of t given by $\mathcal{P}[A(t)](dA/dt)\,dt$. In particular, the variance in the measured values of t will be given by

$$(\delta t)^2 = \int dt\,\mathcal{P}[A(t)](dA/dt)[t-\bar{t}\,]^2\,, \tag{2.1}$$

where $\bar{t}$ is the mean value of t with respect to the above distribution. Other moments can be similarly calculated.

To proceed, we must relate $(\delta t)^2$ back to the state of the system $\widehat{\rho}$ and the operator $\widehat{A}$. To do so, we rewrite Eq. (2.1) as

$$(\delta t)^2 = \int dA\,\mathcal{P}(A)[t(A)-\bar{t}\,]^2\,, \tag{2.2}$$

and note that $\mathcal{P}(A)\,dA$ is simply the probability of finding the measured value of $\widehat{A}$ in the interval dA. But then using Eq. (1.1), we have $\mathcal{P}(A)\,dA = tr\,\widehat{\rho}\,\widehat{\pi}(dA)$, where $\widehat{\pi}(dA)$ is the projection operator corresponding to the spectral interval dA

centered around A. Thus we can rewrite Eq. (2.2) in the form

$$(\delta t)^2 = \int [t(A) - \bar{t}\,]^2 \,\mathrm{tr}\, \hat{\rho}\hat{\pi}(dA) \ . \tag{2.3}$$

Since $\hat{A}\hat{\pi}(dA) = A\hat{\pi}(dA)$, we can use the completeness property $\int \hat{\pi}(dA) = 1$ to recast (2.3) into the final form

$$(\delta t)^2 = tr\, \hat{\rho}\,[t(\hat{A}) - \bar{t}\,]^2 \ . \tag{2.4}$$

Note that $t(\hat{A})$, now a function of an operator, is itself an operator. Equation (2.4) is a remarkably simple formula expressing the dispersion in the measured values of time in terms of the clock state $\hat{\rho}$ and the chromometric variable $\hat{A}$.

Can $(\delta t)^2$ be made arbitrarily small? A precise answer to this question would of course constitute a precise statement of the time-energy uncertainty principle. To answer this question, we first use the Heisenberg inequality

$$\delta D\, \delta B \geq \tfrac{1}{2} \left| tr\, \hat{\rho}\,[\hat{D}, \hat{B}] \right| \ , \tag{2.5}$$

and the identifications $\hat{D} = t(\hat{A})$, $\hat{B} = \hat{H}$, to write

$$2\delta t\, \delta H \geq \left| tr\, \hat{\rho}\,[\hat{H}, t(\hat{A}) - \bar{t}\,] \right| \ . \tag{2.6}$$

Next, using X to denote the right-hand side of (2.6), we seek to minimize it by requiring that its first-order variation vanish. This requirement leads to the condition $[\hat{H}, dt(\hat{A}_0)/d\hat{A}_0] = 0$ for the optimal variable $\hat{A}_0$. This last condition essentially requires that either $\hat{A}_0$ commute with $\hat{H}$ or that $dt(\hat{A}_0)/d\hat{A}_0$ vanish. The first possibility actually maximizes X, since in that case $\hat{A}_0$ would be a constant of the motion, resulting in a clock that is stuck on a fixed value! The second possibility forces $t(\overline{A}_0)$, or $\overline{A}_0(t)$, to be a linear function, so that $tr\, \hat{\rho}\, \hat{A}_0$ is required to be a linear function of t. An immediate consequence of this linearity is that $X = 1$, and (Partovi and Blankenbecler, 1986a)

$$\delta t\, \delta H \geq \tfrac{1}{2} \ . \tag{2.7}$$

In other words, when a quantum system is used as a clock, the dispersion in the measured values of the time of an event cannot be reduced below $1/2\delta H$. In particular, a quantum system which is almost in a stationery state would make a very poor clock. Conversely, a system required to measure time accurately must have a correspondingly large uncertainty in the value of its energy.

Note that the optimal chronometric variable $\widehat{A}_0$ is precisely what we designated as $\widehat{C}$ earlier and characterized as an ideal standard of time. We also stated that such an operator does not in general exist, a well-known fact that follows from the non-existence of a well-behaved canonical conjugate to the Hamiltonian operator. Strictly speaking then, the lower limit in (2.7) is not realizable (even in principle) in actual measurements.

3 *MULTITIME MEASUREMENTS*

In Section 2 we described how quantum systems may be used as clocks, and established the fact that one can in principle construct clocks of arbitrary accuracy by allowing δH to be sufficiently large. Our next task is to generalize the formalism described in Section 1 to multitime measurements assuming the existence of such clocks of arbitrary accuracy.

Let us consider a measurement of a quantum system involving the observables $\widehat{A}^\nu(t_r^\nu)$, where ν labels different observables as in Section 1, and where the additional label t_r^ν denotes the time at which the measurement was carried out. As before, the results of these measurements are summarized in a set of frequencies $\mathcal{P}_{ir}^\nu$, where

$$\mathcal{P}_{ir}^\nu = tr\left[\widehat{\rho}\,\widehat{\pi}_i^\nu(t_r^\nu)\right] . \tag{3.1}$$

Note that $\widehat{\rho}$ does not carry a time label as it corresponds to the reference time $t = 0$. Comparing (3.1) with (1.1), we see that the multitime measurement is in essence not different from the single time case, once the constraint conditions (3.1) are rewritten in terms of $\widehat{\pi}_i^\nu \equiv \widehat{\pi}_i^\nu(0)$ so that all projection operators refer to $t = 0$. In terms of the evolution operator $\widehat{U}(t)$, defined by

$$i\,\frac{d}{dt}\,\widehat{U}(t) = \widehat{H}\widehat{U}(t)\,, \quad \widehat{U}(0) = 1\,, \tag{3.2}$$

we have

$$\widehat{\pi}_i^\nu(t) = \widehat{U}^\dagger(t)\,\widehat{\pi}_i^\nu\widehat{U}(t)\,, \tag{3.3}$$

so that the constraint equations (3.1) now read

$$\mathcal{P}_{ir}^\nu = tr\,\left[\widehat{\rho}\,\widehat{U}^\dagger(t_r^\nu)\widehat{\pi}_i^\nu\widehat{U}(t_r^\nu)\right] . \tag{3.4}$$

We can now write the solution for the multitime case as a simple generalization of (1.2):

$$\widehat{\rho} = Z^{-1}\exp\left[-\sum_{\nu ir}\lambda_{ir}^\nu\widehat{\pi}_i^\nu(t_r^\nu)\right] , \tag{3.5}$$

with constraint equations similar to (1.3). Equation (3.5) gives the density matrix of a quantum system subsequent to a general, multitime measurement according to the maximum uncertainty/entropy principle (Partovi and Blankenbecler, 1986a).

To illustrate the use of (3.5), we shall apply it to another long-standing problem in time-energy uncertainty relations: How accurately can the energy of a quantum system be determined if the measurement is to last no longer than T seconds? To answer this question, we shall consider a "canonical" measurement where the (x-component of the) position of a free particle is measured at two different times, say $-T/2$ and $+T/2$. Thus the device is a position measurement apparatus which we take to have bins of uniform size Δ arranged symmetrically along the x-axis so that $\alpha_i^x = [(i-\frac{1}{2})\Delta, (i+\frac{1}{2})\Delta]$, $i = 0, \pm 1, \ldots$ The density matrix resulting from this measurement is, according to (3.5),

$$\widehat{\rho} = Z^{-1} \exp\left\{ -\sum_i \lambda_i^- \widehat{\pi}_i^x \left(-\frac{T}{2}\right) + \lambda_i^+ \widehat{\pi}_i^x \left(+\frac{T}{2}\right) \right\} , \tag{3.6}$$

where $\lambda_i^{\pm}$ are parameters related to the measured frequencies $\{\mathcal{P}_i^{\pm}\}$ in the standard way.

We must now study the dispersion

$$(\delta H)^2 = tr\,(\widehat{\rho}\,\widehat{H}^2) - (tr\,\widehat{\rho}\,\widehat{H})^2 , \tag{3.7}$$

and determine how it is related to T. More specifically, we will determine how small δH can be made when T is considered fixed and the $\{\mathcal{P}_i^{\pm}\}$, and consequently the $\{\lambda_i^{\pm}\}$ are varied so as to produce the state of lowest possible δH. To avoid technical complications, we shall outline the main points of the argument and leave the details to the literature (Partovi and Blankenbecler, 1986a).

It can be shown that the state $\widehat{\rho}_0$ corresponding to the lowest possible δH has certain symmetry properties which imply the conditions $\lambda_i^+ = \lambda_i^-$ and $\lambda_i^{\mp} = \lambda_{-i}^{\mp}$. These conditions in turn imply that $\widehat{\rho}_0$ is self-conjugate under a Fourier transformation that sends $\widehat{x}$ into $(T/2m)\widehat{p}$ and $\widehat{p}$ into $(-2m/T)\widehat{x}$; here $\widehat{x}$ and $\widehat{p}$ are the position and momentum operators, and m is the mass of the particle. This interesting invariance in turn can be used to show that the optimal state $\widehat{\rho}_0$ has an energy dispersion which is no less than $1/2T$. In other words,

$$T\delta H \geq \tfrac{1}{2} . \tag{3.8}$$

This result is a precise statement of the time-energy uncertainty relation in the form that was often used by Bohr. Note that T in Eq. (3.8) is not a dispersion or uncertainty but the time elapsed between the two position measurements, so that it is the duration of the canonical measurement which was performed to determine the state of the quantum system.

We conclude by briefly discussing the result of applying the above formalism to the derivation of a limit known as the *standard quantum limit* (SQL). This

limit naturally arises in connection with the use of laser interforometry in gravitational wave detection (Partovi and Blankenbecler, 1986b). The problem is this: suppose one is trying to detect the presence of a very weak force (the gravitational force resulting from the passage of a gravitational wave in the actual situation) by successively measuring the position of an otherwise free mass m and thereby the acceleration caused by that force. Under these circumstances, optimal sensitivity is obtained when the displacement (i.e., the change in the position) of the mass is measured with the highest possible accuracy. The accuracy, ℓ, with which the position of the mass can be measured for optimal sensitivity in the detection of acceleration (or force) is the limit referred to above, the SQL. Since this type of measurement is essentially the canonical measurement discussed above, the results of our analysis can be applied here. Using these, we have shown that

$$\ell \geq \ell_0 \left\{ 2U_{\min} \left[\frac{m}{2\pi T} (\Delta x)^2 \right] \right\}^{1/2} , \tag{3.9}$$

where T is the time elapsed between the two measurement, Δx is the resolution of the device used to measure position, $\ell_0 = (T/2m)^{1/2}$, and $U_{\min}[(\Delta x)(\Delta p)/2\pi]$ is the minimum possible value of the dispersion product $\delta x \delta p$ for a measuring device whose position and momentum resolutions are, respectively, Δx and Δp.

Clearly, the minimum value of ℓ in Eq. (3.9) is achieved for the lowest possible value of $U_{\min}$. But the latter is the standard Heisenberg result $\frac{1}{2}$, which is possible when the resolutions Δx and Δp are essentially equal to zero. This gives us the absolute lower bound $\ell \geq \ell_0 = (T/2m)^{1/2}$. For comparison, we note that SQL, the result previously quoted in the literature (and the subject of controversy previous to our work), gives an absolute lower bound equal to $\sqrt{2}\,\ell_0$ (Caves, 1985).

This work was supported in part by a grant from the California State University, Sacramento, NSF Grant No. PHY-8513367, and the Department of Energy under Contract No. DE-AC03-76SF00515.

REFERENCES

Blankenbecler, R. & Partovi, M. H. (1985). Phys. Rev. Lett. 54, 373.
Caves, C. (1985). Phys. Rev. Lett. 54, 2465. For a very physical discussion see M.L. Goldberger & K.M. Watson, (1964). Phys. Rev. 134, 919.
Deutsch, D. (1983). Phys. Rev. Lett. 50, 631.
Jaynes, E. T. (1957). Phys. Rev. 106, 620.
Partovi, M. H. (1983). Phys. Rev. Lett. 50, 1883.
Partovi, M. H. & Blankenbecler, R. (1986a). Phys. Rev. Lett. 57, 2887.
Partovi, M. H. & Blankenbecler, R. (1986b). Phys. Rev. Lett. 57, 2891.