SLAC–PUB–4433
October 1987
(A)

# A QUALITATIVE STUDY OF WAKE FIELDS FOR VERY SHORT BUNCHES*

R. B. PALMER
*Stanford Linear Accelerator Center*
*Stanford University, Stanford, California 94305*
*and*
*Brookhaven National Laboratory*
*Upton, New York 11973*

Contributed to the *Proceedings of the Workshop on Impedance Beyond Cutoff, Berkeley CA, 18–21 August 1987*

---

*Work supported by Department of Energy contracts DE–AC03–76SF00515 (SLAC) and DE–AC02–76C0016 (BNL).

# 1. INTRODUCTION

When a charged particle (or bunch of particles) passes through some passive but conducting structure, it induces electromagnetic wake fields that are either reflected from the structure or remain in it after the particle has passed. Clearly the energy needed to establish these fields comes form the particle, and it must always be that the wake fields act back on the particle so as to decelerate it.

It can easily be seen that as the energy (or gamma) of the particle rises, it is able to excite ever higher frequency wake fields, and will thus tend to loose ever more of its energy. The question then is what the dependence is, and whether there is some cutoff. An analogous question concerns bunches of finite length. In this case, provided the energy is high enough, the maximum frequency excited depends on the bunch length. The question, in this case, is the dependency of the energy loss on the bunch length. Does the loss rise without limit as the bunch becomes arbitrarily small?

Before 1972 there was considerable confusion[1] about these questions, but, at least according to J. D. Lawson,[2] the problem was then solved by E. Keil.[3] In Keil's introduction he states:

> "Single cavity models[4,5] show an increase of the radiation loss as $\gamma^{1/2}$ However, their validity breaks down above some value of $\gamma$ when interference between successive cavities becomes an important factor, and periodic models are more appropriate. In cylindrical geometry, they yield a radiation loss independent of $\gamma$, and in planar geometry decrease of the radiation loss[6] as $\gamma^{-1/2}$."

The single cavity result can be obtained[7] from numerical solutions, including calculation using time dependent general purpose programs like TBCF. The single cavity result can also be obtained from diffraction considerations.[4] The multiple cavity result can, in principle, be obtained from numerical calculations but solutions for very short bunches require an analystic extension using an optical resonator model.[8]

Despite E. Keil's paper and Lawson's faith in it, there remains some uncertainly about the conclusion. The applicability of the optical resonator model to this problem is not so obvious, and a true analytical solution of the multiple cavity problem is not yet available.

In this paper I present a qualitative treatment of both single and multiple cavity wakefields. Unfortunately, I am unable to obtain quantitative results, but I confirm E. Keils conclusion as to the energy and bunch length dependencies. And hopefully, I provide some physical insight into why the conclusion should be correct.

## 2. A SINGLE CAVITY

Consider a bunch of charged particles, of length d, moving along the axis with a relatively large $\gamma$. In free space, the lines of electric field will radiate from the bunch in a disc (Fig. 1), whose thickness $w(r)$ will be given approximately by:

$$w(r) \approx d + \frac{r}{\gamma} \tag{2.1}$$

The electric field strength at a radius $r$ will be:

$$E(r) \approx \frac{Q}{2\pi\,\epsilon_o\,r w(r)} \tag{2.2}$$

where the units are mks and $\epsilon_o \approx 8.84\ 10^{-12}$. The energy in an interval $dr$ of radius is:

$$dU_o = \frac{1}{2}\epsilon_o E^2\, dV \approx \frac{Q^2}{4\pi\,\epsilon_o\,r w(r)} dr \tag{2.3}$$

If the bunch is travelling down a pipe of radius $a$ then the fields inside $a$ remain approximately as given by Eq. (2.3), although the longitudinal components vanish at the wall. Now we consider what happens when the pipe suddenly opens up (Fig. 2). The fields near the bunch cannot at first be effected, since there has not yet been time for the information to arrive. The smallest radius $(r-\delta)$ at which the fields can be modified will be given when the following is satisfied: a signal that started when the front of the bunch passed the iris edge (O in Fig. 2) must arrive at the same time (at $P$ in Fig. 2) as the back of the bunch, *i.e.*, when

$$\sqrt{z^2+\delta^2} = \frac{z+w}{\beta} \tag{2.4}$$

Assuming

$$\delta << a \tag{2.5}$$

$$w << z \tag{2.6}$$

$$\gamma^2 >> 1 \tag{2.7}$$

then

$$\delta \approx \sqrt{2zw(a)} \tag{2.8}$$

By a similar argument, there can be no field within the bunch length at a radius larger than $a+\delta$. Qualitatively we would thus expect the field energy to follow Eq. (2.3) up to $a-\delta$, and then fall to a negligible value by $a+\delta$. And if we assume that the energy fall approximately linearly (as in Fig. 3), then the energy within the bunch length, in the field outside radius, $a$ will be given by:

$$U_{outside} \approx \frac{Q^2}{4\pi\epsilon_0 a w(a)} \frac{\delta}{4} . \tag{2.9}$$

The field lines cannot, of course, just stop, they must bend backwards as indicated in Fig. 2b. And the fields behind the bunch will also contain some energy; this will not however be a large contribution since the extent of the fields in $z$ is rapidly increasing, and thus the magnitude of the fields and their stored energy, are small.

If the cavity ends after a distance g, all the energy outside the radius $a$ will be reflected at the wall and will be lost to the bunch. In addition Lawson (Ref. 4) has shown that there is an equal energy inside $a$ that has been retarded so that it cannot catch up with the bunch; that too will be lost. So the total energy lost by the bunch will be twice that given by Eq. (2.9). With the cavity length $g$ substituted for $z$, and some factor $F_1$ to correct for our approximate treatment, the total energy loss will be:

$$U = 2F_1 \frac{Q^2}{4\pi\epsilon_o a w(a)} \frac{\delta}{4} \tag{2.10}$$

$$U = F_1 \frac{Q^2}{4\pi\epsilon_o a} \sqrt{\frac{g}{2w(a)}} \tag{2.11}$$

We can now apply this result to three cases:

Point Charge

If we consider the case of a bunch of negligible length, then from Eq. (2.1) we substitute $a/\gamma$ for $w(a)$ in Eq. (2.11), and obtain:

$$U = \frac{F_1}{\sqrt{2}} \frac{Q^2}{4\pi\epsilon_o} \frac{g^{\frac{1}{2}}\gamma^{\frac{1}{2}}}{a^{\frac{3}{2}}} \tag{2.12}$$

In Ref. 4, this same calculation is performed exactly using a Fourier sum of frequencies to form the bunch, and Fresnel integrals to determine the diffraction of the waves at the edge. The result then obtained is identical to the above but for the factor $F_1/\sqrt{2}$ which is replaced by 0.6, *i.e.*,

$$F_1 \approx .85 \tag{2.13}$$

Thus we see that the approximations used, though crude, give a give a result that is correct qualitatively, and is even within fifteen percent of a more accurate determination.

We note that the energy loss rises as $\sqrt{\gamma}$, as quoted in our introduction from Ref. 3. This apparent divergence of the energy loss does not, of course, violate energy conservation: the fraction of energy lost falls as $1/\sqrt{\gamma}$.

Infinite Gamma

In this case we can use Eq. (2.11) with $w(a)$ replaced by the bunch length $d$, and $F_1 = .85$. For a gaussian bunch, we should obtain an approximately correct result by replacing $w$ by 2.5 $\sigma_z$. The energy loss in this case is commonly expressed by a loss factor $k(\sigma_z)$, where

$$U = Q^2 \; k(\sigma_z) \tag{2.14}$$

so we have

$$k(\sigma_z) \approx \frac{.38}{4\pi\epsilon_0 a}\sqrt{\frac{g}{\sigma_z}}. \tag{2.15}$$

Calculations using the program TBCI (Ref. 7) agree quite closely with this result, providing $\sigma_z$ is sufficiently small; *i.e.*, providing Eq. (2.5) and Eq.(2.6) are well satisfied.

It is convenient to note here that the loss factor is related to a wake function $W(z)$ by

$$k(\sigma_z) = \frac{1}{Q^2}\int\limits_{-\infty}^{+\infty}\left[\int\limits_{-\infty}^{s} W(s-l)q(l)\; dl\right] \; q(s)ds \tag{2.16}$$

where $q(z)$ is the charge density along the bunch.

Impedance

Instead of considering a single bunch, we now consider a continuous but periodic current, with a periodicity $\lambda$. Simple diffraction theory (Ref. 4) shows that the field would be little effected inside $r - \delta$, and negligible above $r + \delta$. Where $\delta$ is now given by

$$\delta \approx \sqrt{\frac{\lambda z}{2\pi}} \tag{2.17}$$

The field energy in an interval of $dr$ must now be expressed per unit of length $dl$ and charge density $q$, and will be

$$\frac{dU}{dl} = \frac{q^2 dl}{4\pi r \epsilon_o} dr \tag{2.18}$$

and the resulting impedance is

$$Z \approx \frac{F_2}{4\pi\epsilon_o r}\sqrt{\frac{g}{\omega c}} \tag{2.19}$$

where $F_2$ is a new factor to cover the approximations. Eq. (2.19) shows the familiar $\omega^{\frac{1}{2}}$ fall off of impedance with frequency, that is not sufficient to stop the divergence of the wake field for short bunches.

## 3. MULTIPLE CAVITIES

Consider a sequence of $n$ identical cavities with vertical thin iris walls at a spacing of $g$ (Fig. 4). Just before the $n^{th}$ iris, the fields will not extend out beyond $r + \delta_{out}$, where $\delta_{out}$ is the same as the distance $\delta$ that we calculated in Eq. (2.8). But the fields on the inside of the iris will be depleted down to $r - \delta_{in}$, where $\delta_{in}$ is larger than our old $\delta$. This follows from the causality argument used above. The waves have travelled a longer distance $z = ng$ since the first disturbance, and since the extent of the disturbance goes as the root of this distance [Eq. (2.8)], so one obtains

$$\delta_{in} = \sqrt{n}\ \delta_{out} \tag{3.1}$$

and the total distance over which the field energy, within the bunch length, must fall (Fig. 5) will be

$$\delta_{in} + \delta_{out} = (1 + \sqrt{n})\ \delta_{out} \tag{3.2}$$

If again we assume the field energy to fall linearly between $r - \delta_{in}$ and $r + \delta_{out}$ (see Fig. 5a), then the energy reflected by the $n$th iris will be

$$U_n \approx \frac{2F_1Q^2}{4\pi\epsilon_o a w(a)} \frac{1}{2} \frac{\delta_{out}^2}{\delta_{out} + \delta_{in}} \tag{3.3}$$

$$U_n \approx \frac{2F_1Q^2}{4\pi\epsilon_o a w(a)} \frac{\delta}{2(1+\sqrt{n})} \tag{3.4}$$

and referring back to Eq. (2.10) gives

$$U_n \approx U_{single\ cavity} \frac{2}{1+\sqrt{n}} \tag{3.5}$$

Thus we see that as the number of cavities rises the loss per cavity falls. But this cannot go on for ever. We had to assume [Eq. (2.5)] that $\delta_{in}$ remain small compared to $a$. At some value of n this has to be violated. At that point the field pattern will settle to an equilibrium in which the field is continually being eaten at the irises while it is being replenished on the axis by the deceleration of the beam. An estimate of the number of cavities $n_{crit}$ before equilibrium is reached, will be when (Fig. 5b)

$$\delta_{in} \approx a \tag{3.6}$$

and thus:

$$n_{crit} \approx \frac{a^2}{2gw(a)} \ . \tag{3.7}$$

Providing

$$n_{crit} >> 1 \ , \tag{3.8}$$

an estimate of the magnitude of this equilibrium loss per cell will be obtained (Fig. 5b) by substituting $a$ for $\delta_{in}$ in Eq. (3.3). Introducing a new constant $F_3$ to allow for the approximations;

$$U_n \approx \frac{2F_3Q^2}{4\pi\epsilon_o aw(a)}\,\frac{1}{2}\,\frac{2w(a)g}{a} \tag{3.9}$$

The $w's$ cancel, and the average energy loss per meter is then

$$< \frac{dU}{dl} > \approx \frac{2F_3Q^2}{4\pi\epsilon_o}\,\frac{1}{a^2} \tag{3.10}$$

In the point charge case, Eq. (3.10) will apply if [from Eq. (3.8)]:

$$\gamma >> \frac{2g}{a} \tag{3.11}$$

beyond that value of $\gamma$ the loss [Eq. 3.10] is independent of $\gamma$, and we have obtained the $\gamma^0$ dependance discussed in the introduction.

In the $\gamma = \infty$ case Eq. (3.10) is valid so long as [from Eq. (3.8)]:

$$d << \frac{a^2}{2g} \tag{3.12}$$

and in this case we can define the loss parameter:

$$k(o) = \frac{2F_3}{4\pi\epsilon_o}\,\frac{1}{a^2} \tag{3.13}$$

The form of the result is interesting. The energy loss per meter is independent of the gap length and the pulse length. The result should not however be too much of a surprise: the situation is analagous to that of the energy loss of a bunch in a resistive wall tube (of radius $a$). In both cases the fields are absorbed at radius $a$, and supplied by the bunch at the axis. And in both cases, for sufficiently short bunches, the energy loss is independent of the bunch length. In the resistive wall case, for an infinitesimal bunch length, (Ref. 9), the wake function per meter is

$$W(o) = 4 \frac{1}{4\pi\epsilon_o} \frac{1}{a^2} \tag{3.14}$$

Since, for small $z$, this is a constant, it follows that the loss parameter, for short bunches, must be a constant. It then follows from Eq. (2.16) that

$$k(o) = \frac{W(o)}{2} \tag{3.15}$$

and thus

$$k(o) = 2 \frac{1}{4\pi\epsilon_o} \frac{1}{a^2} \tag{3.16}$$

which when compared with Eq. (3.13) gives (with some sleight of hand)

$$F_3 = 1 \tag{3.17}$$

<u>An Example</u>

For the SLAC structure ($\lambda = 10.5$ cm, $a = 1.1$, $g = 3.5$ cm) Eq. (3.13) with Eq. (3.17), gives

$$k(o) = 149 \text{ Volts per picocoulomb meter .} \tag{3.18}$$

This may be compared with a calculation[8] using a numerical method with the optical resonator analytical extension. Reference 8 gives for the SLAC structure ($\lambda = 10.5$ cm, $a = 1.1$, $g = 3.5$ cm) a value of the wake field $W(o)$, for an infinitesimal bunch, of 8 Volts per picocoulomb per cell, or approximately 228 Volts per picocoulomb meter. From Eq. (3.15) the value of the loss parameter for a constant $W(z)$ is $k(o) = 0.5\ W(o)$. So, for the SLAC case

$$k(o) = 114 \text{ Volts per picocoulomb meter .} \tag{3.19}$$

which, though a little low, is in reasonable agreement with our expectation.

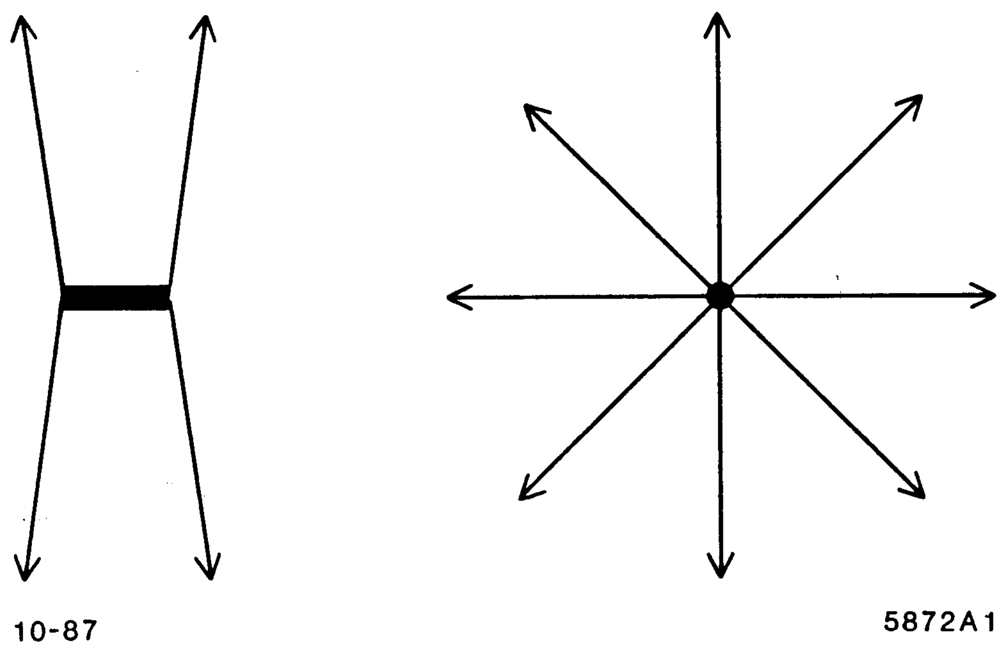


Fig. 1

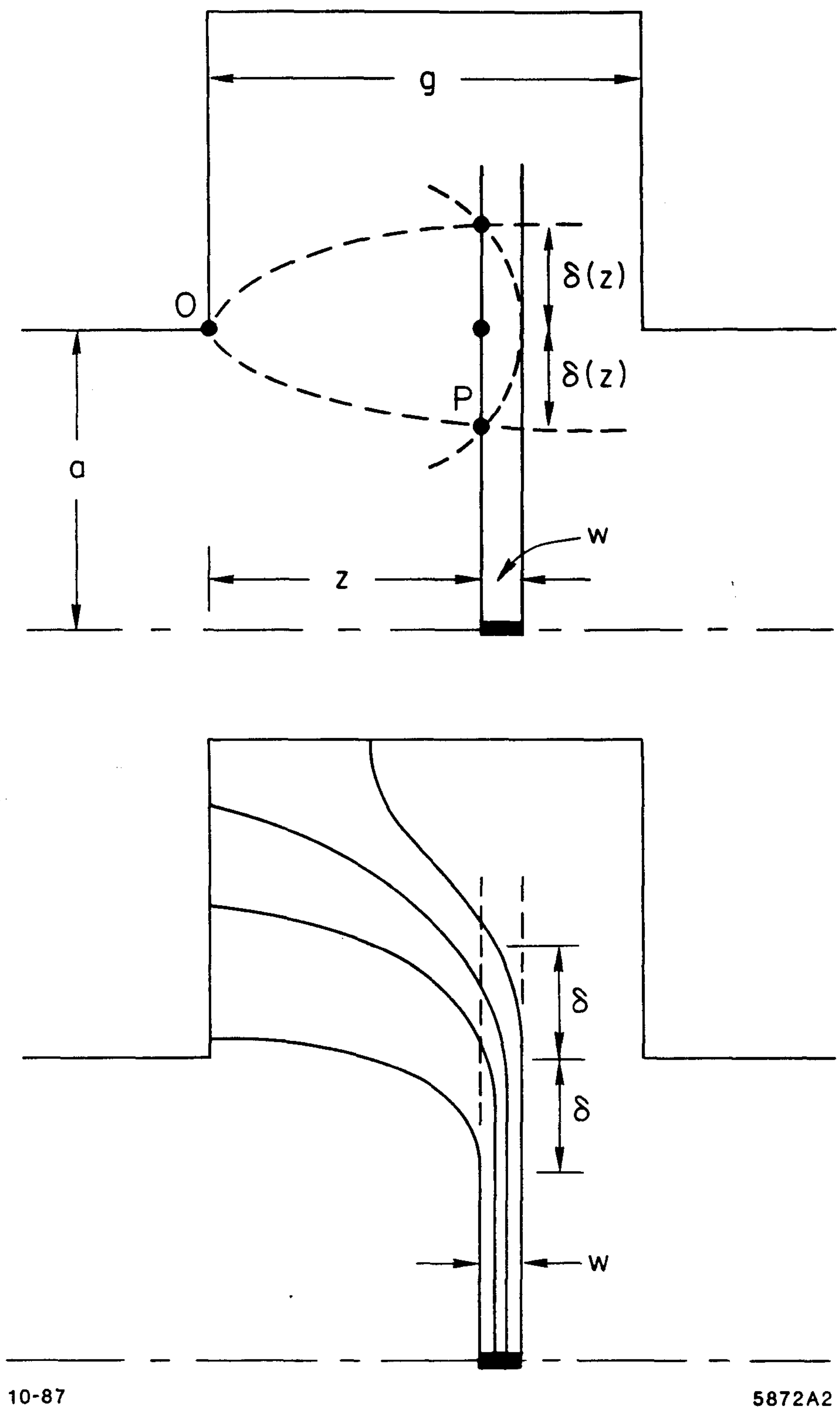


Fig. 2

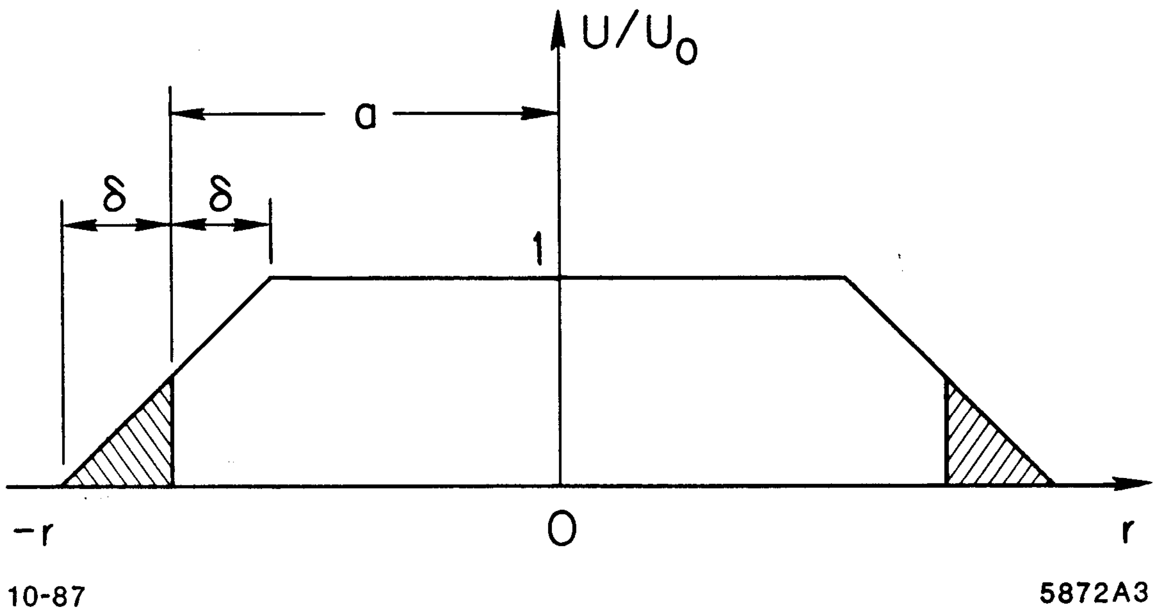


Fig. 3

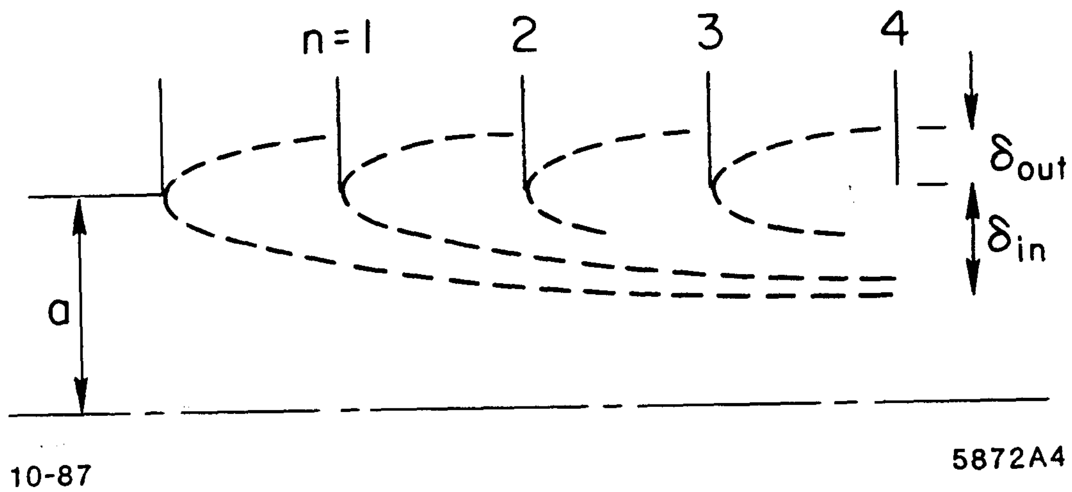


Fig. 4

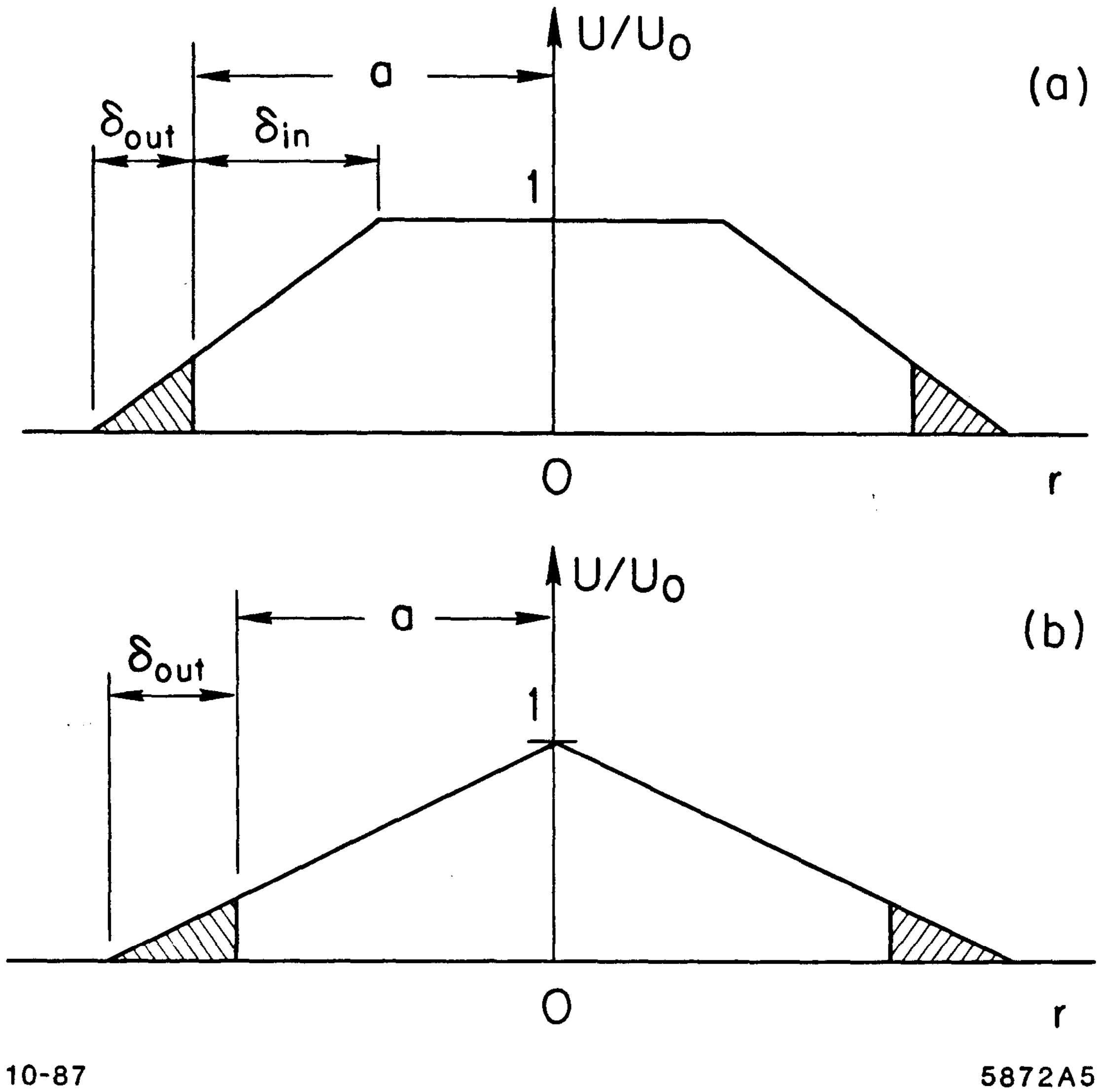


Fig. 5