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Scattering Amplitudes and Contact Interactions in Witten's Superstring Field Theory

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ABSTRACT

The four-massless-particle scattering amplitudes are calculated for Witten's covariant superstring field theory. The picture-changing effect of the Ramond sector propagator and of the three-boson vertex is displayed. The results agree with the first-quantized prescription of Friedan, Martinec and Shenker except for the four-boson amplitude, where an extra divergent contact term appears. The addition of a four-boson counterterm to Witten's action cancels the extra term, and is also necessary for gauge invariance of the action to order g^2.

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1. Introduction

In order to construct a covariant field theory of strings, Witten has proposed simple forms for the actions which describe open bosonic and supersymmetric strings, using the terminology of differential geometry [1,2]. Other than kinetic terms, there is only a cubic interaction which essentially glues three strings together in a symmetric way. A very fundamental check of the field theory is that it should reproduce scattering amplitudes of the dual theory which it seeks to generalize; for Witten's bosonic string field theory Giddings and coworkers [3,4] showed how this comes about in the original picture where string world-surfaces are glued together by 3-string vertices and propagator strips. In this paper, we will calculate four-massless-particle scattering amplitudes in Witten's superstring field theory, with the intent of comparing them with standard dual theory results. The calculational framework will be related to that of Giddings [3]. We will connect the amplitudes given by the field theory with results obtained in the "BRST quantized" formalism of Friedan, Martinec and Shenker (FMS) [5] for duality diagrams, showing in particular that the picture-changing of FMS is accomplished in the field theory. The final formulae agree with those of FMS except for the appearance of a divergent contact term in the four-boson amplitude. Related work has been done by Cai and He [6], although our results will differ.

Witten's field theory actions have been studied from several other angles as well. Explicit constructions of the actions in operator language are exemplified by the work of Gross and Jevicki [7], Suehiro and Kunitomo [8], and Samuel [9]. Recently LeClair, Peskin and Preitschopf have constructed the bosonic string field theory using the language of 2-dimensional conformal field theory, and connected this formulation to the operator constructions [10]; here we will apply some of their ideas to the superstring. The intent of all of the above was to develop an explicit framework for calculations and also to study symmetries of the theory. Additionally, work done by Horowitz, Strominger and Qiu for the bosonic string field theory has brought to light some anomalies in the associativity of the star product which Witten used to define his 3-string vertex, and their proper treatment [11].

The plan for the paper is as follows. In Section 2, we will examine the kinetic energy term of Witten's superstring field theory. In the Neveu-Schwarz sector of the superstring it looks just like that of bosonic string field theory, so we obtain the usual propagator in the Siegel gauge. In the Ramond sector Witten's action includes an extra midpoint insertion of Y, the inverse picture-changing operator of FMS. We will choose the gauge $b_0 = \beta_0 = 0$ and find the propagator subject to this gauge condition. The picture-changing effect of this propagator emerges later.

Section 3 lays out the techniques necessary to evaluate the Feynman diagrams for four-particle scattering of massless bosons and fermions. For simplicity, we treat first a field theory with Witten's three-string vertices replaced by generalized versions of the Caneschi-Schwimmer-Veneziano dual model vertex, the difference lying in the way that the world surface defined by a scattering diagram is mapped onto the complex plane. In any case, there are two types of vertex in the theory: the 2-fermion-1-boson vertex may be viewed simply as a gluing operation among the three string states, but the three-boson vertex includes an extra midpoint insertion of X, the picture-changing operator of FMS. From FMS we do not expect picture changing for the 4-fermion amplitude, and indeed in the field theoretic calculation the propagator and vertices are combined just as in bosonic string field theory. In 2-fermion-2-boson diagrams containing the Ramond propagator, we will find (after overcoming some technical difficulties) that this propagator picture-changes one boson vertex operator insertion — as expected from FMS. In other channels where the Ramond propagator is replaced by the Neveu-Schwarz propagator, the X insertion in the 3-boson vertex accomplishes the same task.

For the 4-boson diagram there are two 3-boson vertices and hence two X insertions, each of which changes the picture of one boson vertex operator. However, extra terms in the amplitude are generated by the necessary manipulations. In these extra terms the factor L_0^{-1} in the propagator is cancelled by an anti-commutator of Q_{BRST} with the b_0 from the propagator. The operator L_0^{-1} is represented by a world-sheet strip whose length is integrated from zero to infinity; in the absence of L_0^{-1} the length effectively degenerates to zero. We can call these terms "contact terms" because the resulting world-sheet looks like the gluing together of two 3-string vertices with no propagator in between. Their effect is to produce an unwanted additional term in the scattering amplitude. However, this is for the field theory vertex which is a generalization of the CSV vertex, not for Witten's vertex. In fact the generalization of the CSV vertex has a fatal flaw, namely that it defines a field theory which is not gauge invariant in order g^2, so it is not too much of a surprise that the desired gauge invariant amplitude does not emerge. For bosonic string field theory, the geometry of Witten's vertex ensures a gauge invariant action whereas the CSV-style vertex does not, so one might expect this to fix the superstring action as well.

In section 4 we point out what modifications of the foregoing analysis are relevant when substituting Witten's vertices for CSV-style ones. We will argue that the previous conclusions still hold, except that something new happens to the 4-boson diagram: in the contact terms, when the propagator strip is removed, operators are brought together on the world-sheet which have a divergent operator product expansion. These operators are X, the picture-changing operator of FMS, and ξ in terms of which X can be defined. It is the geometry of the vertex which makes these operators come exactly together for Witten's vertex, whereas they did not for the CSV-style vertex. Thus the four-boson tree amplitude is not defined until one introduces some regulator into the theory which prevents the midpoint insertions from meeting at the same point. For a simple regulation scheme the extra undesired term in the amplitude appears multiplied by a pure divergence, with no subleading finite piece.

Section 5 shows that the 4-boson counterterm which would remove the unwanted divergence in the amplitude is also required for gauge invariance of the action. That is, the order g^2 gauge variation of Witten's action is ambiguous because two X insertions come together on the world-sheet, and the $X(z)X(w)$ operator product expansion is again divergent as $z \to w$. For our regulation scheme the variation is divergent at order g^2 without the addition of the 4-boson counterterm, after which the order g^2 variation is zero. This regulator also seems to require additional counterterms to ensure that the action is gauge invariant to higher orders in g.

We note that Greensite and Klinkhamer have found it necessary to add 4-string counterterms to the action in the light-cone formulation of superstring field theory [12,13]. Also Green and Seiberg have recently discussed the general issue of contact terms, arguing that in some sense they are ubiquitous in superstring theory [14]. It is not ruled out that some other regulator exists for Witten's action where the addition of contact terms is unnecessary, but in light of these other results one might not be surprised if it does not exist. We will conclude that using the regulated and modified action, at least all the 4-massless-particle scattering amplitudes in Witten's theory agree with FMS.

2. The Ramond propagator

We begin with Witten's action for open superstring field theory [1], which differs in appearance from bosonic string field theory [2] principally because of extra insertions of operators X and Y at points where string segments are joined together. X is the picture changing operator of Friedan, Martinec and Shenker [5] and Y is its inverse. The action is written in terms of a string field A as

$$I = \oint A \star QA + \frac{2}{3} \oint A \star A \star A; \qquad (1)$$

here the $\star$ product glues together two string states to make a third, adding a midpoint insertion of X, and the integration $\oint$ sews two halves of a string together with an insertion of Y. If one separates the string field into pieces describing Neveu-Schwarz (NS) and Ramond (R) sectors, $A = (a, \psi)$, the action breaks up into kinetic energy terms acting separately in the two sectors and into vertices connecting three bosons or two fermions and one boson:

$$I = \int a * Qa + \int Y(\tfrac{\pi}{2})\psi * Q\psi + \frac{2}{3}\int X(\tfrac{\pi}{2})a * a * a + 2\int a * \psi * \psi \qquad (2)$$

In this expression, the symbols $\int$ and $*$ are similar to $\oint$ and $\star$ but with no X or Y insertions.

The first task is to fix the gauge and obtain the propagator by inverting the kinetic energy operator. In the NS sector, the kinetic energy term in Witten's superstring action is just that of bosonic string field theory. If we choose states $|a\rangle$ satisfying the Siegel gauge condition [15] $b_0 = 0$, then the kinetic energy term takes the form*

$$\langle a| Q |a\rangle = \langle a| c_0 L_0 |a\rangle . \qquad (3)$$

The propagator follows from the observation $c_0 b_0 c_0 = c_0$ and is simply $\frac{1}{2} b_0 L_0^{-1}$. In order to prepare the way for the following, note that the factor b_0 in the propagator enforces the gauge condition $b_0 = 0$. Specifically for any state $|\chi\rangle$, where in general $b_0 |\chi\rangle \neq 0$, the propagator satisfies

$$b_0 (b_0 L_0^{-1}) |\chi\rangle = \langle\chi| (b_0 L_0^{-1}) b_0 = 0. \qquad (4)$$

A propagator must always satisfy the gauge condition in this sense.

The Ramond sector kinetic energy given in [1] includes an additional insertion of the inverse picture changing operator $Y(\sigma)$ at the string midpoint $\sigma = \frac{\pi}{2}$. Witten has shown that for states in the gauge $b_0 = \beta_0 = 0$, it may be written as

$$I_R = \langle\psi| 2ce^{-\phi}(\tfrac{\pi}{2}) F_0 |\psi\rangle. \qquad (5)$$

The ϕ which appears here is that used by FMS to "bosonize" the superconformal ghosts, $\beta = e^{-\phi}\partial\xi$ and $\gamma = \eta e^{\phi}$, where ξ and η are new fermionic fields [5]. It is

$\star$ The notation for operators will be that of FMS [5].

important to note that F_0 includes only one term containing ghost zero modes, a term $\gamma_0 b_0$. This term gives zero in the gauge $b_0 = 0$ so we may ignore it. The factor $ce^{-\phi}(\frac{\pi}{2})$ is all that is left to supply the necessary insertions of ghost zero modes. For example, $c(\frac{\pi}{2})$ must supply the c_0 which is required to make a nonzero matrix element between states satisfying $b_0 = 0$, giving $I_R = \langle\psi|2c_0 e^{-\phi}(\frac{\pi}{2})F_0|\psi\rangle$. Now we may ask what the Ramond propagator should look like. The structure of the kinetic energy suggests that it contains a factor $b_0 F_0^{-1} = b_0 F_0 L_0^{-1}$, but this does not invert the factor $e^{-\phi}$. Also, the gauge condition $b_0 = 0$ is enforced by $b_0 F_0 L_0^{-1}$ but so far the other condition $\beta_0 = 0$ is not.

This last requirement would be satisfied if the propagator included a factor $\delta(\beta_0)$. In order to see that this is indeed what happens, recall that Yamron [16] has pointed out that $e^{-\phi}$ is the bosonized version of $\delta(\gamma)$. A useful representation [17] of the algebra of the zero modes $[\gamma_0, \beta_0] = 1$ is $\beta_0 = -d/d\gamma_0$, which operates in a space of functions of γ_0, or vice versa. The factor $e^{-\phi}(\frac{\pi}{2}) = \delta(\gamma(\frac{\pi}{2}))$ may be expanded using

$$\begin{aligned}\delta(\gamma(\sigma)) =& \delta(\gamma_0 + \textstyle\sum_{n\neq 0} \gamma_n e^{-in\sigma}) \\ =& \delta(\gamma_0) + \sum_{n\neq 0} \gamma_n e^{-in\sigma}\delta'(\gamma_0) + \sum_{m,n\neq 0} \gamma_m \gamma_n e^{-i(m+n)\sigma}\delta''(\gamma_0) + \cdots \end{aligned} \tag{6}$$

Sandwiched between states obeying $\beta_0 = 0$, all the derivatives of $\delta(\gamma_0)$ vanish because $\delta'(\gamma_0) = -[\beta_0, \delta(\gamma_0)]$, $\delta''(\gamma_0) = [\beta_0, [\beta_0, \delta(\gamma_0)]]$, and so on. Therefore the kinetic energy is simply

$$I_R = \langle\psi|2c_0\delta(\gamma_0)F_0|\psi\rangle. \tag{7}$$

We may say that the factor $\delta(\gamma_0)$ required between two states satisfying $\beta_0 = 0$ had to be supplied by $e^{-\phi}(\frac{\pi}{2})$, just as c_0 had to come from $c(\frac{\pi}{2})$. To form the propagator from this we only need the identity satisfied by the operators x and p in quantum mechanics, $\delta(p)\delta(x)\delta(p) = \delta(p)/2\pi$. The zero modes β_0 and γ_0

behave like x and ip, so this becomes

$$\delta(\gamma_0)\delta(\beta_0)\delta(\gamma_0) = \frac{1}{2\pi i}\delta(\gamma_0). \tag{8}$$

(This mimics the relation $c_0 b_0 c_0 = c_0$ since a Grassmann quantity such as b_0 obeys $\delta(b_0) = b_0$.) Therefore the Ramond propagator is $\frac{1}{2}\pi i\delta(\beta_0)b_0 F_0 L_0^{-1}$, where now $\delta(\beta_0)$ appears as the inverse of $e^{-\phi}$ in the gauge we have chosen. Again the only ghost zero mode contribution to F_0 is annihilated by the factor b_0, so this propagator ensures $\beta_0 = 0$ as well as $b_0 = 0$.

3. Scattering for a CSV-like vertex

For the following, let us adopt the point of view toward string field theory advocated by the authors of ref. [10]. (This can be considered to be an explicit formulation of Witten's bosonic string field theory; other constructions have been given in refs. [7,8,9].) It would of course be sufficient to construct the string scattering worldsheet in Witten's field theory and then to use Giddings' map onto the upper half plane [3]. However the language of [10] gives us a unified way of visualizing a number of different string field theory vertices in terms of mappings onto the complex plane. For bosonic string field theory, it was shown there how to calculate amplitudes in a simple and transparent fashion for the string field theory vertex which generalizes the dual model vertex of Caneschi, Schwimmer and Veneziano (CSV) [18]. (The CSV vertex was generalized to string field theory by DiVecchia, *et al.*[19].) Even though the field theory based on this vertex does not correctly reproduce the Koba-Nielsen integration region in diagrams with over four particles, one does obtain the expected Veneziano amplitude for 4-particle scattering. In Witten's theory, the mapping onto the complex plane is different and leads to some complication, although it gives the correct answer for all diagrams [4]. With this in mind, let us develop and explain the calculations for the superstring using CSV-style vertices and afterwards modify them for Witten's theory.

First we should review some ideas from ref. [10]. As usual, string states are specified by a field functional along some cross section of the string. For simplicity, use the standard trick of doubling the open string world sheet and considering only the analytic sector. (This trick works because the resulting mode algebra is the same.) An external on-shell particle state is represented by the insertion of a BRST invariant vertex operator at the center of a unit circle in the complex plane, with the boundary of the circle serving as the cross section where field functionals are evaluated. Massless bosons and fermions are described by cV_{-1} and $cV_{-1/2}$, respectively, where the subscripts refer to the ϕ charge of FMS [5] and thus label the "picture". We will use the notation of FMS for vertex operators:

$$
\begin{aligned}
V_{-1} &= e^{-\phi}\psi^{\mu}e^{ik\cdot X} \\
V_0 &= (\partial X^{\mu} + ik\cdot\psi\psi^{\mu})e^{ik\cdot X} \\
V_{-1/2} &= e^{-\phi/2}S^{\alpha}e^{ik\cdot X}
\end{aligned}
\tag{9}
$$

(As explained there, V_0 is a picture-changed version of V_{-1}.) A more general state could be represented by additional insertions, or perhaps it would include folds in the surface interior to the boundary of the unit circle. Suppose there are three string states defined by operator insertions $\mathcal{O}_A$,$\mathcal{O}_B$ and $\mathcal{O}_C$ inside flat unit circles. We may then define the CSV-style three-string vertex acting on these states as the conformal field theory matrix element

$$
V^{CSV}(A,B,C) = \Big\langle T^2[\mathcal{O}_A]\,T[\mathcal{O}_B]\,\mathcal{O}_C\Big\rangle,
\tag{10}
$$

where T is a projective transformation which satisfies $T^3 z = z$ for cyclic symmetry. For example, $T : z \to 1/(1-z)$ is such a mapping. This leads to a picture where the original unit circles representing the three strings are mapped into one complex plane, along the unit circles centered on $z = 0$ and $z = 1$ and along $\mathrm{Re}\, z = \frac{1}{2}$ which is a circle centered at $z = \infty$ (fig. 1a). (This picture of the

CSV vertex originates with Lovelace [20].) In order to make it look like Witten's theory, the CSV-style 3-boson superstring vertex should include an insertion of the picture changing operator X at the three-fold symmetric point $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Note that just by adding this insertion to the CSV vertex, we are not guaranteed to make a superstring vertex which is sensible from any other point of view; the motivation for this definition is only that the calculations using this vertex and those using Witten's vertex should be similar.

Witten's three-string vertex can be represented by a similar matrix element where the unit circles defining the three states are mapped into 120° wedges which touch at $z = 0$ and the unit circle is the image of the open string boundary:

$$V(A, B, C) = \Big\langle T^2 h[\mathcal{O}_A]\, Th[\mathcal{O}_B]\, h[\mathcal{O}_C] \Big\rangle, \tag{11}$$

where h is a map from the unit circle into a 120° wedge (fig. 1b). The joining point is flattened out now, so the only insertion at $z = 0$ is for the superstring version of the three-boson vertex where there is a picture changing operator X. The SL(2,C) invariance of the conformal field theory matrix element allows an additional projective transformation which places the states at $z = 0, 1, \infty$ (fig. 1c).

Objects which act between two string states are analogously defined. For example, the propagator evaluated between states in the Neveu-Schwarz sector is written

$$K^{-1}(A, B) = \Big\langle I[\mathcal{O}_B] b_0 L_0^{-1} \mathcal{O}_A \Big\rangle. \tag{12}$$

Here $I : z \to -1/z$ is the inversion of the complex plane through the unit circle, so that state B, defined by operators inside the unit circle, appears as the image of those operators outside the unit circle. The state A appears inside, and the operator $b_0 L_0^{-1}$ sits between them.

The 4-fermion (4F) diagram is the simplest one, and is most like the diagrams in bosonic string field theory. There are two field theory diagrams which together will make up one dual theory diagram (fig. 2); these may be labelled as s- and t-channel exchanges. For the CSV style vertex, the s-channel amplitude is computed as the correlation function on the complex plane (fig. 3),

$$A_s = \left\langle cV^D_{-1/2}(\infty)cV^C_{-1/2}(1)b_0L_0^{-1}cV^B_{-1/2}(0)cV^A_{-1/2}(-1)\right\rangle \tag{13}$$

where A, B, C, D label the quantum numbers of the particles. The Neveu-Schwarz propagator is pictured as acting between states defined by boundary conditions just outside and just inside the unit circle. One three-string vertex connects the outer boundary of the unit circle to two of the external on-shell states, and likewise for the inner boundary. The effect of the factor

$$L_0^{-1} = \int_0^1 d\lambda\, \lambda^{L_0-1} \tag{14}$$

is to shrink everything inside the unit circle by the factor λ and then to integrate over the thickness of the annulus formed between the shrunken unit circle and the unit circle. Since all of the shrunken operators have zero conformal weight, there are no additional prefactors. After taking account of this, the b_0 contour may be contracted inwards, where it converts $cV^A_{-1/2}(-\lambda)$ into $-\lambda V^A_{-1/2}(-\lambda)$. The remaining bc algebra contributes unity. A projective transformation and change of integration variable yields

$$A_s = -\int_0^{\frac{1}{2}} d\varsigma \left\langle V^D_{-1/2}(\infty)V^C_{-1/2}(1)V^B_{-1/2}(\varsigma)V^A_{-1/2}(0)\right\rangle. \tag{15}$$

The t-channel gives a similar result, and after one more projective transformation one obtains the other half of the Koba-Nielsen integration: $\frac{1}{2} < \varsigma < 1$. The whole argument follows directly the analogous computation for the bosonic string [10]. These basic steps are repeated for all diagrams, since both the Ramond and the Neveu-Schwarz propagators contain $b_0L_0^{-1}$.

The 2-fermion-2-boson (2F2B) amplitude is complicated by the appearance of the Ramond propagator $\frac{1}{2}\pi i\delta(\beta_0)b_0F_0L_0^{-1}$in some channels. FMS [5] have shown that the correct combination of vertex operators needed to calculate the 2F2B amplitude is found by ensuring that the total ϕ charge on the world-sheet is -2. This includes the combination, for example, $V_{-1}V_0V_{-1/2}V_{-1/2}$ but not $V_{-1}V_{-1}V_{-1/2}V_{-1/2}$. The effect of $b_0L_0^{-1}$ is already understood — it provides the Koba-Nielsen integration with the correct measure. One might expect that the effect of the rest of the Ramond propagator is simply to accomplish the necessary picture changing to convert one V_{-1} to a V_0. It will turn out that this is correct.

An awkward feature of these diagrams is that if the mode operators β_0 and F_0 are expressed as contour integrals,

$$\beta_0 = \oint \frac{dz}{2\pi i} z^{1/2}\beta(z) \qquad \text{and} \qquad F_0 = \oint \frac{dz}{2\pi i} z^{1/2} T_{z\theta}(z), \tag{16}$$

the integrands are seen to be double valued around $z = 0$ (due to the factor $z^{1/2}$) and around the locations of insertions $V_{-1/2}$ (due to $\beta(z)$ or $T_{z\theta}(z)$). This means that if one tries to deform the contour onto the objects inside it, it gets hung up around the square root branch cut from $z = 0$ to the position of the inner $V_{-1/2}$ (fig. 4). One may not obtain the value of the integral from a single term in the operator product of two operators, as is often the case. In another view of the same phenomenon, one cannot simply evaluate the amplitude using the algebra of mode operators because the factor $e^{-\phi/2}$ in $V_{-1/2}$ is easily defined only in bosonized language. In fact it connects the half-integer modes with the integer modes. In V_{-1} there is a piece $e^{-\phi}$ which can be represented without bosonization as $\delta(\gamma)$, and $\delta(\beta(z))$ has a bosonized version $e^{\phi(z)}$, but it is not known how to unbosonize $e^{-\phi/2}$. Also, trying to bosonize everything by writing the β_0 in $\delta(\beta_0)$ in bosonized form is not revealing.

In the present case where there are only two insertions of $V_{-1/2}$, we can proceed by performing some projective transformations underneath the Koba-Nielsen integral. The conformal field theory matrix element separates into pieces describing conformal ghosts, superconformal ghosts and space fields. Consider first the superconformal ghosts. If the locations of the $e^{-\phi/2}$ insertions are mapped to $z = 0$ and $z = \infty$ then the effect is to evaluate all the other operators between Ramond vacua. The superconformal ghosts are represented everywhere by mode operators in the Ramond sector, and we may evaluate the amplitude by using the mode algebra. One complication of the projective transformation is that a mode number zero operator gets turned into an infinite sum of mode operators: if the mapping is $z \to w(z)$, then

$$\begin{aligned} \beta_0 &\to \oint \frac{dz}{2\pi i} z^{1/2} \left(\frac{dw}{dz}\right)^{3/2} \beta(w(z)) \\ &= \sum_n \beta_n \oint \frac{dz}{2\pi i} z^{1/2} \left(\frac{dw}{dz}\right)^{3/2} w(z)^{-n-3/2} \\ &\equiv \sum_n a_n \beta_n . \end{aligned} \tag{17}$$

Note that this relation holds as well for general conformal transformations $w(z)$ — a fact which will be relevant for showing that the method applies equally well to Witten's vertex. The coefficients a_n are contour integrals in the cut plane, but we shall see in a moment that it is unnecessary to calculate them.

Let the locations of the vertex operators for states A-D be denoted by z_1 through z_4. These points will be in different places for the various diagrams. The only part of F_0 which survives ghost charge counting is that which contains no ghosts at all, so for the superconformal ghost part of the Koba-Nielsen integrand we are left with the task of computing

$$\begin{aligned} A_{\beta\gamma} &= \langle e^{-\phi/2}(z_4) e^{-\phi}(z_3) \frac{\pi i}{2} \delta(\beta_0) e^{-\phi}(z_2) e^{-\phi/2}(z_1) \rangle \\ &= \frac{\pi i}{2} \frac{z_{41}^{1/4}}{z_{31} z_{42}} \langle 0_R | \delta(\gamma(1)) \delta(\sum_n a_n \beta_n) \delta(\gamma(\rho)) | 0_R \rangle , \end{aligned} \tag{18}$$

where the second line is obtained by using SL(2,C) to map $z_4 \to \infty$, $z_3 \to 1$

and $z_1 \to 0$: $\rho \equiv z_{21}z_{43}/z_{42}z_{31}$, $z_{ij} \equiv z_i - z_j$. Also, $|0_R\rangle = e^{-\phi/2}(0)|0\rangle$ is the superconformal ghost part of the Ramond vacuum.

The algebra of β_0 and γ_0 is just that of x and ip in quantum mechanics, so we may separate out the dependence on β_0 and γ_0 to obtain

$$A_{\beta\gamma} = \frac{z_{41}^{1/4}}{z_{31}z_{42}}\frac{\frac{1}{2}\pi i}{\rho^{1/2}a_0}\langle 0_R|\delta(\gamma_0 + \sum_{n\neq 0}\gamma_n)\delta(\beta_0 + \sum_{n\neq 0}a_n\beta_n)\delta(\gamma_0 + \sum_{n\neq 0}\rho^{-n}\gamma_n)|0_R\rangle. \tag{19}$$

The nonzero mode algebra is to be evaluated between Ramond vacua satisfying $\beta_n|0_R\rangle = \gamma_n|0_R\rangle = 0$ for $n > 0$; it is with respect to this vacuum that normal ordering of $n \neq 0$ operators will be defined in the following. The zero mode part has the structure

$$A_{\beta_0\gamma_0} \sim \langle x = 0|\delta(p + P_1)\delta(x + X)\delta(p + P_2)|x = 0\rangle \tag{20}$$

where P_1 and P_2 commute with each other but not with X.* The x-space wave function of $\delta(p + P_2)\,|x = 0\rangle$ is e^{-iP_2x}, so this becomes

$$\begin{aligned}
A_{\beta_0\gamma_0} &\sim \int dx\, e^{iP_1x}\delta(x + X)e^{-iP_2x} \\
&= \int dx \int \frac{da}{2\pi}\ :e^{iP_1x+ia(x+X)-iP_2x}:\ \exp\left[-ax(\overbrace{P_1X} - \overbrace{XP_2})\right] \\
&= \int dx\ :e^{iP_1x-iP_2x}\delta(i(x+X) - x(\overbrace{P_1X} - \overbrace{XP_2})):\\
&= \frac{-i}{1 + i(\overbrace{P_1X} - \overbrace{XP_2})}\ :\exp\left[\frac{-i(P_1 - P_2)X}{1 + i(\overbrace{P_1X} - \overbrace{XP_2})}\right]:
\end{aligned} \tag{21}$$

in which the symbol $\overbrace{ab}$ denotes a Wick contraction. Sandwiched between Ramond vacua, the exponential term in this object contributes a factor of unity, so

* In this context X is not the picture-changing operator.

that the superconformal ghost part of the amplitude is

$$A_{\beta\gamma} = \frac{1}{4}\left(\frac{z_{41}^{1/4}}{z_{31}z_{42}}\right)\frac{\rho^{-1/2}}{a_0 + \sum\limits_{n<0} a_n + \sum\limits_{n>0} a_n\rho^n} \quad . \tag{22}$$

It remains to evaluate the space part of the Koba-Nielsen integrand, for which we will use the same projective transformation which puts the spin operators at $z = 0$ and $z = \infty$. It appears as (fig. 5):

$$\begin{aligned} A_{X\psi} = \frac{4}{z_{41}^{1/4} z_{31} z_{42}} \Big\langle S_{\alpha_4} e^{ik_4\cdot X}(\infty)\psi^{\mu_3} e^{ik_3\cdot X}(1)\Big(\sum_{n=-\infty}^{\infty} a_n \hat{F}_n\Big) \\ \times \psi^{\mu_2} e^{ik_2\cdot X}(\rho) S_{\alpha_1} e^{ik_1\cdot X}(0)\Big\rangle, \end{aligned} \tag{23}$$

where $\hat{F}_n = \oint \frac{dz}{2\pi i} z^{n+1/2}\psi\cdot\partial X(z)$ is the part of F_n which survives ghost charge counting in this amplitude. For $n \geq 0$ it is useful to deform the $\hat{F}_n$ contour inward. The contribution from the vertex operator sitting at $z = 0$ vanishes, leaving the portion of the contour encircling $z = \rho$ which effectively changes $V_{-1}(\rho)$ into $\rho^{n+1/2}V_0(\rho)$:

$$\begin{aligned} &\oint \frac{dz}{2\pi i} z^{n+1/2}\psi\cdot\partial X(z)\psi^{\mu_2} e^{ik_2\cdot X}(\rho) \\ &\qquad = \oint \frac{dz}{2\pi i} z^{n+1/2}\left(-\frac{g^{\lambda\mu_2}}{z-\rho} + :\psi^{\lambda}\psi^{\mu_2}(\rho): + \cdots\right) \\ &\qquad\qquad \times \left(-\frac{ik_2^{\lambda}}{z-\rho} e^{ik_2\cdot X}(\rho) + :\partial X^{\lambda} e^{ik_2\cdot X}(\rho): + \cdots\right) \\ &\qquad = -\rho^{n+1/2} : \left(\partial X^{\mu_2} + ik_2\cdot\psi\psi^{\mu_2}\right) e^{ik_2\cdot X}(\rho) : \end{aligned} \tag{24}$$

Similarly for $n \leq 0$, deforming the contour outwards has the effect of changing $V_{-1}(1)$ into $V_0(1)$. (So far it is only the space part of these vertex operators which are being turned into each other.) Now we have separate expressions for terms

$n \geq 0$ and $n \leq 0$. In one the $V_{-1}(\rho)$ is picture changed, and for the other it is $V_{-1}(1)$. An easy way to relate them is to observe that the expressions must agree for $n = 0$. This yields the identity

$$\left\langle S_{\alpha_4} e^{ik_4\cdot X}(\infty)(\partial X^{\mu_3} + ik_3 \cdot \psi\psi^{\mu_3}) e^{ik_3\cdot X}(1)\psi^{\mu_2} e^{ik_2\cdot X}(\rho) S_{\alpha_1} e^{ik_1\cdot X}(0)\right\rangle$$
$$= \rho^{1/2}\left\langle S_{\alpha_4} e^{ik_4\cdot X}(\infty)\psi^{\mu_3} e^{ik_3\cdot X}(1)(\partial X^{\mu_2} + ik_2 \cdot \psi\psi^{\mu_2}) e^{ik_2\cdot X}(\rho) S_{\alpha_1} e^{ik_1\cdot X}(0)\right\rangle. \tag{25}$$

All terms can now be written in the same form,

$$A_{X\psi} = -\frac{4}{z_{41}^{1/4} z_{31} z_{42}}\Big(a_0 + \sum_{n<0} a_n + \sum_{n>0} a_n \rho^n\Big)$$
$$\times \left\langle S_{\alpha_4} e^{ik_4\cdot X}(\infty)(\partial X^{\mu_3} + ik_3 \cdot \psi\psi^{\mu_3}) e^{ik_3\cdot X}(1)\psi^{\mu_2} e^{ik_2\cdot X}(\rho) S_{\alpha_1} e^{ik_1\cdot X}(0)\right\rangle. \tag{26}$$

When we multiply the space part by the superconformal ghost part of the amplitude, the factors containing infinite sums of a_n cancel, so that we never need the explicit form of a_n. (This cancellation between factors arising from $\delta(\beta_0)$ and F_0 is reminiscent of a cancellation found by Giddings for bosonic string field theory scattering; the counterparts there were b_0 and L_0^{-1}.) The answer is expressed in terms of the space part of a correlation function of vertex operators $V_{-1/2}V_0V_1V_{-1/2}$; if written in terms of the space plus superconformal ghost part of the same correlation function it simplifies further. Returning to the untransformed coordinates, what we have just shown is that

$$\begin{aligned}&\left\langle V^D_{-1/2}(z_4) V^C_{-1}(z_3) 2\pi i F_0 \delta(\beta_0) V^B_{-1}(z_2) V^A_{-1/2}(z_1)\right\rangle \\ &\qquad = -\left\langle V^D_{-1/2}(z_4) V^C_0(z_3) V^B_{-1}(z_2) V^A_{-1/2}(z_1)\right\rangle.\end{aligned} \tag{27}$$

That is, the Ramond propagator includes a piece which accomplishes the picture changing of FMS. So we know how to calculate those channels of 2F2B diagrams which contain the Ramond propagator. Note that it may be hard to generalize this procedure to diagrams with more than four particles; maybe in some gauge other than $\beta_0 = 0$ the propagator would be simpler to deal with.

The other channel of some 2F2B diagrams and both channels of the 4-boson (4B) diagram involve the 3-boson vertex. This vertex includes an insertion of $X = \{Q_{BRST}, \xi\}$, the picture changing operator of FMS. In each of these diagrams we would end up with a string of vertex operators which agrees with that set down by FMS if each X could be used to accomplish the picture changing of one vertex operator. This might happen, for instance, if X could be freely moved from its original position over to the position of some cV_{-1}. In some circumstances, it is true that insertions of X may be moved around on the complex plane with impunity [1]. To see when, consider a general correlation function containing $X(z)$, written in the "small algebra" of FMS. (Recall that X contains only derivatives of ξ so that it exists within the small algebra.) As long as we are restricted to the small algebra, X cannot be written in its BRST commutator form. This is in fact why it is not zero in a correlation function with other BRST invariant objects. However, the effect of moving it from z_1 to z_2 can be written as a BRST commutator within the small algebra,

$$X(z_2) - X(z_1) = \int_{z_1}^{z_2} dz\, \{Q_{BRST}, \partial\xi(z)\}. \tag{28}$$

If everything else in a correlation function is BRST invariant, the Q_{BRST} contour may be deformed away from $\int \partial\xi$ onto the other operators, giving zero. This is why the X insertion in the 3-boson vertex can be defined equally well in terms of the right movers or the left movers, at least for BRST invariant states. In the present application however, there is an operator b_0 from the propagator which does not commute with Q_{BRST},

$$\{Q_{BRST}, b_0\} = L_0, \tag{29}$$

so any attempt to move X will generate extra terms. Although we will not actually move X around in the following, we will perform an equivalent procedure which leads to these extra terms.

First consider a 2F2B diagram which has the 3-boson vertex and a Neveu-Schwarz propagator (fig. 6). There is a single $X(u)$ at the joining point of the 3-boson vertex. In the large algebra, we may manipulate X using the definition $X = \{Q, \xi\}$. So attach ξ to $cV_{-1}(1)$, passing to the large algebra, and then pull the Q_{BRST} contour off of $\xi(u)$. When Q_{BRST} encircles $\xi cV_{-1}(1)$, it converts it into $\frac{1}{2}cV_0(1)$ plus something which gives zero in this correlation function. The contour annihilates everything else, except that an anticommutator with b_0 is generated; however in this case the extra term vanishes. After passing to the small algebra the proper structure emerges to yield the other half of the Koba-Nielsen integration, with one V_{-1} picture changed to a V_0.

We should mention that when adding together the two channels of 2F2B diagrams, it becomes necessary to add apparently dissimilar terms

$$\int_0^{\frac{1}{2}} d\varsigma \Big\langle V_{-1}^D(\infty) V_0^C(1) V_{-1/2}^B(\varsigma) V_{-1/2}^A(0) \Big\rangle + \int_{\frac{1}{2}}^{1} d\varsigma \Big\langle V_0^D(\infty) V_{-1}^C(1) V_{-1/2}^B(\varsigma) V_{-1/2}^A(0) \Big\rangle . \tag{30}$$

One may use the picture changing operation of ref. [5] to switch the pictures of V^C and V^D, modulo a few extra pieces. When dealing with the full range of the Koba-Nielsen integral, these pieces go away because they are total derivatives. In the present situation a given term includes only part of the integration region so that argument does not apply. Luckily, for the 2F2B amplitude the extra terms give zero because they have the wrong number of conformal or superconformal ghosts. The complete 2F2B amplitudes look like

$$A_{2F2B} = \int_0^1 d\varsigma \Big\langle V_{-1}^D(\infty) V_0^C(1) V_{-1/2}^B(\varsigma) V_{-1/2}^A(0) \Big\rangle . \tag{31}$$

Lastly there is the 4-boson diagram, which yields a little surprise. Each

4-boson field theory diagram can be written like (fig. 7)

$$A_s = 4 \int_{-1}^{0} \frac{dz}{z} \Big\langle cV_{-1}^D(\infty) cV_{-1}^C(1) X(u) \oint \frac{dw}{2\pi i} w b(w) X(z/u) cV_{-1}^B(0) cV_{-1}^A(z/u) \Big\rangle. \tag{32}$$

There are now two insertions of X coming from the two 3-boson vertices. We know from ref. [5] that one obtains the correct amplitude using a set of vertex operators $V_{-1}V_{-1}V_0V_0$, so obviously we should try to use the picture changing operators to convert two V_{-1}'s into V_0's. First let us pass to the large algebra by attaching $\xi(0)$. This will allow X to be manipulated as a BRST commutator. Now write

$$X(z/u) = \oint \frac{ds}{2\pi i} j_{BRST}(s) \xi(z/u), \tag{33}$$

and deform the s contour onto the other operators, giving two nonzero terms. When the contour encircles $z = 0$ it produces a picture changed version of $cV_{-1}(0)$,

$$[Q_{BRST}, \xi cV_{-1}(y)] = \frac{1}{2} cV_0 - \frac{1}{4} \eta e^{\phi} \psi^{\mu} e^{ik\cdot X}(y). \tag{34}$$

One may think of this object as the BRST invariant picture changed version of cV_{-1}. (Note cV_0 is not by itself BRST invariant.) The commutator with b_0 generates another term with b_0 replaced by L_0. This cancels the L_0^{-1} which gave rise to the integral over z, resulting in a contact term. In the first term, the leftover $\xi(z/u)$ must provide the ξ_0 for the large algebra, so we may freely move it onto $cV_{-1}(z)$. Then moving the BRST contour off of $X(u)$ leads to one piece with a picture changed $cV_{-1}(z)$, plus one more piece where L_0^{-1} has been canceled by $\{Q_{BRST}, b_0\}$. Fig. 8 shows the form of the three pieces we have found after discarding terms with the wrong bc charge.

The structure of the result emerges after we evaluate the conformal and superconformal ghost parts of the correlation functions. If the space parts of vertex operators are written $V^{X\psi}$, we have

$$
\begin{aligned}
A_s = & \int_0^{\frac{1}{2}} dx \left\langle V_{-1}^D(\infty) V_{-1}^C(1) V_0^B(x) V_0^A(0) \right\rangle \\
& + \frac{1}{2}(1-4u) \left\langle V_{-1}^{D,X\psi}(\infty) V_{-1}^{C,X\psi}(1) V_{-1}^{B,X\psi}(\tfrac{1}{2}) V_{-1}^{A,X\psi}(0) \right\rangle .
\end{aligned}
\tag{35}
$$

The first term looks like half of the Koba-Nielsen integral we need to get the expected amplitude. There is also a troublesome contact term. Since we will encounter it again, define

$$
F_{ABCD}(x) \equiv \left\langle V_{-1}^{D,X\psi}(\infty) V_{-1}^{C,X\psi}(1) V_{-1}^{B,X\psi}(x) V_{-1}^{A,X\psi}(0) \right\rangle . \tag{36}
$$

The t-channel amplitude is similar, except that labels $ABCD$ are permuted to $BCDA$ so that it is V^A and V^D which are in the minus-one picture. In order that the first terms in A_s and A_t fit neatly together to make up the expected amplitude, the pictures of V^A and V^C must be switched in A_t. For this one may use the picture-changing operation of FMS to obtain the identity

$$
\begin{aligned}
\left\langle V_{-1}^D(\infty) V_{-1}^C(1) V_0^B(x) V_0^A(0) \right\rangle = & \left\langle V_0^D(\infty) V_{-1}^C(1) V_{-1}^B(x) V_0^A(0) \right\rangle \\
& + \frac{d}{dx} F_{ABCD}(x) .
\end{aligned}
\tag{37}
$$

(An easy proof makes use of the BRST invariant object defined in Equation (34).) When integrated over $0 < x < \frac{1}{2}$ the second term in this identity gives $F_{ABCD}(\frac{1}{2})$, another contribution to the contact term. Noting that $F_{BCDA}(x) = F_{ABCD}(1-x)$, we find the result

$$
\begin{aligned}
A_s + A_t = & \int_0^1 dx \left\langle V_{-1}^D(\infty) V_{-1}^C(1) V_0^B(x) V_0^A(0) \right\rangle \\
& + 4(\frac{1}{2} - u) F_{ABCD}(\frac{1}{2}) .
\end{aligned}
\tag{38}
$$

What should we make of the extra term in the 4B amplitude? For bosonic string field theory, the CSV-style vertex gives the correct 4B amplitude. This was traced in ref. [10] to the fact that on-shell, the four-string vertex constructed out of two three-string vertices is cyclically symmetric — a condition needed in the proof of gauge invariance to order g^2. The condition is not true off-shell, so the CSV vertex does not lead to an acceptable string field theory. In the present case, the presence of the X insertions spoils the four-fold symmetry even for on-shell states. So again it seems that the condition of gauge invariance is related to getting the right answer for the scattering amplitude. This connection will appear again when we discuss Witten's form of the vertex. The geometry of the latter is such as to give a four-fold symmetric figure when two three-string vertices are glued together, so it has a better chance to give the right amplitude. (One might ask whether the extra contact term cancels out if the CSV-style superstring vertex is defined with a symmetrical combination of X insertions involving left and right movers; in fact this does not help.)

4. Scattering for Witten's vertex

Now we will discuss what part of the above analysis is modified when Witten style vertices are substituted for CSV ones. One complication arises as follows in the picture where states are mapped into 120° wedges [10]. Using the three-string vertex, we join a pair of external states into a single string state, which is represented by the unused wedge in the vertex. Suppose that we think of the propagator as acting between one such state defined inside the unit circle and another defined outside. The unused wedges must then be conformally mapped into the regions inside and outside the unit circle. Because these mappings are not projective transformations, the inverse mapping from the new coordinates onto the string surface will have branch cuts which demonstrate that the string surface has not been smoothed out into a flat conformal plane. An additional conformal transformation will smooth out the folds, but at the cost of converting a mode number zero operator like $b_0 = \oint \frac{dz}{2\pi i} z b(z)$ into a much more complicated integral. Now that the folds are smoothed out, the conformal field theory matrix element may be evaluated using the simple contractions of fields characteristic of the flat z plane. Equivalently, we could take the viewpoint of Giddings [3] and start with a vertex which includes a curvature singularity at the joining point. In any case the result is the same map of the surface onto the upper half plane. For bosonic string field theory, Giddings showed that the effect of L_0^{-1} and the image of b_0 under this mapping is to give the Koba-Nielsen integration with the right measure. This conclusion is still valid for the superstring, since $b_0 L_0^{-1}$ is present in both Ramond and Neveu-Schwarz propagators.

In diagrams containing the Ramond propagator the resmoothing will also convert β_0 and F_0 into complicated integrals. However a small modification to the CSV-style calculation will take care of this. Expand the transformed versions of β_0 and F_0 in Laurent series, so that they become infinite sums $\sum A_n \beta_n$ and $\sum A_n F_n$. (Again the coefficients A_n are the same for each series because β and $T_{z\theta}$ have the same dimension.) In our argument which showed that $\delta(\beta_0)F_0$ has the effect of picture changing one of the boson vertex operators in the 2F2B amplitude, we immediately did a projective transformation in order to put the spin operators at $z = 0$ and $z = \infty$. After that, we always worked with the transformed versions $\sum a_n \beta_n$ and $\sum a_n F_n$. The argument may be transcribed exactly provided that we understand a_n to describe the combination of the two mappings: the first map is a conformal transformation which smoothes out the plane and the second is a projective transformation which moves the spin operators to where we want them. The conclusion is the same.

The 4F and 2F2B scattering diagrams thus give nothing new when the Witten style vertex is used. For the 4B amplitude however, recall that the CSV-style vertex gave the Veneziano amplitude plus an extra contact term. The extra term shows up when the Q_{BRST} contour is pulled off of one X insertion and anticommuted with the b_0 from the propagator. The anticommutator cancels the L_0^{-1} part of the propagator, so that the term looks like a contact interaction.

Since the geometry of the Witten vertex is different, we might expect that the contact term is changed or maybe absent.

There is however a new complication in finding the contact term. When the strip representing L_0^{-1} is taken away, the ξ left over from the first X insertion is placed right on top of the X insertion from the other 3-string vertex. This gives a divergent result due to the singular operator product between X and ξ,

$$X(z)\xi(w) \sim -\frac{1}{2}\frac{1}{(z-w)^2}be^{2\phi}(\frac{z+w}{2}) + o(1). \tag{39}$$

(For the CSV style vertex, this is not a problem because the joining points of the 3-string vertices do not come together when the propagator degenerates.) There is in addition a nonsingular contact term generated in a similar fashion (*cf.* fig. 8).

In order to control the divergence, some regulator is necessary. Suppose that when gluing together two string states, we insert a thin strip of width ϵ and then take $\epsilon \to 0$ at the end. This results in a picture like that shown in fig. 9 for the divergent contact term. Using Giddings' map [3] from this string surface onto the upper half plane, the joining points are mapped to $z = i\gamma$ and $z = i\delta = i/\gamma$, which approach each other at $z = i$ in the $\epsilon \to 0$ limit. The string scattering states are represented by vertex operator insertions at $z = -\beta, -\alpha, \alpha, \beta$ where $\alpha\beta = 1$ and

$$\lim_{\epsilon\to 0} \alpha \equiv \alpha_0 = \sqrt{2} - 1. \tag{40}$$

Giddings showed how to determine α, β, γ and δ from the value of ϵ using some elliptic integrals. By making an expansion near $\epsilon = 0$, one finds that α and γ are related by

$$\alpha = \alpha_0 - \frac{\alpha_0}{2\sqrt{2}}\left(\frac{1}{2} + \ln\frac{4}{c}\right)c^2 + o(c^3), \tag{41}$$

where $c = 1 - \gamma$ will be a convenient expansion parameter, related to ϵ by

$$\epsilon = \frac{\pi}{2}c^2 + \frac{\pi}{2}c^3 + \frac{3\pi}{8}c^4 + o(c^5). \tag{42}$$

First evaluate the ghost factor of the contact term; it gives something proportional to $(\gamma - \delta)^{-2}$ plus less singular terms. It is independent of the quantum numbers of the external states, so the same factor will occur in s and t channels. This should be multiplied by the space part of the correlator, which depends on ϵ through α. We may convert the space part into a standard form using a projective transformation:

$$\left\langle V_{-1}^{C,X\psi}(\beta)V_{-1}^{B,X\psi}(\alpha)V_{-1}^{A,X\psi}(-\alpha)V_{-1}^{D,X\psi}(-\beta)\right\rangle = \frac{1}{(\alpha+\beta)^2}F_{ABCD}(x_\epsilon), \tag{43}$$

where $x_\epsilon = 4/(\alpha+\beta)^2$ approaches one-half as the regulator goes to zero. Assembling the contact terms with the rest of the amplitude, the s-channel gives

$$\begin{aligned} A_s = &\int_0^{\frac{1}{2}} dx \left\langle V_{-1}^{D}(\infty)V_{-1}^{C}(1)V_0^{B}(x)V_0^{A}(0)\right\rangle \\ &+ \left(-\frac{1}{2c^2} + \frac{1}{2c} - \frac{5}{8} + o(c)\right) F_{ABCD}(x_\epsilon). \end{aligned} \tag{44}$$

Now make explicit the dependence of $F_{ABCD}(x_\epsilon)$ on c, expanding near $c = 0$:

$$\begin{aligned} F_{ABCD}(x_\epsilon) = F_{ABCD}(\tfrac{1}{2}) &+ (x_\epsilon - \tfrac{1}{2})\frac{d}{dx}F_{ABCD}(x)\bigg|_{\frac{1}{2}} \\ &+ \frac{1}{2}(x_\epsilon - \tfrac{1}{2})^2\frac{d^2}{dx^2}F_{ABCD}(x)\bigg|_{\frac{1}{2}} + \cdots \end{aligned} \tag{45}$$

where

$$x_\epsilon - \frac{1}{2} = -\left(\frac{1}{8} + \frac{1}{4}\ln\frac{4}{c}\right)c^2 + o(c^3). \tag{46}$$

Note that the worst divergence from the ghost part of the contact term was $1/c^2$, whereas the coefficient of d^2F/dx^2 in the above is $o(c^4)$. Thus in A_s, only $F(\frac{1}{2})$

and $dF/dx(\frac{1}{2})$ survive the limiting procedure. The t-channel contact terms give exactly the same thing, except that since $F_{BCDA}(x) = F_{ABCD}(1-x)$, the sign of the dF/dx term is changed and it cancels between s and t channels. Finally, making use of the identity (eqn. (37)) for combining the integrals in A_s and A_t, we have

$$A_s + A_t = \int_0^1 dx \Big\langle V_{-1}^D(\infty) V_{-1}^C(1) V_0^B(x) V_0^A(0) \Big\rangle$$

$$+ \left(-\frac{1}{c^2} + \frac{1}{c} - \frac{1}{4} + o(c) \right) F_{ABCD}(\tfrac{1}{2}) \qquad (47)$$

$$= \int_0^1 dx \Big\langle V_{-1}^D(\infty) V_{-1}^C(1) V_0^B(x) V_0^A(0) \Big\rangle - \Big(\frac{\pi}{2\epsilon} + o(\epsilon) \Big) F_{ABCD}(\tfrac{1}{2}).$$

5. Gauge invariance of Witten's action

A related problem appears when trying to prove that Witten's superstring action is gauge invariant to order g^2. That proof hinges on the associativity of the modified star product, which is used to define the three-string vertex. The associativity is problematic because the modified star product of two string fields in the Neveu-Schwarz sector involves an X insertion at the midpoint, $a \star b = X(\frac{\pi}{2}) a * b$, and these insertions come together in a string of several star products. The operator product of $X(z)X(w)$ is quadratically divergent just like $X(z)\xi(w)$, so some regulation scheme is called for. (The connection between these operator products may be seen by deforming the BRST contour off of ξ in one X to enclose both ξ and the other X.) To see why associativity matters, recall the form of the three-boson interaction [1]

$$I_{3B} = \tfrac{2}{3} g \oint a \star a \star a \qquad (48)$$

and the variation of the string field a under a gauge transformation parametrized by λ

$$\delta a = Q\lambda + g(a \star \lambda - \lambda \star a). \tag{49}$$

The order g^2 terms generated in δI_{3B} would cancel if $\star$ were associative and if $\oint(a \star a \star a \star \lambda - \lambda \star a \star a \star a)$ were zero.

If the operator insertions specified by $\oint$ and $\star$ are written explicitly then the three-boson interaction appears as

$$I_{3B} = \tfrac{2}{3} g \int X(\tfrac{\pi}{2}) a * a * a. \tag{50}$$

Here, two X insertions (from $\star$) and one Y insertion (from $\oint$) have come together at the same point to produce a single X insertion. The action is gauge invariant to order g, but there is a problem at order g^2 because the $X(\frac{\pi}{2})$ from the variation δa comes together with the $X(\frac{\pi}{2})$ in the interaction. (The 2-fermion-1-boson interaction, see eqn. (2), has no insertion so it is not afflicted with this problem.) In the 4B scattering amplitude, we inserted a strip of finite width to prevent the midpoint insertions from coming exactly together; perhaps we could do something similar here. Such strips could for example be included in the definition of the three string vertex, a device used by Horowitz, Strominger and Qiu [11] to analyze some questions in bosonic string field theory.

Whatever regulator scheme is chosen, one must ensure that the theory is gauge invariant even before the regulator ϵ shrinks to zero, for the following reason. Consider an action which is gauge invariant for nonzero ϵ. When amplitudes are computed in perturbation theory, terms will appear which are singular as $\epsilon \to 0$, as discovered above. However, whatever amplitude comes out, it will be gauge invariant for every ϵ, including $\epsilon \to 0$ if divergent terms cancel so that the limit is defined. If now we add extra non-gauge invariant terms to this action, but which vanish for $\epsilon \to 0$, they will conspire with the singularities to produce extra non-gauge invariant terms in the amplitude which do not vanish in the limit. In order not to introduce these into the theory accidentally, the action should be gauge invariant to all orders in ϵ.

There exists a simple regulation scheme where a strip of width ϵ appears both in the propagator and in the definition of the gauge variation. This scheme seems natural when Witten's theory is rewritten in a more traditional language.* In such a language the bosonic part of Witten's action is

$$I_B = \langle I_{12}|\, |a\rangle_1 \otimes |a\rangle_2 + \frac{2}{3} g \langle XV_{123}|\, |a\rangle_1 \otimes |a\rangle_2 \otimes |a\rangle_3 \tag{51}$$

and the gauge variation law is

$$\delta |a\rangle_1 = Q|\lambda\rangle_1 + g\langle XV_{234}|\, |I_{12}\rangle \otimes \left(|a\rangle_3 \otimes |\lambda\rangle_4 - |\lambda\rangle_3 \otimes |a\rangle_4\right). \tag{52}$$

Here the three-string vertex including the X insertion, $\langle XV_{123}|$, acts on the direct product of the Fock spaces of three strings; the integer subscripts label the different Fock spaces. There are also gluing operators $\langle I_{12}|$ and $|I_{12}\rangle$, which in the original formulation are intended to behave as $\langle a|b\rangle = \langle I_{12}|\, |a\rangle_1 \otimes |b\rangle_2$. For the present case we will alter the definition of $|I_{12}\rangle$ in order to regulate the theory. Redefine $|I_{12}\rangle$ to insert a strip of width ϵ between the states it glues together, and $\langle I_{12}|$ to be the inverse $\langle I_{12}|I_{23}\rangle = 1_{13}$. Then the action (eqn. (51)) is invariant under the gauge transformation (eqn. (52)) for any ϵ up to order g, and the propagator will contain the extra strip as advertised. In Witten's language the effect is to define the gauge variation in the NS sector as

$$\delta a = Q\lambda + g e^{-\epsilon L_0} X(\tfrac{\pi}{2})(a * \lambda - \lambda * a). \tag{53}$$

Fig. 10 depicts the regulated variation of I_{3B} in order g^2.

Of course the variation in order g^2 is not zero because of the singular operator product in the limit $\epsilon \to 0$. Is it possible to fix up the action within this regulation scheme so that it is gauge invariant to order g^2? We can answer this question in a restricted sense. If the state a is BRST invariant, then there exists a simple

* This suggestion is due to C. Preitschopf.

four boson counterterm whose gauge variation cancels that of the three boson interaction to order g^2 (fig. 11),

$$I_{4B} = g^2 \int \overline{\xi}(\tfrac{\pi}{2}) \left[X(\tfrac{\pi}{2})(a * a)\right] * e^{-\epsilon L_0} \left[\xi(\tfrac{\pi}{2})(a * a)\right] + o(\epsilon). \tag{54}$$

Here $\overline{\xi}$ is the right mover if ξ and X are left movers, and the conformal field theory matrix element must be evaluated in the "large algebra" of FMS. (The divergent part of this counterterm is simply a four-boson interaction where the operator $be^{2\phi}$ sits at the joining point.) Remarkably, the addition of this counterterm also exactly cancels the extra contact term we found in the 4B amplitude, including divergent and finite pieces. In fact this form for the counterterm was roughly inspired by the extra piece of that amplitude. There may well be a generalization of I_{4B} which also works for off-shell states, but so far we have found only the on-shell restriction of it. Certainly it is only that restriction which contributes to the 4B scattering amplitude, which involves physical states. One would also like to find the $o(\epsilon)$ terms in I_{4B} which make the action gauge invariant for finite ϵ, but again these do not affect the 4B scattering amplitude in the limit $\epsilon \to 0$. Also in another respect the situation is not satisfactory, because with the above form for I_{4B} there will be an order g^3 term in the gauge variation, which is apparently also divergent, suggesting that even more counterterms are needed to ensure gauge invariance to higher orders in g.

6. Conclusion

We have seen how to calculate four-particle scattering amplitudes in Witten's superstring field theory. The Ramond sector propagator was constructed in the $\beta_0 = 0$ gauge, which turned out to be awkward for calculations, although workable. It differed from the NS propagator by the presence of $\delta(\beta_0)F_0$, the effect of which was to change the picture of one boson in a 2F2B amplitude. All diagrams gave the same results as the first quantized formalism of Friedan, Martinec and Shenker, except that extra divergent terms appeared in the four-boson amplitude. The divergences occurred when picture changing operator insertions approached each other, which also happens when considering the gauge variation of the action. The implication is that without some regulation scheme, the perturbation series for Witten's action is not well defined, and likewise the gauge variation of the action. Within a simple regulation scheme, the action is gauge invariant to order g^2 only after the addition of a four-boson counterterm; the same counterterm cancels all extra terms in the four-boson amplitude. The form of this counterterm was found only for on-shell states and in the limit where the regulator shrinks to zero. More counterterms may be needed for gauge invariance at higher orders in g.

Perhaps the counterterms would not be necessary in a more cleverly chosen regulation scheme. Or perhaps one may consistently drop all the terms divergent as $1/\epsilon$, both in the amplitudes and in the definition of the gauge transformation law. To this end one may observe that in the 4B amplitude, the contact term includes a contribution only from ϵ^{-1}, and not from ϵ^0. However it is not clear what would justify such a procedure in general. On the other hand such counterterms are necessary in other approaches to superstring theory, so it is certainly possible that there is no regulator where Witten's action is complete without counterterms. For example, Greensite and Klinkhamer have found that in the light-cone formulation of superstring field theory, spacetime supersymmetry requires the action to contain extra contact terms [12]. Very recently they have also shown that four-particle scattering amplitudes are divergent without these extra terms [13], a situation reminiscent of our present results for Witten's theory. Green and Seiberg have discussed how contact terms appear in various guises in different approaches to superstring theory [14]. As they explained, in the first-quantized covariant approach one may often avoid contact terms by analytically continuing in the external momenta, but in calculations where this is not possible they cannot be avoided. They showed that world-sheet supersymmetry can be used to determine their form in this case. (For the contact term we have found here, there is no possibility of analytic continuation because the divergence is due to ghostly operator products on the world-sheet which are independent of momenta.) Even if counterterms are not surprising, Witten's program of describing superstring field theory in the language of differential geometry is an attractive one, and in order fully to implement this program it would seem incumbent to find some way of avoiding the counterterms (unless they too could be shown to fit somehow into this language).

The four particle scattering amplitudes have also been studied by Cai and He [6]. They obtained a different result for the propagator in the Ramond sector, which seems to make the picture-changing effect more apparent, but it does not respect their gauge condition $\beta_0 = 0$. In the 4B amplitude, they have apparently missed the extra contact term.

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FIGURE CAPTIONS

1. Definition of field theory vertices by mappings of three states into the complex plane. (a) CSV-style vertex; (b) Witten's vertex; (c) Witten's vertex after an additional projective transformation.
2. The two field theory diagrams which together will make one dual theory diagram. The arrows are for orientation of Chan-Paton factors.
3. The s-channel four fermion scattering amplitude for the CSV style vertex.
4. A 2F2B diagram with the Ramond propagator, showing the analytic structure of the integrands in β_0 and F_0.
5. Space part of the 2F2B Koba-Nielsen integrand when the propagator is in the Ramond sector, after a projective transformation, showing the transformed $\tilde{F}_0$ contour.
6. A 2F2B diagram containing the 3-boson vertex. The propagator is in the Neveu-Schwarz sector.
7. Four boson scattering amplitude, containing two insertions of the picture changing operator X.
8. The 4B amplitude separates into three terms: (a) is part of the standard Koba-Nielsen integral; (b),(c) are extra contact terms.
9. Regulated contact term of fig. 8c. (a) A strip of width ϵ separates two midpoint insertions. (b) The same figure after mapping onto the upper half plane.
10. World-sheet picture showing regulated variation of I_{3B} to order g^2.
11. The four-boson counterterm which restores gauge invariance on-shell in order g^2.

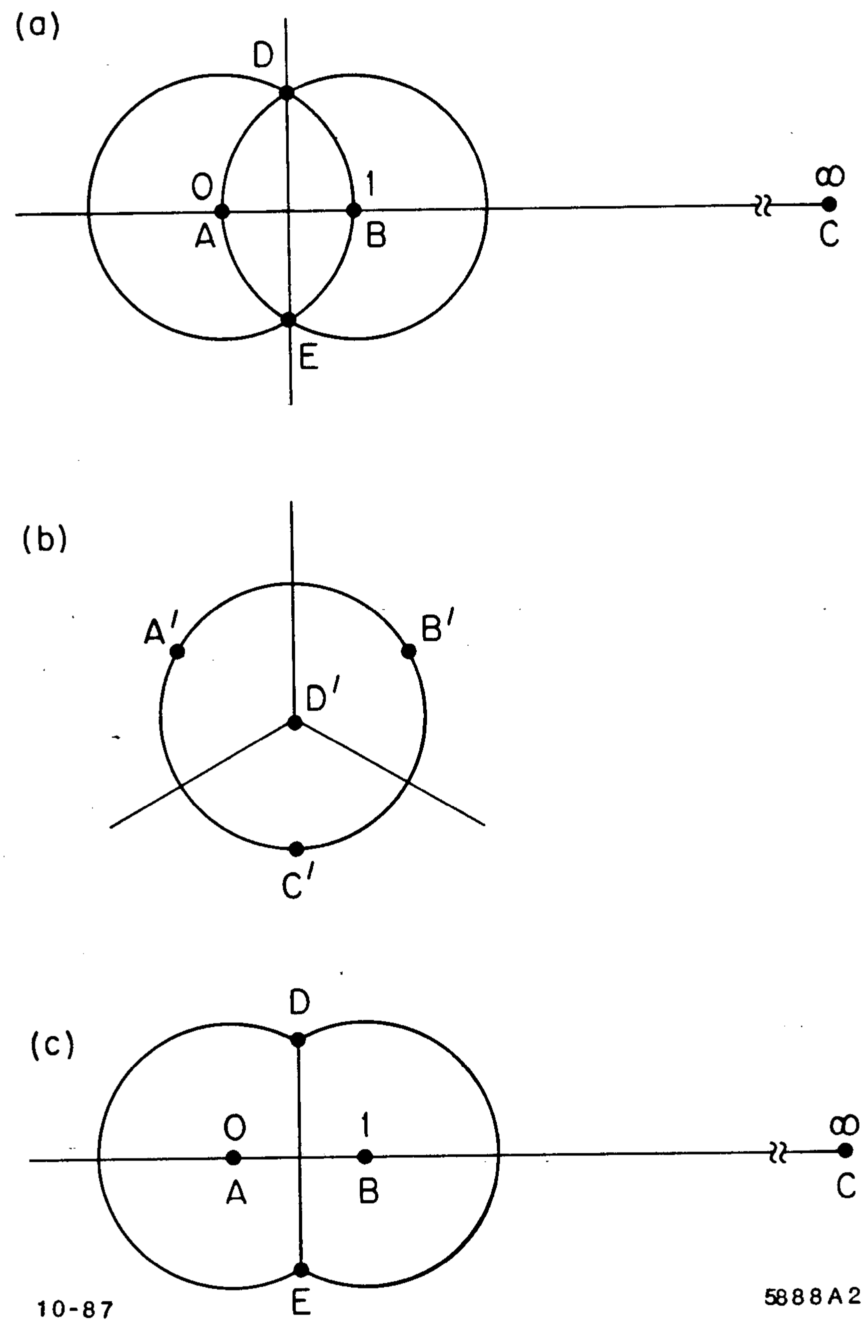


Fig. 1

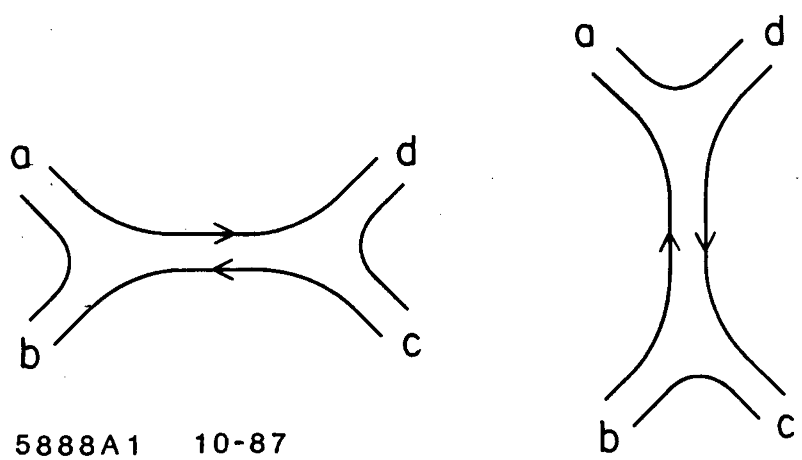
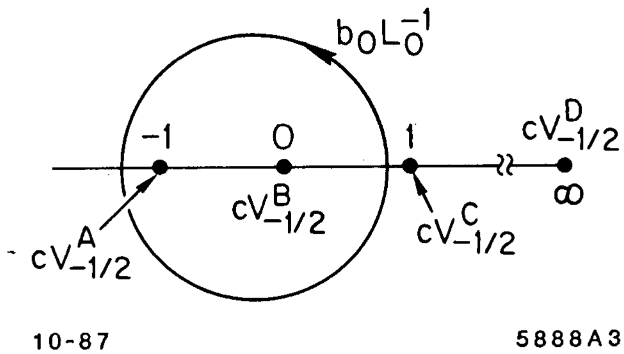


Fig. 2



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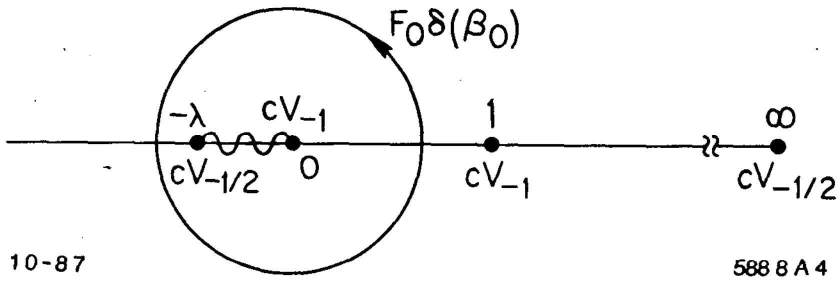

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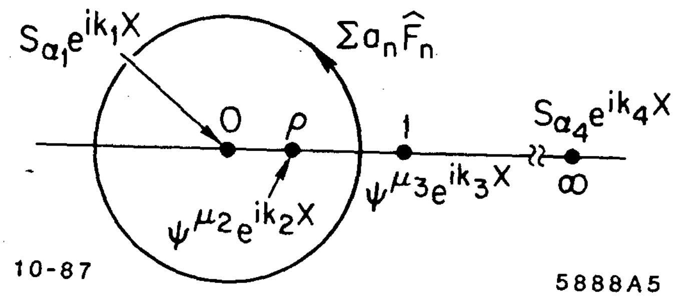

$S_{\alpha_1} e^{ik_1X}$
$\Sigma a_n \hat{F}_n$
0
ρ
1
$S_{\alpha_4} e^{ik_4X}$
∞
$\psi^{\mu_2} e^{ik_2X}$
$\psi^{\mu_3} e^{ik_3X}$
10-87
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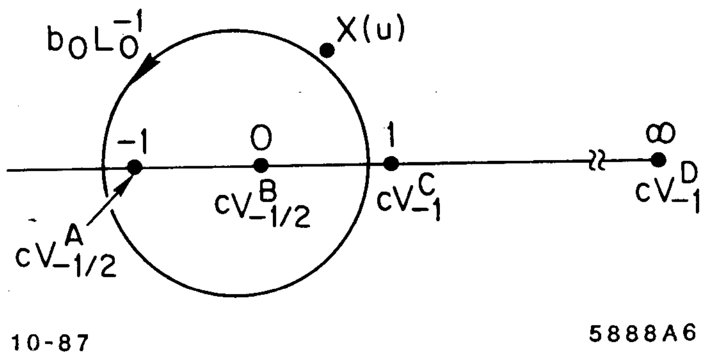
Fig. 5



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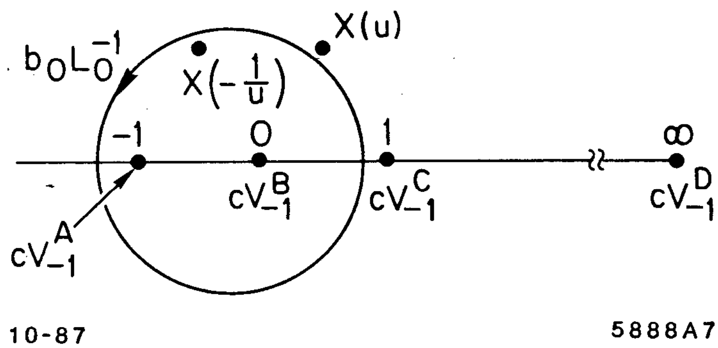

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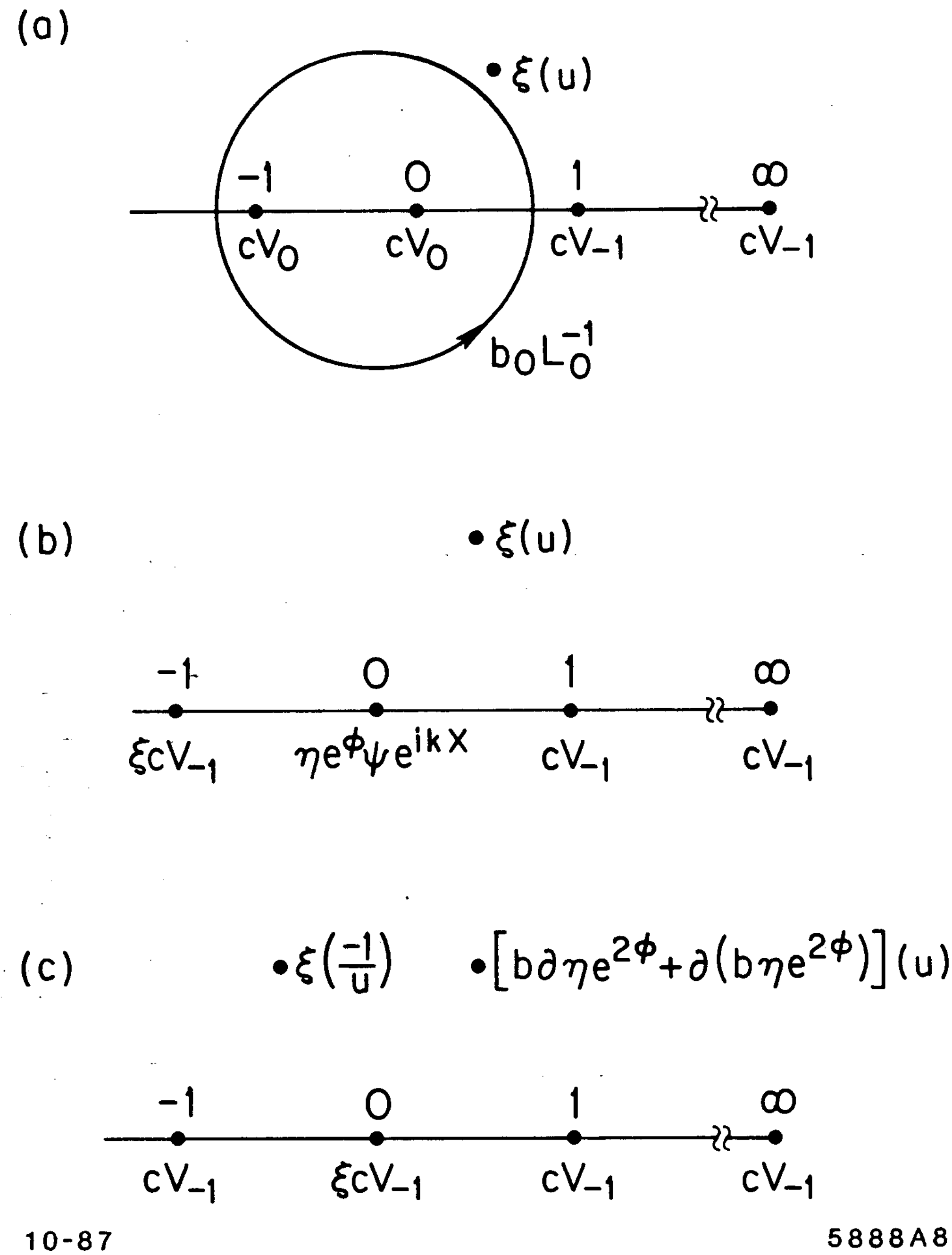

(a)
• ξ(u)
−1
0
1
∞
cV0
cV0
cV−1
cV−1
b0L0−1
(b)
• ξ(u)
−1
0
1
∞
ξcV−1
ηeφψe^{ikX}
cV−1
cV−1
(c)
• ξ(−1/u)
• [b∂ηe^{2φ}+∂(bηe^{2φ})](u)
−1
0
1
∞
cV−1
ξcV−1
cV−1
cV−1
10-87
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Fig. 8

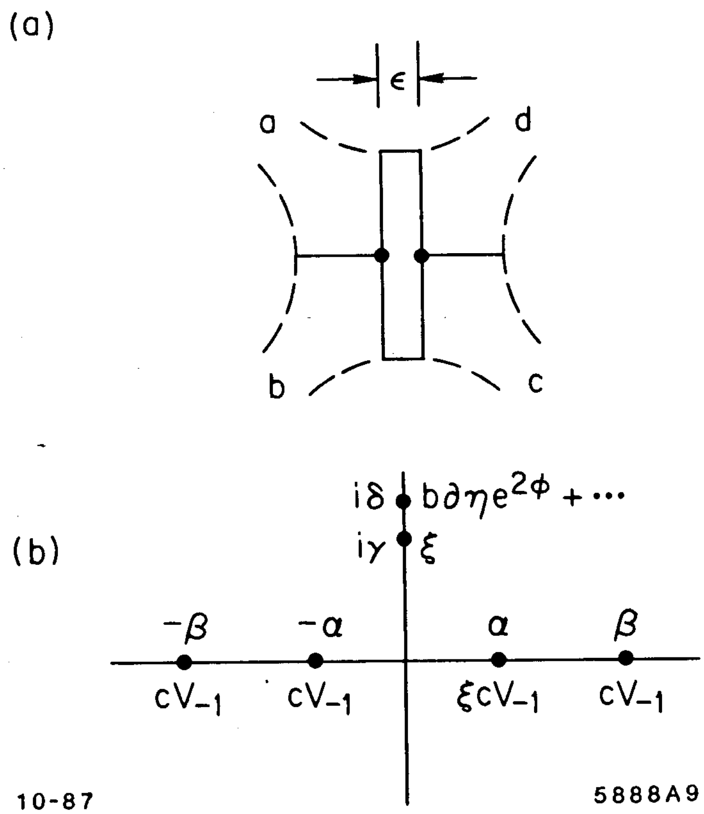

Fig. 9

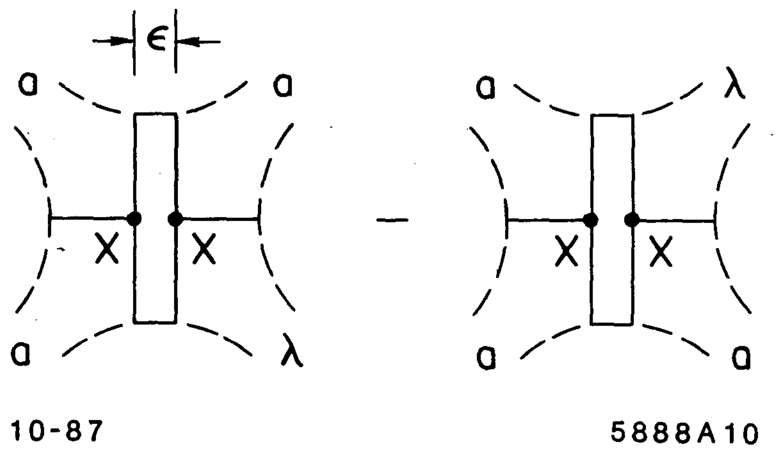

Fig. 10

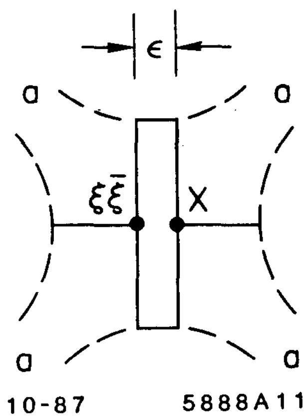

Fig. 11