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EMITTANCE PRESERVATION IN LINEAR COLLIDERS*

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1. INTRODUCTION

The purpose of this paper is to introduce the reader to some of the beam dynamics issues relevant to the preservation of the effective emittance or luminosity of a linear collider. The focus here is on the next generation of linear collider which should have a center of mass energy of about 1 TeV. For this case it is convenient to divide the accelerator into several discrete sections. First we have a damping ring for the production of the transverse and longitudinal emittance at the appropriate intensity. The bunch then must be prepared for injection into the main linac with at least two bunch rotations and a pre-acceleration at some sub-harmonic of the linac frequency. Next comes acceleration by the main linac, and finally the final focus to focus the beams to a small spot for collision. In this paper the focus is almost entirely on the main linac. All of the other sub-systems have similar beam dynamics problems but these will not be discussed here.

In the first section the equations of motion are introduced along with the smooth approximation. The motion is separated in the usual way into betatron oscillations about a central trajectory. In Section 3 we discuss the chromatic effects on the central trajectory. We examine the effect of coherent betatron oscillations in the absence of wakefields and also look at the effects of misaligned quadrupoles and trajectory errors. Transverse wakefields and beam break-up are discussed in Section 4 where the two particle model is used to derive a criterion for 'Landau damping' the instability. In Section 5 we turn pulse to pulse changes or *jitter*. This does not change the emittance of the beam, strictly speaking, but rather has the effect of causing the beams to miss each other at the interaction point. Various possible sources of jitter are considered. In Section 6 we discuss the problem of coupling. It seems advantageous to transport asymmetrical emittances from the damping ring to the final focus. The problem of dilution of the vertical emittance by the horizontal is studied in this case. Finally in the last section we apply the results to a specific self-consistent example.

2. THE EQUATIONS OF MOTION

The equations of motion for the transverse displacement of a particle in a constant magnetic field are given by

$$\frac{d^2x}{ds^2} + \frac{p_0 K(s)x}{p} = \frac{e\Delta B}{pc} \tag{2.1}$$

where $K(s)$ is the focusing function, ΔB is the bending field on the design orbit including all errors and corrections, p_0 is the design momentum, and p is the momentum of the particle. In (2.1) the variation of the momentum due to acceleration has been neglected. This can be taken into account by adiabatic damping and will be included when relevant. It is convenient to define

$$\delta \equiv \frac{p - p_0}{p} \tag{2.2}$$

which is a measure of the deviation of the momentum from the design momentum. With this change Eq. (2.1) becomes

$$x'' + K(s)(1-\delta)x = \frac{(\delta - 1)}{\rho(s)} \ , \tag{2.3}$$

where $\rho(s)$ is the actual instantaneous bending on the design orbit for a particle of the design momentum including all errors and corrections.

The solution of Eq. (2.3) contains all the information necessary to evaluate the effects of errors and their corrections on the effective emittance of the beam in the absence of transverse wakefields.

To motivate the analysis and to make a connection with the standard approach consider the case when $\delta = 0$. In this case the solution of Eq. (2.3) is usually split into two parts, a solution of the inhomogeneous equation plus a solution of the homogeneous equation as follows:

$$x = x_\beta(s) + x_0(s) \tag{2.4}$$

where x_β and x_0 satisfy

$$\begin{aligned} x_0'' + K(s)x_0 &= -\frac{1}{\rho(s)} \\ x_\beta'' + K(s)x_\beta &= 0 \ . \end{aligned} \tag{2.5}$$

In a storage ring x_0 is the closed orbit and is uniquely defined by the requirement that it be periodic. The betatron oscillation, x_β, is centered on the closed orbit. The rms emittance ϵ of the beam of particles is related to the betatron amplitude by

$$(x_\beta)^2_{\rm rms} \equiv \sigma_\beta^2 = \beta(s)\epsilon \tag{2.6}$$

where $\beta(s)$, the Courant-Snyder amplitude function, is a periodic function of s since the particle encounters the magnetic lattice periodically as it circulates in the storage ring.

In the case of a linac one can once again split the solution of Eq. (2.3) into the two parts shown in Eq. (2.5). In this case the solution of the inhomogeneous equation is uniquely specified by the initial position and slope rather than the requirement of periodicity. For this reason we refer to this as the *central trajectory*. However, if the beam is extracted from a damping ring then the initial conditions for the central trajectory are determined from the value of the closed orbit in the damping ring at the extraction point.

For the solution to the homogeneous equation it is sometimes useful to introduce a beta function for the linear accelerator in analogy to the beta function for a circular accelerator. However, in this case the beta function is determined by the initial conditions rather than the requirement of periodicity. This leads to questions of matching and for an ill defined source can lead to some ambiguity. But in the case of a linear accelerator with a damping ring injector, the beta function in the linear accelerator is uniquely defined by the periodic lattice parameters of the damping ring at the extraction point. More precisely, the beta function is uniquely defined by the magnetic elements in the damping ring and linear accelerator independent of the beam being transported.

In the approximation of linear magnetic focusing, $\delta = 0$ and no transverse wake fields, there is in principle no emittance dilution; however, if there are mismatches in the linear optics which are outside the range of correction elements, this can lead to an effective increase in the eventual spot size at the final focus. In addition, if the magnetic elements vary in position or field from pulse to pulse then this causes the central trajectory to vary from pulse to pulse and thus causes an effective increase in the emittance. The effects of this *jitter* will be treated in Section 5.

For the case of nonzero δ it is useful to deviate slightly from the standard practice of defining a dispersion function $D(s)$. In this case we simply split the motion as before except that x_0 and x_β satisfy

$$\begin{aligned} x_0'' + K(s)(1-\delta)x_0 &= \frac{(\delta - 1)}{\rho(s)} \\ x_\beta'' + K(s)(1-\delta)x_\beta &= 0 \ . \end{aligned} \tag{2.7}$$

Thus, the central trajectory has a chromatic dependence (as does the betatron oscillation about that orbit). In the next section we examine the effective emittance dilution from these chromatic effects.

3. CHROMATIC EFFECTS

The chromatic effect on the betatron oscillations, the homogeneous equation, is rather small in the linac. For this reason we treat only the equation for the central trajectory in this section. To study the chromatic effects on the central trajectory it is useful to smooth the focusing system while keeping the discrete nature of the bending errors and corrections. Thus, Eq. (2.7) is replaced by

$$x_0'' + k^2(1-\delta)^2 x_0 = \frac{(\delta - 1)}{\rho(s)} \tag{3.1}$$

where k is the betatron wave number and is related to the average beta function by

$$k = \frac{1}{\beta} \; . \tag{3.2}$$

3.1 The Chromatic Effect of a Coherent Betatron Oscillation

The solution of Eq. (3.1) leads to a chromatic central trajectory. Particles of different momentum travel on different orbits down the linac. As we shall see most of the chromatic dilution of the emittance comes from the variation of the betatron phase advance with momentum rather than from the spectrometer effect of the bending fields on the design orbit. To see this consider the effect of one single misplaced quadrupole (or some other localized bending field). For this single kick the bending radius is given by

$$\frac{1}{\rho(s)} = \theta_i \delta(s - s_i) \; , \tag{3.3}$$

where $\delta(s)$ is the Dirac delta function. If we assume an unperturbed trajectory upstream of the kick, then the solution of Eq. (3.1) is given by

$$x_i = \Theta(s - s_i)\frac{\theta_i}{k(1-\delta)} \sin\left[k(1-\delta)(s - s_i)\right] \; , \tag{3.4}$$

where $\Theta(s)$ is the step function.

To understand the basic mechanism of the dilution of emittance due to the chromatic central trajectory, it is useful to study this simplified model somewhat. To do this consider three test slices of the beam in momentum at $\delta = 0,\ -\sigma_\delta,\ \sigma_\delta$. In the absence of chromatic effects the central trajectory is simply a coherent betatron oscillation. In the presence of energy spread the different slices in momentum slowly get out of phase since the trajectories oscillate at slightly different frequencies. To illustrate this consider Fig. 1. This is a plot of normalized phase space. The transverse emittance of each slice in the figure is simply given by the dashed circle. To avoid emittance dilution all of the dashed circles must line up. The central circle shows the unperturbed case. The three slices mentioned above are plotted at a distance down the linac given by

$$s = s_i + \frac{\pi}{4k\sigma_\delta} . \tag{3.5}$$

In this case the slices at $\pm\sigma_\delta$ have moved $\mp\pi/4$ in phase, and thus if the initial orbit size is larger than the beam size, the beam emittance is greatly diluted.

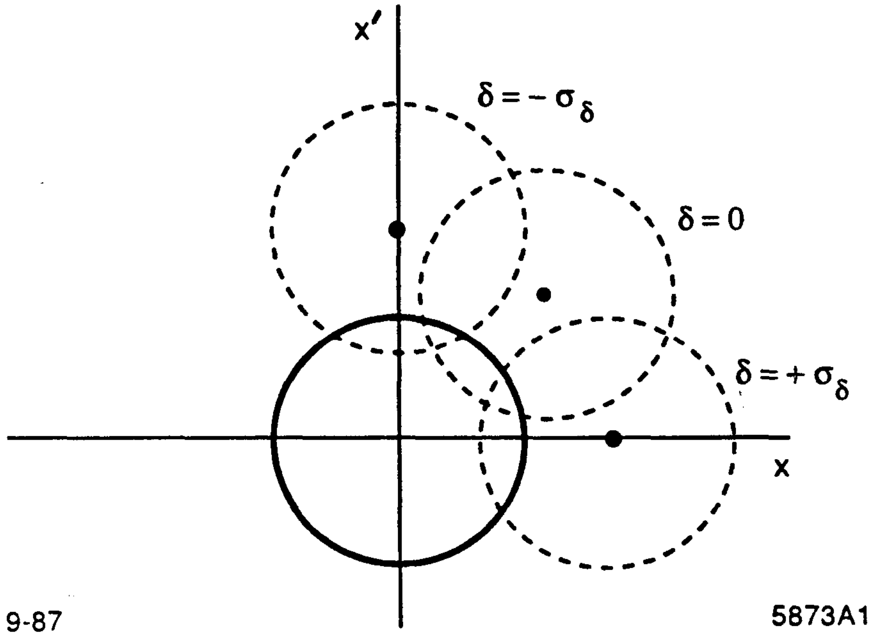

Fig. 1 The Chromatic Effect of a Coherent Betatron Oscillation

Before going any further it is possible specify a tolerance condition on an uncorrected betatron oscillation of the central trajectory. If the *chromatic* variation of the phase advance of the beam is greater than or the order of unity, then the peak of an uncorrected betatron oscillation, $\hat{x}_0$, must be locally smaller than the beam size to avoid emittance dilution. Generally,

$$\text{If} \quad \delta \int_0^L \frac{ds}{\beta(s)} >> 1 , \qquad \text{then} \quad \hat{x}_0 \ll \sigma_\beta . \tag{3.6}$$

In the case of a small chromatic effect we can estimate the tolerance to first order in δ by using Eq. (3.4). The chromatic part of the orbit is then given by

$$x_i(\delta) - x_i(0) \simeq \Theta(s - s_i)\hat{x}\delta k(s - s_i) \cos k(s - s_i) . \tag{3.7}$$

At the end of the linac this effect must be small compared to the beam size to avoid dilution of the effective emittance. This leads to the tolerance for an uncorrected betatron oscillation for weak chromatic effects:

$$\text{If} \quad \delta \int_0^L \frac{ds}{\beta(s)} < 1 \,, \qquad \text{then} \quad \hat{x}_0 \ll \frac{2\sigma_\beta(L)}{\delta\psi N_q} \,. \tag{3.8}$$

where ψ is the phase advance per cell, N_q is the number of quadrupoles in the linac, and $\sigma_\beta(L)$ is the beam size at the end of the linac. In this case the chromatic tolerance has been weakened. In addition, since by assumption the effect is small and therefore linear in δ, it can be measured and possibly corrected.

3.2 THE CHROMATIC EFFECT OF A CORRECTED TRAJECTORY

Now we would like to calculate the effect of a set of random misalignments of quadrupoles which, however, have been corrected so that the beam trajectory is within some tolerance. It is important to emphasize that these are fixed misalignment errors corrected by fixed correctors. The case of pulse to pulse alignment jitter, which cannot be corrected by fixed corrector settings, will be discussed in Section 5.

A distinguishing characteristic of the corrected trajectory is that it does not grow as we proceed down the linac because correctors are used to suppress the growth. This is not true of an uncorrected random alignment error which yields an ever growing central trajectory. The key difference here is that the errors and correctors are correlated.

To model this effect, consider first the following problem. Consider a model linac (no acceleration) in which quadrupoles have a beam position monitor (BPM) in them and a corrector superimposed. In one quadrupole, let the BPM be misplaced relative to the quad center, and let all the quads be aligned perfectly. Steer the beam to zero the measured trajectory. The perceived trajectory is then a straight line; however, the actual trajectory has a bump in it at the location of the misplaced BPM as shown in Fig. 2.

To achieve this apparently zero trajectory, it was necessary to use 3 correctors, one at the bad BPM and the two adjacent to it. A particle with the nominal beam energy has a zero trajectory for the rest of the linac; however, an off momentum particle has a trajectory given to first order in δ by

$$x_\delta \equiv x(\delta) - x(0) = (\Delta x)\psi\delta \sin(ks(1-\delta) - \psi/2), \quad ks > \psi \tag{3.9}$$

where Δx is the BPM placement error and ψ is the phase advance for one cell.

Thus, off-momentum particles follow a non-zero trajectory which can lead to emittance dilution. To estimate the effect for a linac, consider a random sequence of misplaced BPM's and their associated orbit bumps and chromatic residuals. This leads to an orbit which because of

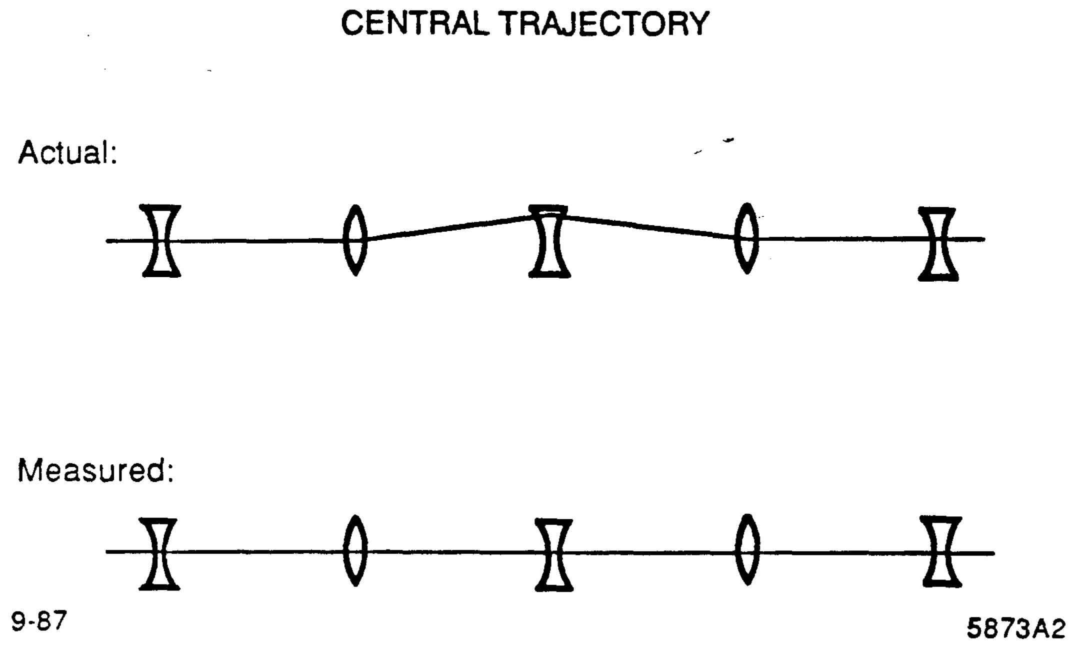

Fig. 2 The Corrected Trajectory With One BPM Error.

the correlations does not grow with s. However, the chromatic effects do not cancel and so yield a net chromatic beam size. The orbit for an off momentum particle is given by

$$x_\delta = \sum_{i=1}^{N_q} (\Delta x)_i \psi \delta \sin(k(s-s_i)(1-\delta) - \psi/2)\Theta(s - s_i - \psi/k) \tag{3.10}$$

where N_q is the number of quadrupoles in the lattice. The square of the beam size is given by

$$x_\delta^2 = \sum_{i,j}^{N} \Delta x_i \Delta x_j \psi^2 \delta^2 S_i S_j \Theta(s - s_i - \psi/2)\Theta(s - s_j - \psi/k) \ , \tag{3.11}$$

where S_i stands for

$$\sin(k(s-s_i)(1-\delta) - \psi/2) \quad . \tag{3.12}$$

Now consider an ensemble average over an uncorrelated sequence of Δx_i's. In this case only terms with $i = j$ in the sum contribute; which yields

$$\langle x_\delta^2 \rangle = \sum_{i=1}^{N_q} \langle \Delta x^2 \rangle \psi^2 \delta^2 \sin^2(k(s-s_i)(1-\delta) - \psi/2)\Theta(s - s_i - \psi/2) \ . \tag{3.13}$$

Replacing the sum with an integral, we obtain

$$(x_\delta)_{rms} = (\Delta x)_{rms} \psi \delta \sqrt{\frac{N}{2}} \ . \tag{3.14}$$

For the particular model chosen $(\Delta x)_{rms}$ is the rms BPM misalignment; however, it could have been either the quadrupole placement error or the rms error in the position measurement.

To specify a tolerance, we require that this effect be small compared to the beam size at the end of the linac, $\sigma_\beta(L)$. This yields a tolerance on Δx_{rms} given by

$$\Delta x_{rms} < \frac{\sigma_\beta(L)}{\psi\delta}\sqrt{\frac{2}{N}} \ . \tag{3.15}$$

Comparing with Eq. (3.8) the effect of a sequence of random corrected errors leads to a smaller dilution of the emittance than an uncorrected betatron oscillation and thus to much weaker orbit tolerances. Once again if the deviation of the phase advance for the entire linac is small due to energy spread as in Eq. (3.8), the corrected orbit yields a superposition of linear effects, and it may be possible to measure and correct this 'dilution'. However, Eq. (3.15) is valid for weak or strong chromatic effects and is modified only slightly if adiabatic damping and taper of the beta function with energy are included. These latter effects are included in Section 5.

4. THE TRANSVERSE WAKEFIELD AND BEAM BREAK-UP[1]

4.1 The Transverse Wake

Consider two particles travelling down a linac structure as shown in Fig. 3. If the leading particle is offset transversely, it induces a deflecting field behind it. This deflecting force is characterized by the transverse wakefield, the value of which depends upon the longitudinal distance behind the first particle. The typical shape is shown in Fig. 4 taken from Ref. 1 and consists of an initial rise from zero followed by oscillations. The kick felt by the trailing particle is given by

$$\frac{d^2x_2}{ds^2} = \frac{eQW_x(z_2)x_1}{E} \tag{4.1}$$

where Q is the charge of particle 1, x_1 is the offset of particle 1, z_2 is the longitudinal separation of the two particles, and E is the energy.

To scale the effect for different wavelengths, note that if we scale all dimensions

$$W_x(z) = \left[\frac{\lambda_0}{\lambda}\right]^3 W_x^0(z\lambda_0/\lambda) \quad . \tag{4.2}$$

However, due to the shape of the longitudinal wake, the wakefield also depends sensitively on the separation of the two particles or more precisely on the length of the bunch being studied. Frequently, over a limited range, this dependence is approximated by a linear variation in the longitudinal separation or bunch length.

This decrease in the transverse wake for small distances can be exploited. If we scale to shorter wavelengths, the increase in transverse wakefield can be partially offset by a decrease in the bunch length (beyond simple scaling).

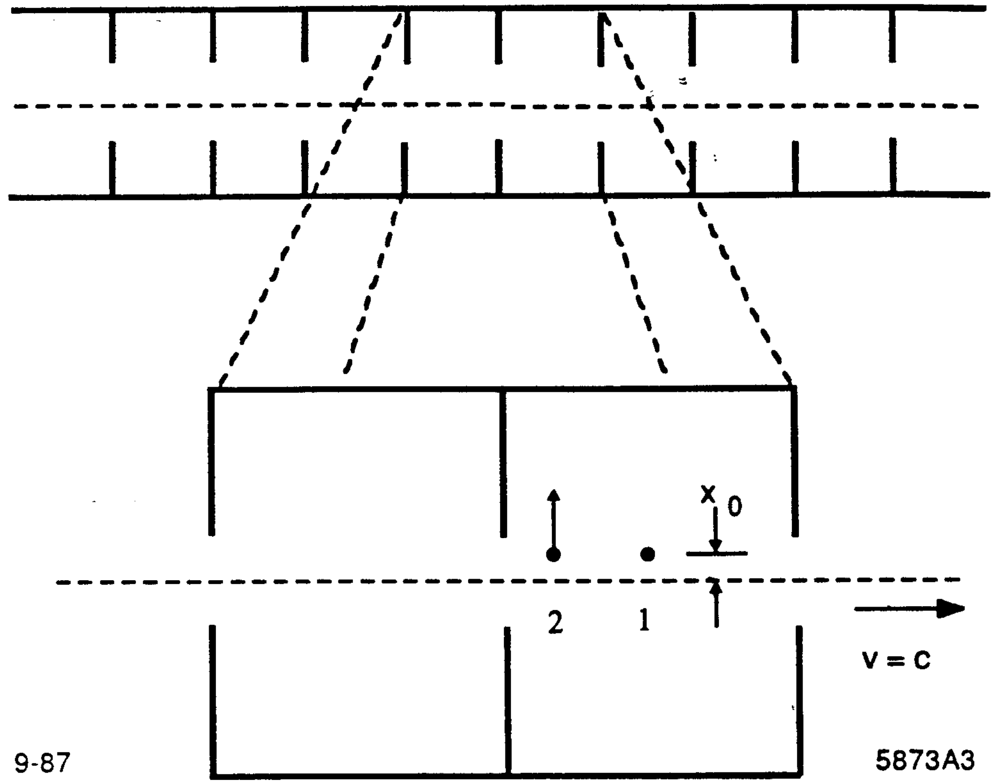

Fig. 3 Two Particles Off Axis in a Linac

Another method for decreasing the transverse wake consists of opening the iris holes in the linac structure. The wakefield at short distance behind the leading particle is dominated by the closest piece of metal and is independent of the distance to the outer wall of the cavity. Therefore, for short bunches one can open the iris holes while keeping the wavelength fixed. However, this decreases the effectiveness of the accelerating structure; therefore, one must balance the transverse benefit of increasing the iris size with the increased rf power necessary to drive the structure.

4.2 Transverse Beam Break-up and 'Landau Damping'

To see the effects of the transverse wake, let us consider a two-particle model as shown in Fig. 3. We place 1/2 of the charge in the bunch into each macro-particle and separate the particles by a distance ℓ which should be set to about $2\sigma_z$ when comparing to actual bunch distributions. The distance between the particles is fixed since they both travel at c, the speed of light; therefore, the wakefield at the trailing particle is fixed. The equations of motion for the two particles in the presence of the external focusing system are

$$\begin{aligned} x_1'' + k^2 x_1 &= 0 \\ x_2'' + (k+\Delta k)^2 x_2 &= \frac{e^2 N W(\ell) x_1}{2E} \,, \end{aligned} \tag{4.3}$$

where N is the total number of particles in the 2 macro particles, and $W(\ell)$ is the wakefield at

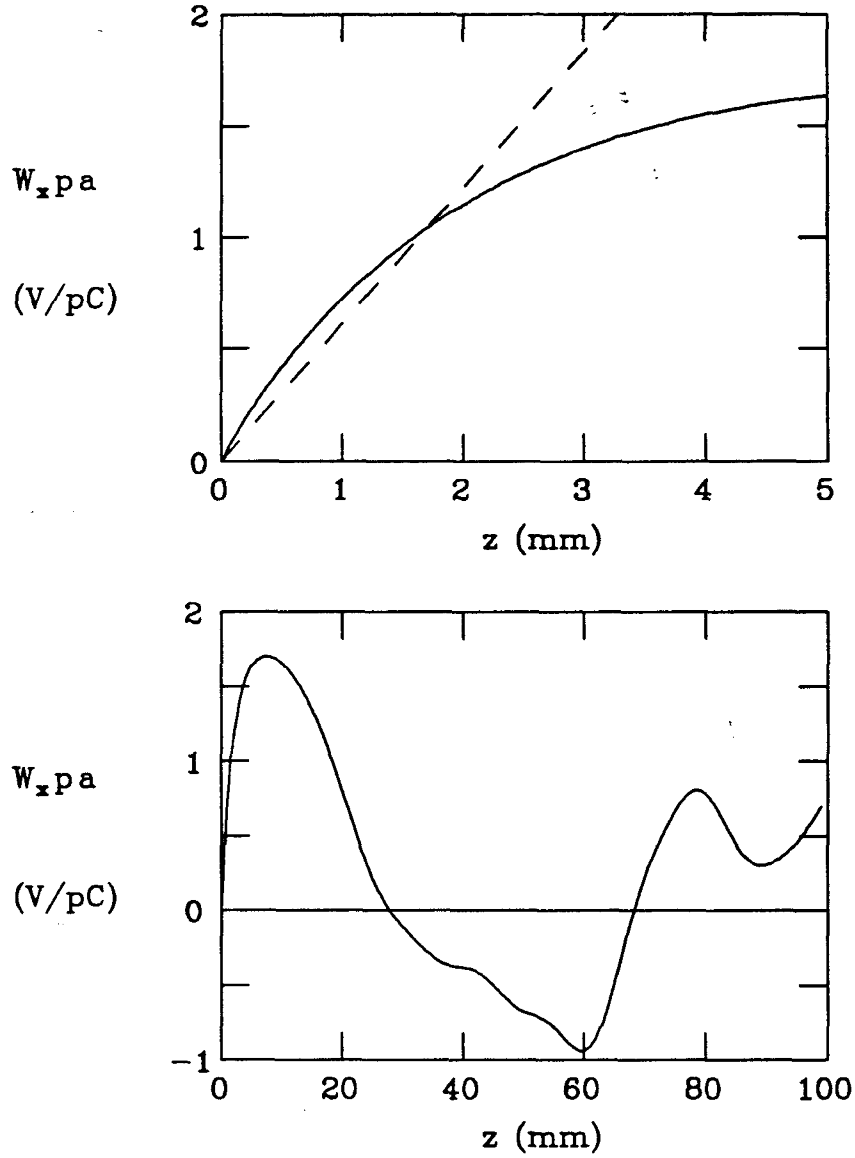

Fig. 4 Transverse Wake for the SLAC structure. p is the cell length and a is the iris opening.

the second particle.

Notice that the external focusing has once again been smoothed as in Eq. 2.1, and the second particle feels a different focusing force characterized by the parameter Δk. This might be due to a difference in energy from the front to the back of the bunch; in this case, Eq. 2.1 yields

$$\Delta k = -\delta k \quad . \tag{4.4}$$

More precisely, for a general lattice we need to evaluate an average chromaticity ξ defined by

$$\frac{\Delta k}{k} = \xi \frac{\Delta E}{E} = \xi \delta \quad . \tag{4.5}$$

For typical lattices ξ is close to -1, and thus the smooth approximation is not too bad.

It is also possible to vary the focusing function along the bunch by the use of RF focusing. This decouples the focusing field from the energy spread but couples it to position within the bunch.

Now let us consider the solution of Eq. (4.3) in which both particles have the same initial offset, $\hat{x}$. For small $\Delta k/k$, the solution for the difference between the transverse positions is given by

$$x_2(s) - x_1(s) = \hat{x}\left(2 - \frac{e^2 NW}{2Ek\Delta k}\right) i \sin\left(\frac{\Delta k s}{2}\right) e^{i\left(k+\frac{\Delta k}{2}\right)s} . \tag{4.6}$$

To study Eq. (4.6) it is useful to consider 3 different cases:

Case 1. $\Delta k = 0$

In this case the difference grows linearly

$$x_2(s) - x_1(s) = -i\frac{e^2 NW \hat{x} s}{4Ek} e^{iks} , \tag{4.7}$$

which yields an amplification factor given by

$$\frac{x_2 - x_1}{\hat{x}} = \frac{e^2 NW s}{4Ek} . \tag{4.8}$$

The linear growth is simply due to a linear oscillation driven on resonance. In an actual beam, the growth of the tail of the beam is much faster and has been calculated in Ref. 2.

Case 2. $\Delta k \neq 0$ **Δk very small.**

In this case the linear growth is turned over leading to a maximum amplification factor of

$$\frac{x_2 - x_1}{\hat{x}} = \left(2 - \frac{e^2 NW}{2Ek\Delta k}\right) \simeq -\frac{e^2 NW}{2Ek\Delta k} \tag{4.9}$$

the growth stops at $s = \pi/\Delta k$ and there is a beating at the maximum amplitude.

Case 3. **"Landau Damping"**

In this case, if we examine Eq. (4.9), we see that the amplification can be set to zero provided that

$$\frac{e^2 NW}{4Ek\Delta k} = 1 . \tag{4.10}$$

This yields no growth at all; in fact simulations of actual beam distributions show genuine damping of the oscillation.[1] This effect is loosely referred to as Landau damping; however, it is really only a cousin to Landau damping. Landau damping refers to the lack of growth of coherent oscillations when there is some uncorrelated spread in the oscillation frequencies of the particles in the bunch. In this case, we see the lack of growth of a particular mode of oscillation. Since the bunches in a linac are quite short, it is likely that offsets occur to both the head and tail simultaneously. If, however, the head and tail were offset on opposite sides of the axis, this exact cancellation would not take place, although the amplitude is limited by a factor similar to that in Eq. (4.9).

The lack of growth is simply due to a cancellation of forces. The wakefield force is exactly cancelled by the additional focusing force for a trailing particle of slightly lower momentum. It is useful to rewrite the condition for the case of momentum spread:

$$\frac{e^2 N W \beta^2}{8 E \delta_L} = 1 \ . \tag{4.11}$$

In this case δ_L is the half spread in energy required for Landau damping and the average beta function β has been used rather than the wave number k.

5. PULSE TO PULSE CHANGES: JITTER

In a linear collider the effective spot size can be enhanced by pulse to pulse changes in the central trajectory. It is obvious that these must be kept small compared to the beam size at the final focus or else the beams would never collide. In fact, it is the local beam size which sets the scale for this problem since the final focus de-magnifies the jitter of the spot as well as the spot itself. The time scale for jitter is set by the repetition rate. Slow changes which can be sampled well can be corrected by feedback, a technique which is used extensively at the SLC at SLAC. All those effects which happen too fast to be cured by feedback are lumped into the category of *jitter*.

5.1 Injection Jitter

Let us consider the position jitter as we enter the main linac. Consider an abrupt change of position Δx_0. Then to keep the head of the bunch colliding we require that

$$\Delta x_0 << \sigma_\beta(0), \tag{5.1}$$

where $\sigma_\beta(0)$ is the beam size at the beginning of the linac. If we are using Landau damping to suppress the growth of the tail, then this is the final story. Otherwise the tail effect may be amplified with the amplification factor in Eq. (4.9). If we trace upstream to the source of the jitter in the injector, then it must come from the time variation of some bending field leading to some variation in bending angle $\Delta x'_0$. In order for this to be a small effect it must be small compared to local divergence of the beam, that is

$$\Delta x'_0 << \frac{\sigma_\beta}{\beta} . \tag{5.2}$$

Of course, if there are several sources of jitter, then these must be added together with the appropriate phases. The effect of large numbers of elements is considered in the next section.

5.2 Quadrupole Alignment Jitter

If we consider the changes due to the motion of one quadrupole, then we arrive at the same conclusions as in the previous section since the betatron oscillation just propagates down the linac. Therefore, in the absence of Landau damping the tail is amplified according to Eq. (4.9). If we use Landau damping for the tail, then the growth of the tail is controlled; however, we still must control the head of the bunch.

For a general sequence of misalignments we must superimpose the betatron oscillation of each misalignment to obtain

$$x(s) = \sum_i q_i d_i \beta_i \sin k(s - s_i)\Theta(s - s_i) \tag{5.3}$$

where q_i is the inverse focal length of the i^{th} quadrupole and d_i is the change in the position of the quadrupole on the present pulse. In this equation we have smoothed the focusing and ignored acceleration. To include acceleration we must include the adiabatic damping of the betatron oscillations and also include the profile of quadrupole strengths and beta functions. For a general lattice the trajectory due to a sequence of misalignments is

$$x(s) = \sum_i q_i d_i [\beta(s)\beta(s_i)]^{1/2} \left[\frac{\gamma(s_i)}{\gamma(s)}\right]^{1/2} \sin[\psi(s, s_i)]\Theta(s - s_i) \tag{5.4}$$

where

$$\psi(s, s_i) = \int_{s_i}^{s} \frac{ds'}{\beta(s')} . \tag{5.5}$$

To be specific let us consider a lattice in which

$$\begin{aligned} \beta(s) &= \beta_0 \left[\frac{\gamma(s)}{\gamma_0}\right]^{1/2} \\ q(s) &= q_0 \left[\frac{\gamma_0}{\gamma(s)}\right]^{1/2} . \end{aligned} \tag{5.6}$$

This scaling can be realized by scaling the length of the cell and the length of all quadrupoles as

$$\ell_{\text{cell}} \propto \gamma^{1/2} \tag{5.7}$$

which yields a phase advance per cell which is constant. This scaling of the beta function and integrated focusing strengths was selected because it yields a Landau damping criterion which is independent of energy. The trajectory in this case is given by

$$x(s) = \sum_i q_0 d_i \beta_0 \left[\frac{\gamma(s_i)}{\gamma(s)}\right]^{1/4} \sin[\psi(s,s_i)]\Theta(s-s_i) \ . \tag{5.8}$$

Let us now consider a random sequence of uncorrelated movements d_i. To estimate the effect we perform an ensemble average as in Section 2 to obtain

$$\langle x^2(s)\rangle = \sum_i q_0^2 \langle d^2\rangle \beta_0^2 \left[\frac{\gamma(s_i)}{\gamma(s)}\right]^{1/2} \sin^2[\psi(s,s_i)]\Theta(s-s_i) \ . \tag{5.9}$$

For a large number of magnets we can replace the sum by an integral to obtain

$$x(L)_{rms} = q_0 \beta_0 d_{rms} \sqrt{\frac{N}{3}} \ . \tag{5.10}$$

The displacement at the end of the linac must be small compared to the beam size there. This yields a tolerance on *random magnet-to-magnet jitter* given by

$$d_{rms} \ll \frac{\sigma_\beta(L)}{q_0\beta_0}\sqrt{\frac{3}{N}} \ . \tag{5.11}$$

The tolerance above is probably a very pessimistic one. Ground motion is far from being uncorrelated from magnet to magnet. A more realistic calculation would start from the noise spectrum due to ground motion and filter that with the response function of the magnet supports.

5.3 Jitter of Transverse Kicks in Acceleration Sections

It is well known that due to various errors, an accelerator section in a linac typically gives the beam a small transverse kick. These kicks may be due to coupler asymmetry, fixed misalignments or symmetry errors in construction. In addition, it has recently been pointed out that field emission currents may also cause large enough deflecting fields to cause a problem.[3] If the kicks, from whatever source, are constant in time, the orbit can simply be corrected by dipole magnets to the required tolerance. However, if the kick varies from pulse to pulse, this can cause the beams to miss in the same way that a moving quadrupole can.

As an example consider an acceleration section which is rotated end to end by a small angle α. Then the transverse momentum kick is related to the longitudinal by

$$\Delta p_\perp = \alpha \Delta p \ , \tag{5.12}$$

where Δp is the momentum gain in the section. In an actual accelerator α will vary from section to section but remain fixed in time provided there is little alignment jitter. In this case, the transverse jitter is almost entirely due to the jitter in the energy gain of the acceleration section, that is

$$\delta(\Delta p_\perp) \equiv \delta p_\perp = \alpha \delta(\Delta p) \tag{5.13}$$

Note that due to unfortunate poor planning, δ in this section refers to the jitter of a quantity rather than a relative momentum deviation.

To calculate the effect for a random sequence of transverse kicks we can follow the previous section with only slight modifications. As in Eq. (5.4) the trajectory change due the change in transverse kicks throughout the linac is

$$x(s) = \sum_i^{N_s} \frac{(\delta p_\perp)_i}{p_i} [\beta(s)\beta(s_i)]^{1/2} \left[\frac{\gamma(s_i)}{\gamma(s)}\right]^{1/2} \sin[\psi(s,s_i)]\Theta(s-s_i) , \tag{5.14}$$

where N_s is the number of acceleration sections in the linac. If we scale the lattice as in the previous section, the trajectory is given by

$$x(s) = \sum_i^{N_s} \left[\frac{\gamma_0}{\gamma_f}\right]^{1/2} \beta_0 \frac{(\delta p_\perp)_i}{p_0} \left[\frac{\gamma(s)}{\gamma(s_i)}\right]^{1/4} \sin[\psi(s,s_i)]\Theta(s-s_i) . \tag{5.15}$$

If we now consider an uncorrelated random sequence and ensemble average as in the previous section, we find

$$x(L)_{rms} = \frac{(\delta p_\perp)_{rms}}{p_0} \beta_f \frac{\gamma_0}{\gamma_f} \sqrt{N_s} . \tag{5.16}$$

It is useful to write this in terms of Δp, the momentum change per section, which yields

$$x(L)_{rms} \simeq \frac{(\delta p_\perp)_{rms}}{\Delta p} \frac{\beta_f}{\sqrt{N_s}} , \tag{5.17}$$

for $\gamma_f \gg \gamma_0$. This orbit change must be small compared to the beam size at the end of the linac which yields a tolerance on the jitter of transverse kicks in acceleration sections given by

$$\frac{(\delta p_\perp)_{rms}}{\Delta p} \ll \frac{\sigma_\beta(L)}{\beta_f} \sqrt{N_s} . \tag{5.18}$$

6. COUPLING

From the parameters listed in Table 1, you see that in the example chosen the emittance of the beam is asymmetrical. The vertical emittance is 100 times smaller than the horizontal. Damping rings normally produce asymmetrical emittances and this has been assumed in the example discussed Section 7. Of course this has implications on the orbit tolerances and misalignments in the damping rings; these will not be discussed in detail here. Qualitatively one must keep the vertical dispersion much smaller than the horizontal dispersion while keeping the betatron coupling also quite small. This task is aided typically by the inclusion of skew quadruples in the lattice as compensating elements. A basic problem is the vertical orbit through sextupoles since these give an effective skew quadrupole field.

The primary purpose in this section is to estimate the effect of a set of rotations in the linac which might couple the two planes and so lead to an effective dilution of the vertical emittance. Actually, since this is a linear effect, one could simply take out the coupling with appropriately space skew quadrupoles at the end of the linac. However, since this effect is only one of many, it might be somewhat difficult to separate wakefield tails combined with chromatic effects from simple coupling.

Consider the problem of coupled betatron oscillations in the linac. The equations of motion of an on-momentum particle are

$$\begin{aligned} \frac{d^2x}{ds^2} + K(s)x + M(s)y &= 0 \\ \frac{d^2y}{ds^2} - K(s)y + M(s)x &= 0 \ , \end{aligned} \tag{6.1}$$

where $M(s)$ is the skew focusing function. For a rotated quadrupole with focusing function $K_0(s)$ rotated by a small angle θ, the focusing functions are

$$\begin{aligned} K(s) &\simeq K_0(s) \\ M(s) &\simeq 2\theta K_0(s) \ . \end{aligned}$$

To evaluate the effect of small rotations recall that we are focusing on the dilution of a very small vertical emittance by coupling to a much larger horizontal emittance. For this reason we can make the approximation that the horizontal equation is relatively unperturbed by y while the y equation is very much perturbed by x. This leads to

$$\begin{aligned} \frac{d^2x}{ds^2} + K(s)x &\simeq 0 \\ \frac{d^2y}{ds^2} - K(s)y &\simeq -M(s)x \ , \end{aligned} \tag{6.2}$$

Now we would like to evaluate the effect of a single rotated quadrupole. The skew focusing function is therefore given by

$$M_i(s) = 2\theta_i q \delta(s - s_i) \ , \tag{6.3}$$

where q is the inverse focal length of the quadrupole. The resulting betatron oscillation is given by

$$y(s) = 2\theta_i x(s_i)[\beta(s_i)\beta(s)]^{1/2} \sin \psi_y(s, s_i) \ . \tag{6.4}$$

where $x(s_i)$ is the betatron oscillation in the x direction and $\psi_y(s, s_i)$ is the vertical phase advance. If we scale the quadrupole strengths and beta functions as in Section 5 and include the adiabatic damping of the oscillations in both transverse directions, the resulting vertical orbit induced by a sequence of rotations is given by

$$y(s) = \sum_i \theta_i q_0 \hat{x}_0 \beta_0 \left[\frac{\gamma_0}{\gamma(s)}\right]^{1/4} \cos(\psi_x(s_i) + \phi_x) \sin[\psi_y(s, s_i)]\Theta(s - s_i) \tag{6.5}$$

where $\hat{x}_0$ is the initial amplitude of the horizontal betatron oscillation and ϕ_x is the initial phase.

Now consider a sequence of random misalignments and perform an ensemble average as in Section 3. Then the vertical betatron oscillation at the end of the linac driven by horizontal oscillations is

$$y_{rms} = \theta_{rms} q_0 \beta_0 \sigma_x(L) \frac{\sqrt{N_q}}{2} \ , \tag{6.6}$$

where $\sigma_x(L)$ is the horizontal beam size at the end of the linac. This must be small compared to the vertical beam size at the end of the linac and thus yields a tolerance on the random rotations given by

$$\theta_{rms} \ll \frac{\sigma_y(L)}{\sigma_x(L)} \frac{2}{q_0 \beta_0} \frac{1}{\sqrt{N_q}} \ . \tag{6.7}$$

7. A NUMERICAL EXAMPLE

In this section we evaluate the various tolerances for the example shown in Table 1. This example is taken from Ref. 4 which is a self consistent parameter set although not necessarily the optimum one. Rather than simply plugging in numbers, we will briefly discuss each tolerance and evaluate it in the case given in Table 1. In those cases in which the beam size sets the tolerance, it is quoted for the *vertical* direction; the corresponding tolerance for the horizontal direction is a factor of 10 larger due to the larger emittance.

TABLE 1. Self-Consistent Parameters from Ref. 4

Parameter	Symbol	Value
Center-of-mass energy	E	1.0 TeV
Max Luminosity 10^{33} cm^{-2} sec^{-1}	$\mathcal{L}_{\max}$	1.3
Beamstrahlung E loss	δ_B	.17
Maximum accelerating gradient	$\mathcal{E}_a$	186 MeV/m
Linac length (excluding final focus)	L	3.4 km
Frequency	Hz	220
Final spot size (vertical) (nm)	σ_y	1.0
Final spot size (horizontal) (μm)	σ_x	.19
Particles per bunch (10^{10})	N	.8
Bunch length (mm)	σ_z	.026
Wavelength	λ	17 mm
Transverse wake potential V pC^{-1} m^{-2}	$W_t(2\sigma_z)$	1.2×10^4
Average β at end of linac (m)	β_f	14
Number of quads in linac	N_q	620
Number of acceleration sections	N_s	5230
Injection Energy	$\gamma_0 mc^2$	5 GeV
Normalized emittance (horizontal) 10^{-6} m	ε_{nx}	2.5
Normalized emittance (vertical) 10^{-8} m	ε_{ny}	2.5

7.1 Landau Damping

This section is taken out of turn because we would like to specify the energy spread before the next section on chromatic effects. The condition for Landau damping is given Eq. (4.11). For the parameters shown in Table 1, the correlated half energy spread is

$$\delta_L = 8 \times 10^{-4} . \tag{7.1}$$

The beam also has an uncorrelated part due to residual beam loading energy spread and a residual longitudinal emittance from the damping ring. The longitudinal energy spread left from the initial longitudinal emittance is damped to this level in the first 1/10 of the linac. The spread due to the longitudinal wake is also about the same size as the correlated spread above.[4] Therefore, in the subsequent sections we assume a δ given by that for Landau damping.

7.2 Chromatic Effects

First we need to evaluate the chromatic phase advance. This is given by

$$\delta\psi_T = \delta \int_0^L \frac{ds}{\beta(s)} , \tag{7.2}$$

which for the variation of the beta function in Eq. (5.6) yields

$$\delta\psi_T \simeq \frac{2\delta L}{\beta_f} , \tag{7.3}$$

where β_f is the beta function at the end of the linac. In the case shown in Table 1 the chromatic phase advance is

$$\delta\psi_T \simeq 0.4 , \tag{7.4}$$

which is small enough to use Eq. (3.8). This yields the tolerance on an uncorrected betatron oscillation given by

$$\hat{x}_0 \ll 2.6\sigma_\beta = 1.5\mu\text{m} . \tag{7.5}$$

To estimate the tolerance on quadrupole misalignment or errors in position measurement we turn to Eq. (3.15) to find

$$\Delta x_{rms} \ll 45\sigma_\beta = 27\mu\text{m} . \tag{7.6}$$

With careful measurement at the end of the linac, we may be able to control the coherent oscillation effects by controlling the launch at the beginning of the linac. This should also be useful in correcting the linear part of the chromatic dilution due to quadrupole misalignment.

7.3 Injection Jitter

From Eq. (5.1) the jitter in position must be much less than the spot size

$$\Delta x_0 \ll 2\,\mu . \tag{7.7}$$

To make much more precise statements on injection jitter we really need to know more details about the injection system. However, if we look up-stream to a possible source of jitter, we find the kicker magnet in the damping ring as an important candidate. To ameliorate the problem there, it is desirable that the kick angle be as small as possible. The jitter of the kick angle is dominated by power supply jitter which gives a fixed fraction of the total angular kick. The kick angle should be reduced until the angular jitter is small compared to the beam divergence at the kicker. It is still possible to extract the beam in this case. If the jitter is 10^{-3} of the kick and we set this to be 1/10 of the beam divergence, then we can kick the beam by $100\sigma_\beta$ if we look $\pi/2$ downstream in betatron phase.

7.4 QUADRUPOLE ALIGNMENT JITTER

To calculate the tolerance on quadrupole jitter we use Eq. (5.11). In a thin lens cell with a $\pi/2$ phase advance, the product of $q_0\beta_0$ is given by

$$q_0\beta_0 = 2\sqrt{2} \ , \tag{7.8}$$

where β_0 here is the average of β_{max} and β_{min} in the cell. This yields a tolerance for the example in Table 1 of

$$d_{rms} \ll 0.03\sigma_\beta(L) \simeq .02\mu\mathrm{m} \ . \tag{7.9}$$

Although this is quite small recall that this motion must occur at high frequency and in an uncorrelated fashion from magnet to magnet.

7.5 JITTER OF TRANSVERSE KICKS IN ACCELERATION SECTIONS

We evaluate the jitter in transverse momentum kick relative to the momentum gain in a typical section using Eq. (5.18). This yields

$$\frac{(\delta p_\perp)_{rms}}{\Delta p} \ll 3 \times 10^{-6} \ . \tag{7.10}$$

This same tolerance evaluated for the SLC at SLAC is

$$\frac{(\delta p_\perp)_{rms}}{\Delta p} \ll 6 \times 10^{-5} \ . \tag{7.11}$$

This sets tight tolerances on section alignment and asymmetries induced by asymmetric couplers or construction errors.

7.6 COUPLING

To evaluate the effect of random rotations of quadrupoles we use Eq. (6.7). This formula is only valid for $\sigma_y \ll \sigma_x$; however, in the example in Table 1 the beam sizes differ by an order of magnitude so this is well satisfied. The tolerance for the example shown is

$$\theta_{rms} \ll 3\,\mathrm{mrad} \ . \tag{7.12}$$

This tolerance is easily achieved with conventional survey techniques.

8. CONCLUSION and ACKNOWLEDGEMENTS

Various beam dynamics issues have been discussed here to introduce the reader to many tolerance requirements for the main linac in a linear collider. This treatment has not been exhaustive, but highlights the main effects. One important effect which has not been discussed is the tolerance on alignment of the acceleration sections. For typical parameters this tolerance is usually much weaker than those presented here.

In addition, nothing has been said of tolerances or beam dynamics issues in other sub-systems. In the damping ring the requirement of small vertical emittance puts tight tolerances on the vertical orbit. However, PEP at SLAC, the VUV ring at BNL, and others have achieved the emittance ratio in Table 1; therefore, it is not unreasonable. In the bunch compressors, emittance dilution due to chromatic effects must be avoided. Finally, in the final focus there will be tight tolerances on orbits through sextupoles to avoid coupling and also tight tolerances on the jitter of the final quadrupoles.

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