SLAC – PUB – 4423
October 1987
[T/AS]

Direct and Indirect Capture of WIMPs By The Earth

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ABSTRACT

The capture of weakly interacting massive particles (WIMPs) by the Earth is calculated taking account of corrections due to the Earth's motion deep within the potential well of the sun. Two distinct types of capture are evaluated. First, direct capture of WIMPs out of the Galactic halo. Second, indirect capture of WIMPs which had previously interacted with the Earth and had lost enough energy to go into solar orbit, but not enough to be directly captured. It is found that the capture "resonances" (peaked where the WIMP mass is matched to the mass of some nucleus well-represented in the Earth) are enlarged and broadened by these corrections. General methods are developed for analyzing the propagation of the anisotropies of a non-relativistic distribution in a central potential.

Submitted to *Astrophysical Journal*

⋆ Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.

1. Introduction

Weakly interacting massive particles (WIMPs) may make up some or all of the "dark matter" in the Milky Way (Faulkner and Gilliland 1985; Spergel and Press 1985; Press and Spergel 1985). If they do, they will have been captured by the Sun, the Earth, and other bodies, possibly giving rise to observable consequences. These may include "annihilation signals" from the cores of these bodies or a solution to the solar neutrino problem. In order to accurately predict these effects, one must understand the statistical behavior of the WIMPs, including their rate of capture into, their steady state distribution within, and their evaporation rate from these bodies.

I have previously written two papers addressing these questions (Gould 1987a; Gould 1987b), hereafter referred to as Paper I and Paper II. In particular, in Paper II, I evaluated the rate of capture of WIMPs by a massive body in free space. For a spherical shell of such a body, the capture rate per unit volume is

$$\frac{dC}{dV} = \int du \frac{f(u)}{u} w \Omega^{-}_{v_e}(w) \tag{1.1}$$

where $f(u)$ is the speed distribution of the WIMPs (integrated over all angles) far from the body, v_e is the escape velocity at the shell,

$$w \equiv (u^2 + v_e^2)^{\frac{1}{2}}, \tag{1.2}$$

and $\Omega^{-}_{v_e}(w)$ is the rate at which a WIMP with velocity w scatters to less than velocity v_e while it is travelling through the shell. The capture rate for the whole body is found by integrating equation (1.1) over the body.

Equation (1.1) must, of course, be evaluated separately for each element represented in the body, and the results summed. I will regard this type of summation as implicit in all capture formulae.

In Paper II, I treated the Earth as though it were a body in free space, and applied equation (1.1) to evaluate the capture rate. I found that this rate rises dramatically when the mass of the WIMP is nearly matched to the mass of an element well-represented in the Earth. When such matching occurs, the particular Earth element utterly dominates the capture rate. These peaks were dubbed "resonances" and are apparent in Figure 1.

In this paper I evaluate the corrections to this earlier calculation due to the fact that the Earth is not in free space, but is moving with velocity

$$v_{\oplus} \sim 30 \text{ km/sec} \tag{1.3}$$

deep within the potential well of the sun.

This motion in a potential well leads to two distinct types of corrections. First, the calculation of the capture rate of WIMPs *directly* out of the Galactic halo must be modified from what is given by equation (1.1). Second, an entirely new source of capture, *indirect* capture, must be evaluated. A WIMP may weakly interact with the Earth and, while not losing enough energy to be bound to the Earth, might still lose enough to be in a bound solar orbit. If this happens it will be said to be *orbit-captured.* Once a WIMP is orbit-captured, it may again weakly interact with the Earth, resulting in its indirect capture into the Earth's core.

The first correction, the correction to direct capture, is comparatively easy to evaluate. The Galactic velocity distribution of WIMPs gives rise to a speed distribution $f_e(u)du$ in the neighborhood of (and relative to the frame of) the Earth. To calculate the revised direct capture rate, it is "only" necessary to calculate $f_e(u)$ and plug it into equation (1.1). I have enclosed the word "only" in quotes because, in general, $f_e(u)du$ bears no simple relation to the Galactic speed distribution, $f_g(t)dt$. However, in Section II, I demonstrate the existence of an accidental symmetry of the problem which allows such a simple relation to be established.

The second correction, the calculation of indirect capture is both more difficult and less accurate than the first. It may be broken down into three distinct steps. First, find the rate of orbit-capture. Second, determine the evolution of the distribution of the orbit-captured WIMPs. Third, find the rate of indirect capture from the resulting distribution.

Orbit-captured WIMPs may be divided into two categories according to whether the WIMP's relative speed at orbit-capture is less or greater than

$$(2^{\frac{1}{2}} - 1)v_{\oplus}. \tag{1.4}$$

Each of the above three steps is comparatively straightforward for WIMPs below this threshold and is quite difficult for those above it.

In Section III, I analyze indirect capture for the lower range of WIMPs. This calculation involves a critical assumption regarding the evolution of the orbit parameters of the orbit-captured WIMPs. While this assumption is a reasonable zeroth order approximation, it undoubtedly introduces significant errors. The calculation of direct capture, by contrast, is accurate to within a few percent or less.

In Section IV, I estimate capture in the upper range. In this calculation, the errors introduced in Section III are compounded by new, and less well-founded assumptions.

I find that, for Dirac neutrinos of mass 10-80 GeV, direct and indirect capture are of the same order of magnitude. Over most of the range, direct capture is somewhat greater than indirect capture. Further, indirect capture from the lower range is generally greater than that from the upper range. Indeed, for most masses, indirect capture from the upper range is small or negligible.

Thus, generally speaking, the various types of capture contribute to the total capture rate in the same order as their respective calculations are reliable. This enhances the credibility of the net result. In any event, the *direct* capture rate alone serves as a reliable *minimum* for total capture.

Other types of WIMPs which have spin-independent cross-sections will behave similarly to Dirac neutrinos. For WIMPs with spin-dependent cross-sections, such as photinos and Majorana neutrinos, indirect capture represents an extremely small fractional correction to an already small direct capture rate.

In Appendix A, I analyze the propagation of the anisotropies in the WIMP distribution as it approaches the region of the Sun. From the standpoint of the present paper, this analysis is important only because it allows me to explicitly demonstrate that anisotropies are not important in the computation of capture rates. For this reason, the calculation has been relegated to an appendix. However, the question of the propagation of anisotropies in a central potential is of interest in its own right, and the results of this calculation are not easily anticipated.

Paper II contains an extensive list of references relevant to WIMP capture by the Earth. In this paper I have included only such *additional* references as are required for the evaluation of the corrections to that paper.

2. Direct Capture

In this section, I calculate the rate of direct capture of WIMPs by the Earth. The net result is most simply presented as a ratio of the true direct capture rate to what the capture rate would be if the earth were in free space. That is, I will evaluate a correction factor

$$\eta \equiv \left[\int du \frac{f_e(u)}{u} w\Omega^-_{v_e}(w)\right] \Big/ \left[\int du \frac{f_g(u)}{u} w\Omega^-_{v_e}(w)\right]. \tag{2.1}$$

All the quantities in the above expression were defined in Section I. In particular, $f_g(u)$ and $f_e(u)$ are respectively the Galactic distribution of WIMPs in the frame of the Sun, and the induced distribution of WIMPs in the frame and neighborhood of the Earth. The denominator in equation (2.1) was evaluated in Paper II.

Equation (2.1) could be easily evaluated if the Galactic velocity distribution were isotropic in the frame of the sun. In this case, the analysis of Paper II would apply without modification, and one finds that $f_s(s)ds$, the distribution in the neighborhood of the Earth (but in the frame of the Sun), would be given by

$$\frac{f_s(s)ds}{s} = \frac{f_g(t)dt}{t} \tag{2.2}$$

where

$$s^2 = t^2 + 2v_\oplus^2, \tag{2.3}$$

and $2^{\frac{1}{2}}v_\oplus$ is the escape velocity from the Sun at the position of the Earth. Moreover, this distribution would be isotropic. A simple Galilean transformation would then yield $f_e(u)$. [Eqn. (2.2) follows directly from eqn. (2.7) of Paper II. A more direct argument is given in Appendix A; in particular, eqn. (2.2) is a special (isotropic) case of eqn (A4).]

However, if the Galactic distribution is anisotropic, then $f_s(s)ds$ has a very complicated dependence on the angular distribution of the Galactic WIMPs. It is, in addition, anisotropic.

The actual Galactic distribution must be highly anisotropic. It may possess anisotropies in the frame of the Galaxy; but even if it doesn't, the motion of the Sun about the Galactic center induces anisotropies in the distribution as seen from the frame of the Sun. I will show below, however, that an accidental symmetry makes it possible to ignore these anisotropies. It will then be possible to evaluate η to within a very small fractional error,

$$\frac{\mid \Delta\eta \mid}{\eta} < \frac{v_\oplus^2}{v_\odot^2} \sim 1.5\%. \tag{2.4}$$

I will assume that the Galactic velocity distribution satisfies the following conditions.

1. At the position of the Sun, the distribution is symmetric with respect to reflections through the plane of the Galaxy.
2. The distribution is the same at every point along the Sun's orbit.
3. The scale of the anisotropies is $v_\odot$, the speed of the Sun in its Galactic orbit,
$$v_\odot \sim 250 \text{ km/sec}. \tag{2.5}$$
4. The scale of the speed distribution, $f_g(t)dt$, is also $v_\odot$.

These assumptions are physically reasonable. In particular, they apply to the case of a WIMP distribution which was "originally" isothermal but which was subsequently "disrupted" by the infall of the baryonic matter, as well as to anisotropies due to the Sun's motion.

The third condition may be expressed mathematically as follows: Let the velocity distribution of the WIMPs be expanded

$$\begin{aligned} f_g(t)dt\Big[1+\sum_{l=2,4\ldots} a_l(t)\left(\frac{t}{v_\odot}\right)^l P_l(\cos\theta)+\sum_{l=1,3\ldots} a_l(t)\left(\frac{t}{v_\odot}\right)^l P_l(\cos\theta) \\ +\sum_{l,m\neq 0} a_{lm}(t)\left(\frac{t}{v_\odot}\right)^l P_l^m(\cos\theta)e^{im\phi}\Big]\frac{d\cos\theta\, d\phi}{4\pi} \end{aligned} \tag{2.6}$$

where θ is the angle measured from the Galactic pole, the P_l are Legendre polynomials, the P_l^m are associated Legendre polynomials (Jackson 1975) and the coefficients a_l and a_{lm} are functions of t. The third condition means that these coefficients are of order unity or less,

$$| a_l | \lesssim 1, \quad | a_{lm} | \lesssim 1. \tag{2.7}$$

In most applications, one is concerned only with the long-term capture of WIMPs and thus only with the distribution averaged over long time periods.

This is the point of view I will adopt in this section. (At the end of Appendix A, I analyze capture without assuming long-term time-averaging of the distribution.) Now, consider a frame which travels with the Sun as it circles the Galactic center but whose orientation remains fixed (relative to the distant galaxies - or to the Earth-Sun angular momentum vector). In this frame, the position of the center of the Galaxy rotates through an angle 2π during the $\sim$ 200 million years of the Sun's orbit.

By the second condition, the time averaged distribution in this frame is azimuthally symmetric. This means that the last term in equation (2.6) may be dropped. By the first condition, the third term may also be dropped. Thus to next to leading order, the angular distribution may be written

$$f_g(t)dt\left[1 + a_2 \frac{t^2}{v_\odot^2} P_2(\cos\theta)\right] \frac{d\cos\theta}{2}, \tag{2.8}$$

where

$$P_2(x) \equiv \frac{1}{2}(3x^2 - 1). \tag{2.9}$$

The coefficient of P_2 is restricted by the condition that the velocity distribution be positive. For a distribution strictly of the form (2.8), the coefficient of P_2 must be in the interval $(-1, 2)$. Allowing for higher Legendre polynomials, the absolute limits are $(-5/2, 5)$.

The function a_2 is easily evaluated in specific cases. In Paper II, I used the example of a distribution which was Maxwell-Boltzmann in the Galactic frame with a velocity dispersion $\bar{v}$,

$$\bar{v}^2 = \frac{3}{2} v_\odot^2. \tag{2.10}$$

In the Sun's frame this distribution is anisotropic and

$$a_2(t) = \frac{30}{(2t/v_\odot)^2}\left[\frac{\coth(2t/v_\odot)}{(2t/v_\odot)} - \frac{1}{(2t/v_\odot)^2} - \frac{1}{3}\right] = -\frac{2}{3} + \mathcal{O}\left(\frac{t^2}{v_\odot^2}\right). \tag{2.11}$$

Note that in this case, a_2 satisfies

$$| a_2(t) | < \frac{2}{3}, \qquad -\frac{5}{2} < a_2(t)\frac{t^2}{v_\odot^2} < 0. \tag{2.12}$$

Expression (2.8) describes the time averaged distribution experienced by the *Earth-Sun system*, as measured from the Galactic pole. To find the time-averaged distribution experienced by the *Earth* as it orbits the Sun, one must first transform coordinates so that the polar angle is measured from the pole of the Earth's orbit. [The axis of the Earth's orbit is inclined at 60° relative to the Galactic axis (Mihalis and Binney 1981).] Thus, one should rewrite expression (2.8)

$$f_g(t)dt\left[1 + a_2'\frac{t^2}{v_\odot^2}P_2(\cos\theta') + \sum_{m\neq 0} a_{2m}'\frac{t^2}{v_\odot^2}P_2^m(\cos\theta')e^{im\phi'}\right]\frac{d\cos\theta' d\phi'}{4\pi} \tag{2.13}$$

where θ' is the angle measured from the axis of the Earth's orbit, ϕ' is the azimuthal angle measured from some arbitrary fixed direction, and a_2' and the a_{2m}' are new functions of t.

By the spherical harmonics addition theorem,

$$a_2' = a_2 \cdot P_2(\cos 60°) = -\frac{a_2}{8}. \tag{2.14}$$

Taking the time average of the distribution (2.13) (over a year) eliminates the $m \neq 0$ terms. The result is

$$f_g(t)dt\left[1 - \frac{a_2}{8}\frac{t^2}{v_\odot^2}P_2(\cos\theta')\right]\frac{d\cos\theta'}{2}. \tag{2.15}$$

Thus, the time averaged distribution is nearly isotropic. The dipole term vanishes identically and the quadrupole term is highly suppressed. This implies, according to Appendix A, that the WIMP distribution at the Earth (integrated over all angles) is given by equation (2.2) to an extremely good approximation,

$$\left| \frac{f_s(s)tds}{f_g(t)sdt} - 1 \right| < \frac{|a_2'|}{2}\frac{t^2}{v_\odot^2} \cdot \frac{v_\oplus^2}{t^2} = \frac{|a_2|}{16}\frac{v_\oplus^2}{v_\odot^2}. \tag{2.16}$$

The errors generated by ignoring this are well below the limits given by equation (2.4). The anisotropies in f_g will, as mentioned above, also induce anisotropies in f_e. However, in Appendix A I show that the errors generated by ignoring this are less than

$$\frac{7}{30} \mid a_2' \mid \frac{v_\oplus^2}{v_\odot^2} = \frac{7}{240} \mid a_2 \mid \frac{v_\oplus^2}{v_\odot^2}. \tag{2.17}$$

Thus, equation (2.2) effectively holds and the distribution may be regarded as isotropic. [Corrections from higher l moments are of $\mathcal{O}(v_\oplus^l/v_\odot^l)$, and are thus completely negligible.]

I proceed now to the determination of $f_e(u)du$. If $f_g(t)dt$ is regular at zero, then for speeds much less than the characteristic speed of the distribution, $\sim v_\odot$,

$$f_g(t) = \kappa t^2 \qquad t \ll v_\odot, \tag{2.18}$$

where κ is a constant. This implies, by equations (2.2) and (2.3),

$$f_s(s) = \kappa s^2\ \theta(s^2 - 2v_\oplus^2) \qquad s \ll v_\odot. \tag{2.19}$$

Making a Galilean transformation to the frame of the Earth, one finds

$$f_e(u) = 0 \qquad\qquad u < (2^{\frac{1}{2}} - 1)v_\oplus, \tag{2.20}$$

$$f_e(u) = \frac{\kappa}{4}\frac{u}{v_\oplus}(u^2 + 2uv_\oplus - v_\oplus^2) \qquad (2^{\frac{1}{2}} - 1)v_\oplus < u < (2^{\frac{1}{2}} + 1)v_\oplus, \tag{2.21}$$

$$f_e(u) = \kappa u^2 \qquad (2^{\frac{1}{2}} + 1)v_\oplus < u \ll v_\odot. \tag{2.22}$$

It is now possible to evaluate equation (2.1). In Paper II, I showed that for a velocity-independent isotropic cross-section, σ,

$$w\Omega_{v_e}^{-}(w) = \sigma n v_e^2 \left(1 - \frac{u^2}{v_c^2}\right), \tag{2.23}$$

where

$$v_c \equiv \frac{\mu^{\frac{1}{2}}}{\mu_-} v_e \tag{2.24}$$

is the "cut-off velocity", the velocity-at-infinity above which capture is impossible. In the above two equations, n is the density of nuclei in the Earth,

$$\mu \equiv \frac{M}{m}, \qquad \mu_\pm \equiv \frac{\mu \pm 1}{2}, \tag{2.25}$$

and M and m are the masses of the WIMP and nuclei respectively. Using equations (2.20) through (2.23), equation (2.1) may be evaluated in three cases:

$$\eta = 0 \qquad\qquad v_c < (2^{\frac{1}{2}} - 1)v_\oplus \tag{2.26}$$

$$\eta = \frac{2}{15}\xi^{-1} + \frac{1}{2} - \frac{2}{3}\xi + \frac{1}{3}(4 \cdot 2^{\frac{1}{2}} - 5)\xi^2 - \frac{1}{30}(56 \cdot 2^{\frac{1}{2}} - 79)\xi^4 \qquad (2^{\frac{1}{2}} - 1)v_\oplus < v_c < (2^{\frac{1}{2}} + 1)v_\oplus \tag{2.27}$$

$$\eta = 1 - \frac{10}{3}\xi^2 + \frac{79}{15}\xi^4 \qquad\qquad (2^{\frac{1}{2}} + 1)v_\oplus < v_c \ll v_\odot \tag{2.28}$$

where

$$\xi \equiv \frac{v_\oplus}{v_c}. \tag{2.29}$$

In Appendix B, I show that in the opposite limit

$$\eta = 1 - \frac{7}{3}\xi^2 \qquad\qquad v_c \gg v_\odot. \tag{2.30}$$

Since the limiting forms of equations (2.28) and (2.30) are almost the same, and since $\eta \sim 1$ in the transition region between them, one might guess that a smooth transition from one to the other could be effected without introducing fractional errors greater than $v_\oplus^2/v_\odot^2$. A more detailed analysis, given in Appendix B, bears out this conjecture and shows that the transition should *begin* at $\sim v_\odot$.

Therefore, to within 2%, equations (2.26) through (2.30) give the correction factor η, independent of the particular distribution $f_g(t)$!

Using these equations, I have evaluated the direct capture of Dirac neutrinos by the Earth. This is shown by the dashed line in Figure 1. The solid line shows the uncorrected result. The principal effect of the correction is to sharply cut off the capture rate away from the resonances, that is for $|\mu - 1| \gtrsim 0.25$. In the WIMP mass range of 12-65 GeV, this correction has relatively little effect because, for these WIMPs, capture is dominated by resonances. Outside this range the effect is considerable.

Note that I have not included the effects of "lack of coherence" in the above analysis. As I discussed in Paper II, equation (2.23) is strictly valid only if the entire nucleus interacts coherently with the WIMP, that is, in the limit where the nuclear radius is small compared to the inverse momentum transfer. When this condition is not satisfied, the lack of coherence may considerably affect the capture rate. However, by working through the above derivation and including the form factor given in Paper II which takes account of lack of coherence, the reader may verify that the correction factor, η, is unaltered from its value as given by equations (2.26) through (2.30), within the limits set by expression (2.4).

3. Indirect Capture: I

In this section I analyze the indirect capture of WIMPs whose orbit-capture speed, u, is in the range

$$0 < u < (2^{\frac{1}{2}} - 1)v_{\oplus} \sim 12 \text{ km/sec.} \tag{3.1}$$

The orbit-capture speed is the speed at which the WIMP initially escapes the Earth's gravitational field. All of the WIMPs in the range given by expression (3.1) are in bound solar orbits; even the WIMPs which scatter with the maximum allowed velocity and in the direction of the Earth's motion, do not have enough energy to escape the Sun. Because of this, no attention need be paid to the angle at which the WIMP scatters from the Earth. It is only necessary to take account of the *magnitude* of the recoil velocity. Consequently, the derivation of the orbit-capture rate is completely analogous to the derivation of direct capture given in Paper II. For a given shell at escape velocity v_e, the capture rate per unit of orbit-capture velocity is

$$c(u)du = \sigma N \frac{\mu_+^2}{\mu} du^2 \int\limits_{u}^{\left(v_c^2 + \frac{\mu_+^2}{\mu^2} u^2\right)^{\frac{1}{2}}} du' \frac{f_e(u')}{u'} \left\{ \exp\left[-\frac{M(u'^2 - u^2)}{2E_0} \right] \right\}, \tag{3.2}$$

where N is the number of nuclei in the shell. The cut-off velocity, v_c, and the distribution of WIMPs in the neighborhood of the Earth, $f_e(u')du'$, are given in Section II. The term in curly brackets takes account of lack of coherence; it plays very little role except at the iron resonance ($M \sim 56$ amu). The coherence energy, E_0, is discussed in the appendix to Paper II. The orbit-captured WIMPs (like the Galactic WIMPs) are the source of a distribution of WIMPs in the neighborhood of the Earth, which are available for capture. This means that u is the appropriate name for the argument of the orbit-capture function, $c(u)du$. Hence, I have renamed the dummy variable in the distribution, f_e, from u (which was used in the last section) to u'.

Equation (3.2), like equation (1.1), must be evaluated for each element separately and the results summed. Again this sum is not shown explicitly.

To find the indirect capture due to the whole Earth, one simply sums over all shells. While this summing poses no great difficulties, for present purposes it suffices to take N to be the total number of nuclei (of a given type) in the Earth, and v_e^2 to be the average of the square of the escape velocity over these nuclei. Because all elements in the Earth are concentrated either mainly in the core or mainly in the mantle, and because the gravitational potential varies so little over each of these regions (see Paper II), this approximation introduces only very small errors. Not only are these errors absolutely small in the sense that they are of order 1%, they are small compared to errors which will be introduced by later approximations.

Comparing the *differential* orbit-capture [eqn. (3.2)] with the formula for *total* direct capture, C_d, [given by the volume integral of eqn. (1.1)], one finds that for two limiting regions

$$\frac{c(u)du}{C_d} = \frac{du^2}{v_e^2} \qquad v_c \gg v_\odot \tag{3.3}$$

$$\frac{c(u)du}{C_d} = 2\left(1 + \frac{u^2}{v_e^2}\right)\frac{du^2}{v_e^2} \qquad v_\oplus \ll v_c \ll v_\odot. \tag{3.4}$$

Making use of the fact (a matter of complete coincidence) that

$$(2^{\frac{1}{2}} - 1)v_\oplus \sim v_e, \tag{3.5}$$

one sees immediately that at a resonance, total orbit-capture (in the lower velocity range) is about equal to total direct capture. On the "shoulder" of a resonance, [the region described in eqn. (3.4)], total orbit-capture is ~ 3 times total direct capture. For lower cut-off velocities, $v_c \sim v_\oplus$, the corresponding formulae are more complicated, but one finds that as the cut-off velocity falls, the total orbit-capture grows as a fraction of direct capture. Thus indirect capture from this population of the orbit-captured WIMPs is potentially very significant.

The whole problem now turns on determining what fraction of orbit-captured WIMPs are subsequently indirectly captured by the Earth. In principle, this fraction could be obtained by numerically integrating the orbits of a statistical sample of WIMPs and allowing them to weakly scatter off Earth nuclei when their orbits intersected the Earth. This problem is similar to (but more complicated than) those which arise in asteroid physics (Everhart 1979; Greenberg and Scholl 1979). Unfortunately, the best efforts of asteroid physicists have succeeded in numerically integrating over only tens of millions of revolutions. What is required here is billions of revolutions.

I turn instead to an approximation which will render the problem analytically tractable. A WIMP will from time to time suffer a close gravitational encounter with the Earth. Since the geometric cross section of the Earth is 40 to 400 times greater than its weak cross-section (depending on the mass of the WIMP), the WIMP will weakly interact with the nuclei in the Earth in only a small fraction of these gravitational encounters. In between gravitational encounters, the WIMP will be in an elliptic orbit about the Sun. The parameters of this ellipse will slowly evolve in response to the long-range action of the Earth, Jupiter, and to a lesser degree other planets. I will assume that, during these periods between gravitational encounters

a. the angle between the WIMP orbit plane and the Earth orbit plane (ecliptic) does not change, and

b. the eccentricity and length of the semi-major axis of the WIMP's orbit does not change.

That is, I am assuming that the only effects of long-range interactions on the WIMP's orbit are to cause the precession of its perihelion about the Sun and to cause the precession of its angular momentum vector about the axis of the ecliptic.

This assumption certainly does not rigorously reflect the exact behavior of the WIMPs. It must be regarded as a zeroth order approximation to the WIMPs' *average* behavior. It undoubtedly introduces errors, and these errors will be more significant, the greater is indirect capture compared to direct capture. At the end of this section I will make numerical and analytic estimates of this ratio.

Consider a WIMP which has just interacted with the Earth (during either its initial orbit-capture or a subsequent close gravitational encounter). It will leave the Earth's gravitational field with some relative velocity, $\mathbf{u}$,

$$\mathbf{u} \equiv (u, \theta, \phi). \tag{3.6}$$

Here θ is the polar angle measured from the direction of the Earth's motion and ϕ is the azimuthal angle measured from the half-plane determined by the Earth's direction and the north pole of the ecliptic. For definiteness, I will take the WIMP to be headed away from the Sun. It follows from the above assumption that when the WIMP again encounters the Earth, it will again have speed u, and velocity

$$\mathbf{u}' = (u, \theta, \pm\phi), \tag{3.7}$$

where the upper (lower) sign corresponds to the case of an outbound (inbound) encounter.

It is appropriate, then, to parameterize the velocity of each WIMP according to (u, θ, ϕ), the coordinates of its velocity in an outbound encounter. These coordinates do not change between encounters.

How are these coordinates affected by close gravitational encounters? The magnitude of the velocity does not change because the interaction region is small compared to the radius of the Earth's orbit. The angle will be considerably changed. Even the fastest WIMPs under consideration will suffer $\sim 40°$ deflections if they graze the Earth. Slower WIMPs may be deflected much more. I will assume that during each such encounter (or each several such encounters) the WIMP's direction is randomly reoriented.

I now introduce two time scales, $T_{gr}(u,\theta,\phi)$ and $T(u,\theta,\phi)$. $1/T_{gr}(u,\theta,\phi)$ is defined to be the rate at which a WIMP in state (u,θ,ϕ) experiences a close gravitational encounter with the Earth. $1/T(u,\theta,\phi)$ is the rate at which it interacts weakly with the Earth. Of course, $T_{gr}(u,\theta,\phi)$ depends on how one defines "close", which is somewhat arbitrary. However, it will turn out that this arbitrariness plays no role. For now I will simply assume that if the impact parameter is less than some distance $b(u)$, the encounter is close, and otherwise it is not. Once this definition is settled upon, it follows that, of all WIMPs of speed u suffering gravitational encounters, some definite fraction, $B(u)$, will weakly interact with the Earth. Thus

$$T_{gr}(u,\theta,\phi) = B(u)T(u,\theta,\phi). \tag{3.8}$$

The function $B(u)$ will depend both on the function $b(u)$ and on the WIMP cross-sections (and hence on its mass). However, in no case will $B(u)$ be more than 1/100, and usually it will be much less. Because $B(u)$ is so small, one is justified in assuming that the WIMPs come to equilibrium through gravitational encounters before weak interactions take place. From the above description and from detailed balance, it follows that the number of WIMPs in a given state (u,θ,ϕ) is proportional to the interaction time, $T_{gr}(u,\theta,\phi)$, of that state. In order to put this statement in mathematical form, I introduce $n(u,\theta,\phi)dud\Omega'$, the density of WIMPs per state, and

$$n(u) \equiv \int d\Omega' n(u,\theta,\phi), \tag{3.9}$$

the density of WIMPs per unit speed. The spheroidal angle, Ω', is primed to indicate that ϕ is allowed to assume only values in the interval $(0,\pi)$.

I also introduce

$$\bar{T}_{gr}(u) \equiv \frac{1}{2\pi} \int d\Omega' T_{gr}(u,\theta,\phi), \qquad \bar{T}(u) \equiv \frac{1}{2\pi} \int d\Omega' T(u,\theta,\phi). \tag{3.10}$$

In terms of these variables, the detailed balance result can be expressed

$$n(u,\theta,\phi) = \frac{T_{gr}(u,\theta,\phi)}{\bar{T}_{gr}(u)} \frac{n(u)}{2\pi}. \tag{3.11}$$

By equation (3.8), equation (3.11) may be rewritten

$$n(u,\theta,\phi) = \frac{T(u,\theta,\phi)}{\bar{T}(u)} \frac{n(u)}{2\pi}. \tag{3.12}$$

Now it can be seen explicitly that the imprecision in defining "close" gravitational encounters plays no role in the final result.

To simplify the argument, I will initially assume that if a WIMP weakly interacts with the Earth, it is captured by it. Later, I will adjust the results to take account of the fact that this is not always the case.

It is now possible to write down the differential equation for WIMP density. WIMPs of speed u are constantly entering the orbit-captured distribution at a rate $c(u)du$. By the definition of $T(u,\theta,\phi)$, they are constantly leaving this distribution through indirect capture at a rate

$$du \int d\Omega' \frac{n(u,\theta,\phi)}{T(u,\theta,\phi)}. \tag{3.13}$$

Using equation (3.12), equation (3.13) may be evaluated as

$$\frac{n(u)}{\bar{T}(u)} du. \tag{3.14}$$

Thus the differential equation describing $n(u)$ is

$$\frac{dn(u)}{dt} = c(u) - \frac{n(u)}{\bar{T}(u)}. \tag{3.15}$$

This is easily solved. The subject of interest is not actually the number of WIMPs left in orbit, but rather the number indirectly captured. For convenience of comparison with direct capture, I express this as $c_i(u)du$, the rate of indirect capture *averaged* over the lifetime of the Earth. From equation (3.15), this is found to be

$$c_i(u) = c(u)\left\{1 - \frac{1 - \exp[-\alpha(u)]}{\alpha(u)}\right\} \tag{3.16}$$

where

$$\alpha(u) \equiv \frac{\tau}{\bar{T}(u)} \tag{3.17}$$

is the dimensionless interaction rate and $\tau \sim 4.5$ Gyr is the lifetime of the Earth.

The total indirect capture from the lower velocity range of the orbit-captured WIMPs, C_{i1}, is then

$$C_{i1} = \int_0^{(2^{\frac{1}{2}}-1)v_\oplus} du c_i(u). \tag{3.18}$$

I turn now to the evaluation of $\bar{T}(u)$. The Earth has a slight eccentricity so that, at a given time, it might be found anywhere in a finite ring of area

$$2\pi Rl, \tag{3.19}$$

where R is an astronomical unit and l is the difference between the aphelion and perihelion of the Earth's orbit. I will assume that the Earth is uniformly distributed over this region. (As will be seen below, the actual distribution as well as the actual length, l, make no difference to the derivation.)

Now consider a WIMP in an elliptic orbit. As the perihelion of the orbit precesses 2π around the Sun, the WIMP orbit intersects the Earth ring four times. During each such encounter, it intersects the ring over an angle

$$\frac{l}{R} \mid \tan\Theta_1 \mid \tag{3.20}$$

where Θ_1 is the angle between the WIMP orbit velocity vector and the radial direction, *as seen from the frame of the sun.* Thus, the probability that the

WIMP will intersect this ring in a given revolution is

$$\frac{4}{2\pi}\frac{l}{R} \mid \tan\Theta_1 \mid . \tag{3.21}$$

In truth, this probability should have a maximum of unity, since, except for special cases, the WIMP can intersect the ring no more than once per revolution. However, the effect of this maximum would be to put a floor below the interaction time over the region of phase space to which it applied. For low velocity WIMPs ($u \ll 1$ km/sec) this region of phase space is considerable, but the floor is so low ($\lesssim 10^7$ yrs) compared to the lifetime of the Earth, that it plays no role. On the other hand, in the case of high velocity WIMPs ($u \gtrsim 2$ km/sec) for which this floor is beginning to be significant, the minimum takes effect over only a tiny region of phase space. Since $\bar{T}(u)$ is an average over phase space of $T(u,\theta,\phi)$, the floor also has little effect in this case. For the remaining values of u, the floor is both very low and applicable over very small regions of phase space.

Now suppose for a moment that the WIMP is travelling fast compared to the escape velocity of the Earth and hits the ring orthogonally. Then the probability that it will interact with a nucleus in the Earth is simply

$$\frac{\sigma_E}{2\pi R l} \tag{3.22}$$

where σ_E is the weak cross section of the Earth, the sum of the cross-sections of its nuclei. If the WIMP is travelling at a slower speed, this probability will be enhanced by the "focusing factor" due to the Earth's gravitational field (see Paper II),

$$1 + \frac{v_e^2}{u^2}. \tag{3.23}$$

If, additionally, it does not strike the ring orthogonally, then there will be a further enhancement by a factor

$$\mid \cos\Theta_2 \mid^{-1}, \tag{3.24}$$

where Θ_2 is the angle between the axis of the ecliptic and the WIMP velocity vector *as seen in the Earth's frame.* (One can see that this is the correct frame by moving to it and conducting the scattering experiment there.) Combining expressions (3.21) through (3.24), one finds that the WIMP weakly interacts with the Earth

$$\frac{|\tan\Theta_1|}{|\cos\Theta_2|}\frac{\sigma_E}{\pi^2 R^2}\left(1+\frac{v_e^2}{u^2}\right) \tag{3.25}$$

times per orbit. The period of the WIMP's orbit is proportional to its semi-major axis to the three-halves power. The semi-major axis is in turn inversely proportional to the energy of the orbit. From this one concludes that the orbit period is

$$\left[1-2\frac{u}{v_\oplus}\cos\theta-\frac{u^2}{v_\oplus^2}\right]^{-3/2}\text{yrs.} \tag{3.26}$$

The angle Θ_1 can be evaluated from its definition:

$$\cos\Theta_1 \equiv \frac{\mathbf{R}\cdot(\mathbf{u}+\mathbf{v}_\oplus)}{R\,|\,\mathbf{u}+\mathbf{v}_\oplus\,|} = \frac{u\sin\theta\sin\phi}{\left(u^2+v_\oplus^2+2uv_\oplus\cos\theta\right)^{\frac{1}{2}}}; \tag{3.27}$$

$$\cot\Theta_1 = \frac{u}{v_\oplus}\frac{\sin\theta\sin\phi}{\left[1+2\frac{u}{v_\oplus}\cos\theta+\frac{u^2}{v_\oplus^2}\left(1-\sin^2\theta\sin^2\phi\right)\right]^{\frac{1}{2}}}. \tag{3.28}$$

Similarly, the angle Θ_2 can be evaluated from its definition:

$$\cos\Theta_2 \equiv \frac{\mathbf{u}\cdot(\mathbf{v}_\oplus\times\mathbf{R})}{uv_\oplus R} = \sin\theta\cos\phi. \tag{3.29}$$

Combining expressions (3.25) and (3.26), and substituting in equations (3.28) and (3.29), one may evaluate $T(u,\theta,\phi)$:

$$T(u,\theta,\phi) = \frac{2}{3}\frac{\pi R^2}{\sigma_E}\frac{u}{v_\oplus}\frac{u^2}{u^2+v_e^2}\gamma(u,\theta,\phi)\ \text{yrs}, \tag{3.30}$$

where

$$\gamma(u,\theta,\phi) \equiv \frac{3\pi}{2}\sin^2\theta \mid \sin\phi\cos\phi \mid \left[1 + 2\frac{u}{v_\oplus}\cos\theta + \frac{u^2}{v_\oplus^2}(1-\sin^2\theta\sin^2\phi)\right]^{-\frac{1}{2}}$$
$$\left[1 - 2\frac{u}{v_\oplus}\cos\theta - \frac{u^2}{v_\oplus^2}\right]^{-3/2}. \tag{3.31}$$

Thus

$$\bar{T}(u) = \frac{2}{3}\frac{\pi R^2}{\sigma_E}\frac{u}{v_\oplus}\frac{u^2}{u^2+v_e^2}\bar{\gamma}(u) \text{ yrs}, \tag{3.32}$$

where

$$\bar{\gamma}(u) \equiv \frac{1}{2\pi}\int d\Omega'\gamma(u,\theta,\phi). \tag{3.33}$$

The function $\bar{\gamma}(u)$ evidently plays a crucial role in the evaluation of indirect capture. While I do not think $\bar{\gamma}(u)$ can be evaluated in closed form, its basic behavior is easily understood. In the low velocity limit, $\gamma(u,\theta,\phi)$ can be Taylor expanded, which leads to an expansion for $\bar{\gamma}(u)$,

$$\bar{\gamma}(u) = 1 + \frac{12}{5}\left(\frac{u}{v_\oplus}\right)^2 + \ldots \tag{3.34}$$

Somewhat greater care is required in approaching the opposite limit, $u \to (2^{\frac{1}{2}} - 1)v_\oplus$. The velocity $(u,\theta) = [(2^{\frac{1}{2}} - 1)v_\oplus, 0]$ is a special point in phase space. A WIMP with this velocity has its orbit tangent to the Earth's, with infinite period. Physically, this corresponds to the fact that these WIMPs are on the boundary of the upper velocity range where, as discussed in the next section, the assumptions of this section break down and the whole analysis becomes shakier. Indeed, as mentioned above, the assumptions of this section already tend to break down for colinear orbits. The calculation will therefore be less accurate in this region than in the rest of phase space. It is thus important to check that the integral (3.33) is at least well-behaved in this region, to insure that these inevitable errors do not pollute the entire calculation. To facilitate this check, I parameterize u by ϵ

$$1-\epsilon \equiv \frac{u}{(2^{\frac{1}{2}}-1)v_{\oplus}}, \tag{3.35}$$

and expand $\gamma(u,\theta,\phi)$ about the pole at $\epsilon=\theta=0$. Then, to leading order in ϵ,

$$\begin{aligned}\bar{\gamma}(u) &= \frac{(2^{\frac{1}{2}}+1)^{\frac{3}{2}}}{2^{\frac{1}{2}}} \int_{\theta=0}^{\pi} d\frac{\theta}{2}\cos^3\frac{\theta}{2}\frac{\sin^3\frac{\theta}{2}}{\left(\frac{\epsilon}{2^{\frac{1}{2}}}+\sin^2\frac{\theta}{2}\right)^{\frac{3}{2}}} \times \\ &\int_0^{\pi/2} d\phi 3\sin\phi\cos\phi\left[1+(2^{\frac{1}{2}}-1)\sin^2\frac{\theta}{2}+(2^{\frac{1}{2}}-1)^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\sin^2\phi+\ldots\right] \\ &\simeq (1.14)\frac{(2^{\frac{1}{2}}+1)^{\frac{3}{2}}}{2^{\frac{1}{2}}}\left(1-\frac{3}{2^{\frac{1}{4}}}\epsilon^{\frac{1}{2}}+\ldots\right) \sim 3.02\left(1-2.52\epsilon^{\frac{1}{2}}\right).\end{aligned} \tag{3.36}$$

The integral converges even for $\epsilon \to 0$; thus, the singular region has no special importance. These two limiting behaviors intersect at

$$\epsilon \sim .04, \qquad u \sim .96(2^{\frac{1}{2}}-1)v_{\oplus}, \qquad \bar{\gamma}(u) \sim 1.4. \tag{3.37}$$

Thus, over most of the interval, $\bar{\gamma}(u)$ may be regarded as a gently rising function, not much above 1. It is only near the very upper end of the interval that it rises sharply, and even there it is well-behaved.

The dimensionless interaction rate, $\alpha(u)$, may now be written

$$\alpha(u) = \frac{3}{2}\frac{\sigma_E}{\pi R^2}\frac{v_{\oplus}}{u}\left(1+\frac{v_e^2}{u^2}\right)\frac{1}{\bar{\gamma}(u)}\frac{\tau}{\mathrm{yr}} \sim \frac{\sigma_{E10}}{32}\frac{v_{\oplus}}{u}\left(1+\frac{v_e^2}{u^2}\right)\frac{1}{\bar{\gamma}(u)}, \tag{3.38}$$

where σ_{E10} is the weak cross-section of the Earth in units of its weak cross-section for 10 GeV Dirac neutrino WIMPs. (For 10, 20...60 GeV Dirac neutrinos, σ_{E10} is respectively 1, 2.7, 4.6, 6.4, 7.9, 9.5).

Equation (3.30) [and consequently eqns. (3.32) and (3.38)] should now be modified to take account of the fact that not all WIMPs which scatter off the Earth are actually captured by it. To do this, I define the "critical velocity", $v_\star$, to be the velocity at which an orbit-captured WIMP's scattering rate is the inverse of half an Earth lifetime:

$$v_\star \equiv \alpha^{-1}(2). \tag{3.39}$$

WIMPs which scatter to a velocity less than $v_\star$ will be assumed to be eventually captured and those which scatter to greater velocities will be assumed to remain in solar orbit forever. [The number (2) on the right hand side of eqn. (3.39) is somewhat arbitrary, but since $\alpha(u)$ is such a rapidly falling function of u, the precise number used is not important.]

The critical velocity may be evaluated from equations (3.39) and (3.38):

$$v_\star = \left[\frac{3}{2}\frac{\sigma_E}{\pi R^2} v_\oplus v_e^2 \, \frac{\tau/2}{\mathrm{yr}}\right]^{\frac{1}{3}} \sim 4.5\sigma_{E10}^{\frac{1}{3}} \text{ km/sec.} \tag{3.40}$$

The form of equation (3.40) is an appropriate simplification of equation (3.38) because, at the critical velocity,

$$\frac{1+\frac{{v_\star}^2}{v_e^2}}{\bar{\gamma}(v_\star)} \sim 1. \tag{3.41}$$

Note that the critical velocity is 4.5 km/sec for 10 GeV Dirac neutrinos and 9.0 km/sec for 50 GeV Dirac neutrinos.

With this correction, the dimensionless capture rate can be written

$$\alpha(u) = \frac{3}{2}\frac{\sigma_E}{\pi R^2}\frac{v_\oplus}{u}\frac{v_e^2}{u^2}\frac{\tau}{\mathrm{yr}}\frac{\beta(u)}{\bar{\gamma}(u)} = 2\left(\frac{v_\star}{u}\right)^3\frac{\beta(u)}{\bar{\gamma}(u)} \tag{3.42}$$

where

$$\beta(u) \equiv \sum_{\oplus\text{elem.}} f\left[1 - \frac{u^2}{v_c^2} + \frac{\mu_+^2}{\mu_-^2}\frac{\min\left(u^2, {v_\star}^2\right)}{v_c^2}\right]. \tag{3.43}$$

In the above sum over Earth elements, f is the fraction of the total Earth weak cross-section due to each element and μ and v_c are evaluated for that element.

Using equations (3.42), (3.43), (3.31), (3.33), (3.2), and (3.16), one may evaluate indirect capture numerically. The results of this calculation are plotted in Figure 1 for Dirac neutrino WIMPs of various masses. The Galactic WIMP distribution and first Earth model of Paper II are assumed. The dashed line is for direct capture alone and the dotted line is the sum of direct capture and indirect capture from the lower range.

However, in order to gain physical insight, it is useful to make some analytic estimates. Toward this aim, I will make the following further approximations. First I will evaluate $c_i(u)$ according to the asymptotic behavior of equation (3.16)

$$\begin{aligned} c_i(u) &\to c(u) && u < v_\star \\ c_i(u) &\to \frac{1}{2}\alpha(u)c(u) && u > v_\star. \end{aligned} \tag{3.44}$$

Because $\bar{\gamma}(u)$ does not differ appreciably from 1 except near the upper limit of the integral (3.18), and because [according to equations (3.42), (3.16), (3.3), and (3.4)] this region does not contribute much to the integral anyway, I will take

$$\bar{\gamma}(u) \to 1. \tag{3.45}$$

Next note that for Dirac neutrinos of any mass, iron accounts for well over half of the Earth's total weak cross-section. Since (except for very low mass WIMPs) the cut-off velocity for iron is well above the upper limit of integral (3.18), this means that iron will also dominate the process of indirect capture. (By contrast, orbit-capture as well as direct capture are dominated by the Earth element which is nearest to the WIMP in mass.) In line with this fact, I will approximate

$$\beta(u) \to 1 - \left(\frac{u}{v_c'}\right)^2 \tag{3.46}$$

where v_c' is the iron cut-off velocity. While this characteristic form of $\beta(u)$ will be important for later discussions, it may be replaced by unity without much loss of accuracy in the cases I am about to consider. With these simplifications, indirect capture has the form

$$c_i(u) \to c(u) \cdot \min\Big(1, \frac{{v_\star}^3}{u^3}\Big). \tag{3.47}$$

From equation (3.47) and equations (3.3) and (3.4), and making use of the accidental relation (3.5), it is quite easy to evaluate C_{i1}. [Recall from eqn. (2.24), that at a resonance, the cut-off velocity is infinite.] Then, on a resonance, one finds

$$\frac{C_{i1}}{C_d} \to \frac{{v_\star}^2}{v_e^2}\Big[(1) + \Big(2 - 2\frac{v_\star}{v_e}\Big)\Big] \quad v_c \gg v_\odot; \tag{3.48}$$

on the shoulder of a resonance,

$$\frac{C_{i1}}{C_d} \to \frac{{v_\star}^2}{v_e^2}\Big[\Big(2 + \frac{{v_\star}^2}{v_e^2}\Big) + \Big(4 - 4\frac{{v_\star}^2}{v_e^2}\Big)\Big] \quad v_\oplus \ll v_c \ll v_\odot. \tag{3.49}$$

In each of the above equations, the term in the first parentheses comes from the region below $v_\star$ and the second term comes from the region above it.

Equations (3.48) and (3.49) can be interpreted physically as reflecting a deepening of the *effective* gravitational potential of the Earth. In Paper II, I showed that at a resonance, direct capture is proportional to the gravitational potential, ϕ, and away from a resonance it is proportional to ϕ^2. Further, the width of a resonance is proportional to $\phi^{\frac{1}{2}}$. When indirect capture is taken into account, a WIMP can have a higher kinetic energy at interaction and still *ultimately* be captured. Thus the Earth has an effective gravitational potential ϕ_{eff},

$$\phi \equiv \frac{1}{2}v_e^2 \to \phi_{\text{eff}} \equiv \frac{1}{2}(v_e^2 + {v_\star}^2). \qquad (3.50).$$

If, for a moment, one includes only the first term in equations (3.48) and (3.49), the term which comes from below $v_\star$, then direct plus indirect capture obeys relations similar to those which hold for direct capture alone. The sum is proportional to ϕ_{eff} at a resonance and to ϕ_{eff}^2 on the shoulder of a resonance. The second term in the above equations (as well as other corrections not shown in this simplified analysis) reflect the fact that the potential well is not defined by some sharp cut-off velocity, v_e, but rather by the probability function, (3.16), over a range of velocities in the neighborhood of $(v_e^2 + v_\star{}^2)^{\frac{1}{2}}$.

A more detailed analysis would show that the width of the resonance of the sum is proportional to $\phi_{\text{eff}}^{\frac{1}{2}}$. This increased width can be seen clearly in Figures 1 and 2, especially in the three resonances at Mg, Si, and S. Note that in the direct capture curve, these three resonances are relatively distinct, but in the total capture curve they tend to blend together. It is important to emphasize that the smooth character of the direct and total capture curves is *not* shared by the curve representing indirect capture alone. In Figure 2, I show the direct, indirect, and total capture curves for Dirac neutrinos of mass $12 - 35$ GeV. The indirect curve is rather jagged. Note, however, that the "corners" of this curve blend perfectly with the "dips" in the direct capture curve, to form a smooth total. This is not an accident. It reflects the fact that the Earth's gravitational field has been effectively extended, and that considering the "extension" (that is, indirect capture) alone is somewhat artificial.

According to equations (3.48) and (3.49), indirect capture can be quite significant. At the oxygen, silicon and iron resonances, it is respectively 35%, 50%, and 75% of direct capture. On the shoulders of these resonances, the figures are respectively 90%, 135%, and 200%. Since expression (3.47) overstates the capture rate in the region $u \sim v_\star$, it may be expected that these estimates are on the high side. This observation is confirmed by Figure 1.

4. Indirect Capture: II

In this section, I analyze indirect capture in the upper range,

$$\left(2^{\frac{1}{2}} - 1\right)v_{\oplus} < u < \left(2^{\frac{1}{2}} + 1\right)v_{\oplus}. \tag{4.1}$$

This presents considerably greater difficulties than did the analysis of the last section. Consequently, this calculation should be regarded as a rough estimate. Fortunately for the accuracy of the total capture rate, this estimate indicates that capture from this range is, in most cases, quite small.

The number of WIMPs scattering to a velocity u, is still given by the quantity $c(u)du$ in equation (3.2). However, now not all the WIMPs so scattering are in bound solar orbit. There exists a cone of half-angle

$$\Phi_1 = \cos^{-1}\left(\frac{v_{\oplus}^2 - u^2}{2uv_{\oplus}}\right) \tag{4.2}$$

inside of which WIMPs with velocity u, escape. Thus, only a fraction

$$f_1 = \frac{v_{\oplus}^2 + 2uv_{\oplus} - u^2}{4uv_{\oplus}} \tag{4.3}$$

of all possible directions is available for orbit-capture. If the incoming WIMP distribution were isotropic, then the scattered WIMPs would likewise have an isotropic distribution and orbit-capture would be given by

$$f_1 \cdot c(u)du. \tag{4.4}$$

The incoming distribution is, in fact, basically isotropic in the range $u' > \left(2^{\frac{1}{2}} + 1\right)v_{\oplus}$. But below this limit it is highly anisotropic: At velocity $u' = (2^{\frac{1}{2}} - 1)v_{\oplus}$, for example, all incoming WIMPs are lined up with the Earth's velocity vector. To calculate scattering from an anisotropic distribution properly, one should, for each incoming direction, track the incoming hyperbolic orbit, track the less regular orbit inside the Earth, determine the incoming scattering angle, calculate the outgoing angle (which is highly mass dependent) as a function of outgoing velocity, track the cone of resulting final particles back through the Earth and on another set of hyperbolas, and find what fraction of them are outside the forbidden escape cone described by Φ_1. This could, in principle, be done on a computer; but it is beyond my analytic capabilities. I will simply assume that equation (4.4) is valid. This assumption is very good for WIMPs above 12 GeV. For these, orbit-capture is dominated by resonances (or near-resonances) and hence by incoming WIMPs from the isotropic part of the distribution. For lower mass WIMPs, this approximation introduces some errors.

Next I turn to the time evolution of the orbit-captured WIMPs. Recall that in the last section, the corresponding problem was handled by assuming that the WIMPs thermalized, that is, that they populated orbits in proportion to the orbit's interaction time. This approach must now be radically modified for two reasons. First, if the WIMPs were allowed to thermalize, they would eventually find escape orbits and leave the solar system. Second, WIMPs in the upper range will not generally migrate very far in phase space during half an Earth lifetime. The mean migration angle can be estimated as follows.

Suppose a WIMP passes the Earth with impact parameter between b and $b + db$. Using an impulse approximation, one finds that it will be deflected through an angle, δ,

$$\delta(b) = \frac{R_\oplus}{b}\frac{v_0^2}{u^2}, \tag{4.5}$$

where v_0 is the escape velocity from the surface of the earth, $\sim$ 11.2 km/sec. According to the analysis of section III, the WIMP will make such a pass

$$N(b)db = \frac{3}{2}\,\frac{2\pi b db}{\pi R^2}\frac{v_\oplus}{u}\tilde{\gamma}^{-1}\frac{\tau/2}{\text{yr}} \tag{4.6}$$

times during half an Earth lifetime. Here $\tilde{\gamma}$ is meant to indicate an average of $\gamma(u,\theta,\phi)$ over the orbits sampled by the WIMP. For a "typical" WIMP it is of order unity. Now, strictly speaking, a random walk on a sphere (with a hole in it, no less!) should not be handled in the same way as a random walk on a plane. I will ignore this complication and treat the problem naively. The mean angular random walk, $\Delta(u)$, is then given by

$$[\Delta(u)]^2 = \int_{b_{min}}^{b_{max}} db\ N(b)[\delta(b)]^2 = 3\frac{R_\oplus^2}{R^2}\frac{\tau/2}{\text{yr}}\tilde{\gamma}^{-1}\frac{v_\oplus}{u}\frac{v_0^4}{u^4}\ln\frac{b_{max}}{b_{min}}. \tag{4.7}$$

Clearly $b_{min} = R_\oplus$. The upper limit may be obtained from considerations of unitarity, equation (4.6), and the observation that factors of $\mathcal{O}(10)$ in b_{max} do not make much difference,

$$1 \sim \frac{\pi b_{max}^2}{\pi R^2}; \qquad \ln\frac{b_{max}}{b_{min}} \sim 10. \tag{4.8}$$

From this one finds

$$\Delta(u) \sim 1.5\tilde{\gamma}^{-\frac{1}{2}}\left(\frac{v_\oplus}{u}\right)^{\frac{5}{2}} \text{ radians.} \tag{4.9}$$

This result indicates that a WIMP at the bottom of the upper range will migrate about 10 radians, and a WIMP with speed $u \sim v_\oplus$ will migrate about 1.5 radians. Thus, WIMPs in the lower part of the range will sample much of the sphere and hence are likely to find the escape orbits (or at least the very long ones). On the other hand, WIMPs in the mid part of the range will not find the escape orbits unless they start out fairly close to them. If this result is combined with the previous observation [see eqns. (4.2) and (4.3)] that the hole in phase space is much larger for the higher than the lower speed WIMPs, one concludes that for all speeds, u, only a relatively small fraction of the WIMPs scattering to u, actually stay in orbit long enough to have a chance of being captured. I translate this qualitative observation into a specific estimate as follows. I assume that for WIMPs in the entire range, the fraction of $c(u)$ which are actually orbit-captured *and* remain in orbit long enough to be indirectly captured is

$$f_2 = \frac{1}{2} f_1. \tag{4.10}$$

Further I assume that these WIMPs thermalize over two-thirds of the *available* phase space; that is, *outside* of the forward cone with half-angle Φ_2, such that

$$1 + \cos\Phi_2 = \frac{2}{3}(1 + \cos\Phi_1). \tag{4.11}$$

This is an admittedly crude estimate, but I believe that it reasonably takes into account the two factors which determine long-term WIMP population. It also takes account of the fact that the WIMPs will spread from their initial position without becoming completely thermalized. Using this assumption one may define an appropriate $\bar{\gamma}(u)$,

$$\begin{aligned}\bar{\gamma}(u) &\equiv \frac{1}{f_2} \cdot \frac{2}{1 + \cos\Phi_2} \frac{1}{2\pi} \int_{\theta=\Phi_2}^{\pi} d\cos\theta \int_0^{\pi} d\phi\gamma(u,\theta,\phi) \\ &= \frac{3}{2\pi {f_1}^2} \int_{\cos\theta=-1}^{\frac{4}{3}f_1-1} d\cos\theta \int_0^{\pi} d\phi\gamma(u,\theta,\phi).\end{aligned} \tag{4.12}$$

With this definition of $\bar{\gamma}(u)$, equations (3.42), (3.43), (3.2), and (3.16) can be used to evaluate indirect capture in the upper region. I have done this numerically and the results are given in Figure 1. The dot-dash curve shows the total capture rate including indirect capture from the upper region. The dotted curve shows capture without this contribution.

It is again useful, however, to give an analytic estimate. To do this, I approximate

$$\frac{1}{\bar{\gamma}(u)} \to \frac{1}{3}\,\theta\left(\frac{5}{4}v_{\oplus} - u\right) \sim \frac{1}{3}\,\theta(3v_e - u), \tag{4.13}$$

and $\beta(u)$ as in expression (3.46). Then for $v_c' < 3v_e$ (as is the case near the oxygen and silicon resonances) the contribution of indirect capture from the upper range,

C_{i2}, is given by

$$\frac{C_{i2}}{C_d} \to \frac{2}{3}\frac{v_\star{}^3}{v_e^3}\left(1 - \frac{v_e}{v_c'}\right)^2 \qquad v_c \gg v_\odot \tag{4.14}$$

and

$$\frac{C_{i2}}{C_d} \to \frac{8}{9}\frac{v_\star{}^3}{v_e^3}\left[\frac{v_c'}{v_e} - 3\frac{v_e}{v_c'} + 2\left(\frac{v_e}{v_c'}\right)^2\right] \qquad v_\oplus \ll v_c \ll v_\odot. \tag{4.15}$$

Since the values of v_c'/v_e near oxygen and silicon are respectively ~ 1.5 and ~ 2.8, indirect capture from the upper region at these resonances is respectively 0.6% and 4% of direct capture. On the shoulders of these resonances the figures are 2% and 25%.

On the other hand, $v_c' \sim \infty$ near the iron resonance, so the corresponding formulae are

$$\frac{C_{i2}}{C_d} \to \frac{4}{9}\frac{v_\star{}^3}{v_e^3} \qquad v_c \gg v_\odot \tag{4.16}$$

and

$$\frac{C_{i2}}{C_d} \to \frac{32}{9}\frac{v_\star{}^3}{v_e^3} \qquad v_\oplus \ll v_c \ll v_\odot. \tag{4.17}$$

Thus, at and on the shoulder of the iron resonance, indirect capture from the upper region is respectively 15% and 100% of direct capture. Figure 1 shows that these estimates are roughly accurate.

ACKNOWLEDGEMENTS

I would like to thank M. Peskin for his critical reading of the manuscript.

APPENDIX A: Propagation of Anisotropies In A Central Potential

In this Appendix, I analyze the propagation of the anisotropies of a distribution of non-relativistic particles in a central potential. This analysis is then applied to the quadrupole anisotropy of the time-averaged WIMP distribution in the Sun's (Coulomb) potential. The term "propagation" is used here to describe the changes in the distribution as viewed by an observer who moves from infinity to some finite distance from the Sun. The distribution itself is, of course, regarded as time-independent.

In the discussion of the general problem of a central potential, I will refer to the center of the potential as "the Sun" and the observation point as "the Earth". These terms may be regarded as mnemonic devices and do not compromise the general character of the discussion.

Let $f_g(t,\theta)dtd\cos\theta$ be the WIMP distribution far from the Sun, and let $f_s(s,\delta)dsd\cos\delta$ be the induced distribution at a point in the neighborhood of the Earth but in the frame of the Sun. The angles δ and θ are measured from the same polar axis, the axis linking the Sun and the Earth. The angle θ may be regarded as a function of δ,

$$\theta = \theta(\delta), \tag{A1}$$

since the orbit of every WIMP which approaches the Earth at angle δ can be traced back to its angle of origin, θ. It is useful to parameterize this relation in terms of b, the impact parameter. From conservation of angular momentum it follows that

$$\sin\delta = \frac{b}{Rs/t} \equiv \frac{b}{b_{\mathrm{max}}}, \tag{A2}$$

where R is the distance from the Sun to the Earth. This equation serves to define b_{max}. It is also useful to express the relation between s and t in terms of ϵ,

$$\epsilon \equiv \frac{s^2 - t^2}{2t^2}, \tag{A3}$$

a parameter which vanishes in the limit of high incident velocities.

I now claim that the distributions f_s and f_g are related by the differential form of equation (2.2),

$$\frac{f_s(s,\delta)}{s}ds = \frac{f_g(t,\theta)}{t}dt. \tag{A4}$$

To see this, consider a ring of WIMPs all travelling in the polar direction with impact parameters between b and $b+db$. These will intersect the "Earth shell" (the locus of points one astronomical unit from the Sun) between polar angles θ and $\theta+d\theta$, and with angles-to-the-normal between δ and $\delta+d\delta$. The angles θ and δ are related by equation (A1). Let the WIMP density (integrated over all angles) at infinity and at the shell be described respectively by $f_g(t)dt$ and $f_s(s)ds$. Then by flux conservation

$$2\pi b db f_g(t)t dt = 2\pi R \sin\theta(R d\theta) f_s(s) s ds \cos\delta, \tag{A5}$$

or

$$f_s(s) = f_g(t)\frac{s dt}{t ds}\frac{d\cos\delta}{d\cos\theta}. \tag{A6}$$

In going from equation (A5) to equation (A6), I made use of equation (A2).

Of course the density at θ due to a polar flux is the same as the density at the pole due to an incident flux at angle θ. Consider, then, a distribution of small but finite angular width,

$$f_g(t) = f_g(t,\theta)\Delta\cos\theta. \tag{A7}$$

This will induce a distribution near the Earth which is also of finite width,

$$f_s(s) = f_s(s,\delta)\Delta\cos\delta, \tag{A8}$$

where again, θ and δ are related by equation (A1). Plugging equations (A7) and (A8) into equation (A6) yields equation (A4).

In the foregoing treatment, I ignored the azimuthal angle ϕ. Since, by conservation of angular momentum, the orbits are confined to a plane, the azimuthal angle is conserved up to a possible addition of π. This ambiguity can be overcome by adopting the following convention. If an incoming WIMP would "ordinarily" be described by coordinates (θ, ϕ), but its orbit turns through an angle between π and 2π (or between 3π and 4π etc) then it will instead be described by $(2\pi - \theta, \phi + \pi)$. With this convention, ϕ is strictly conserved and one may write

$$\frac{f_s(s, \delta, \phi)}{s} ds = \frac{f_g(t, \theta, \phi)}{t} dt. \tag{A9}$$

Equations (A1) and (A9), together with the distribution $f_g(t, \theta, \phi)$ completely specify $f_s(s, \delta, \phi)$.

The distributions f_g and f_s may be expanded in terms of the Y_{lm}, the standard spherical harmonics (Jackson 1975), and the components of f_s may then be evaluated in terms of f_g. First expand f_g,

$$f_g(t, \theta, \phi) = \frac{f_g(t)}{(4\pi)^{\frac{1}{2}}} \sum_{lm} A_{lm} Y_{lm}(\theta, \phi). \tag{A10}$$

Note that the distribution is normalized so that $A_{00} \equiv 1$. Also note that the dependence of the A_{lm} on t is not shown explicitly. [There is no need to take explicit account of the convention for ϕ, mentioned above, because the Y_{lm} are invariant under $(\theta, \phi) \rightarrow (2\pi - \theta, \phi + \pi)$.] Next expand f_s,

$$f_s(s, \delta, \phi) = \frac{f_g(t)}{(4\pi)^{\frac{1}{2}}} \frac{sdt}{tds} \sum_{lm} A_{lm}(\epsilon) Y_{lm}(\delta, \phi) \equiv \frac{f_s^0(s)}{(4\pi)^{\frac{1}{2}}} \sum_{lm} A_{lm}(\epsilon) Y_{lm}(\delta, \phi). \tag{A11}$$

Equation (A11) serves to define $f_s^0(s)ds$ as what the total density of WIMPs at the Earth *would be*, if f_g were isotropic. Again the dependence on of the $A_{lm}(\epsilon)$ on t is implicit. Note that in the limit of high incident velocities, $\epsilon \ll 1$,

$$A_{lm}(\epsilon) \rightarrow A_{lm}(0) = A_{lm}. \tag{A12}$$

Using equations (A1), (A9), (A10), and (A11), the components of f_s may be evaluated

$$
\begin{aligned}
A_{lm}(\epsilon) &= \frac{(4\pi)^{\frac{1}{2}}}{f_g(t)}\frac{tds}{sdt}\int_0^{\pi} d\cos\delta \int_0^{2\pi} d\phi Y^*_{lm}(\delta,\phi) f_s(s,\delta,\phi) \\
&= \frac{(4\pi)^{\frac{1}{2}}}{f_g(t)}\int_0^{\pi} d\cos\delta \int_0^{2\pi} d\phi Y^*_{lm}(\delta,\phi) f_g[t,\theta(\delta),\phi] \\
&= \sum_{l'}\sum_{m'} A_{l'm'}\int_0^{\pi} d\cos\delta \int_0^{2\pi} d\phi Y^*_{lm}(\delta,\phi) Y_{l'm'}[\theta(\delta),\phi].
\end{aligned}
\tag{A13}
$$

By factoring the Y_{lm} into their separate θ and ϕ dependence,

$$
Y_{lm}(\theta,\phi) \equiv \bar{Y}_{lm}(\theta)e^{im\phi}, \tag{A14}
$$

the double sum in equation (A13) may be reduced to a single sum,

$$
A_{lm}(\epsilon) = 2\pi\sum_{l'} A_{l'm}\int_0^{\pi} d\cos\delta\ \bar{Y}_{lm}(\delta)\bar{Y}_{l'm}[\theta(\delta)]. \tag{A15}
$$

This simplification reflects the azimuthal symmetry of the Sun's effect on the distribution at the Earth. Using the symmetry properties of the Y_{lm}, equation (A15) may be further simplified

$$
A_{lm}(\epsilon) = 2\pi\sum_{l'} A_{l'm}\int_0^{\pi/2} d\cos\delta\cdot\bar{Y}_{lm}(\delta)\cdot\{\bar{Y}_{l'm}[\theta_+(\delta)]\ \ (-)^{l-m}\ \ \bar{Y}_{l'm}[\theta_-(\delta)]\}, \tag{A16}
$$

where

$$
\theta_+(\delta) \equiv \theta(\delta), \qquad \theta_-(\delta) \equiv \theta(\pi-\delta). \tag{A17}
$$

Equation (A16) is as far as the analysis can proceed without knowledge of the specific form of the potential. For the case of the Sun's Coulomb potential, equation (A1) may be explicitly parameterized:

$$\sin\delta = \frac{b}{Rs/t}, \tag{A18}$$

$$\theta_+ = \alpha - \Phi \qquad 0 < \delta < \frac{\pi}{2}, \tag{A19}$$

$$\theta_- = 2\pi - \alpha - \Phi \qquad \frac{\pi}{2} < \delta < \pi, \tag{A20}$$

$$\tan\Phi = \frac{b}{a}, \tag{A21}$$

$$\cos\alpha = \cos\Phi - \frac{b}{R}\sin\Phi, \tag{A22}$$

$$\frac{a}{R} = \epsilon = \frac{v_\oplus^2}{t^2}. \tag{A23}$$

Here Φ is the complement of half the turning angle, α is the angle between the symmetry axis of the orbit and the Earth-Sun axis, and R is an astronomical unit. Note in particular that ϵ is the dimensionless inverse energy of the WIMP. Equation (A18) is the same as equation (A2). Equations (A19) through (A23) may be verified by writing the equation of a hyperbola with the origin at one focus,

$$\frac{\left[x + \left(a^2 + b^2\right)^{\frac{1}{2}}\right]^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{A24}$$

and then switching to radial coordinates. I now take the distribution f_g to be the one represented in equation (2.15). This distribution must first be rewritten in terms of the present coordinates (in which the Earth-Sun axis rather than the ecliptic axis is the polar axis). To be specific, I take the x, y, and z directions to be defined respectively by the ecliptic axis, the Earth's velocity vector, and the

Earth-Sun axis. Then, by comparing equations (2.15) and (A10), the coefficients A_{lm} may be evaluated

$$A_{00} = 1, \quad A_{1m} = 0, \quad A_{20} = -\left(\frac{1}{20}\right)^{\frac{1}{2}} a_2' \frac{t^2}{v_\odot^2}, \quad A_{2\pm1} = 0, \quad A_{2\pm2} = \left(\frac{3}{40}\right)^{\frac{1}{2}} a_2' \frac{t^2}{v_\odot^2}, \tag{A25}$$

and all higher spin components are ignored.

I wish to evaluate the (0,0), (2,0), and (2,±2) components of $A_{lm}(\epsilon)$. For each of these components, the alternating sign in equation (A16) is positive. This fact, and equation (A25), imply that the non-vanishing terms in equation (A16) are linear in $(\cos^2\theta_+ + \cos^2\theta_-)$, a quantity which has a simple dependence on $\cos\delta$. Using equations (A18) through (A23), this sum may be evaluated,

$$\frac{1}{2}(\cos^2\theta_+ + \cos^2\theta_-) = 1 - \cos^2\Phi + (2\cos^2\Phi - 1)\cos^2\alpha = g_\epsilon(\sin\Phi_m\cos\delta), \tag{A26}$$

where

$$g_\epsilon(z) \equiv 1 - \frac{\epsilon^2/(1+\epsilon)^2}{1-z^2} + \left[\frac{2\epsilon^2/(1+\epsilon)^2}{1-z^2} - 1\right]\frac{\left[1-(\epsilon+1)z^2\right]^2}{1-z^2}, \tag{A27}$$

and Φ_m is the complement of half the minimum turning angle,

$$\tan\Phi_m \equiv \frac{b_{\max}}{a} = \frac{st}{v_\oplus^2} = \frac{(2\epsilon+1)^{\frac{1}{2}}}{\epsilon}. \tag{A28}$$

Thus, equation (A16) for the components of f_s become

$$A_{00}(\epsilon) = \int_0^{\sin\Phi_m} \frac{dz}{\sin\Phi_m} \cdot 1 \cdot \left[A_{00}\cdot 1 + A_{20}\cdot 5^{\frac{1}{2}}\left(\frac{3}{2}g_\epsilon(z) - \frac{1}{2}\right)\right], \tag{A29}$$

$$A_{20}(\epsilon) = \int_0^{\sin\Phi_m} \frac{dz}{\sin\Phi_m} \cdot 5^{\frac{1}{2}}\left(\frac{3}{2}\frac{z^2}{\sin^2\Phi_m} - \frac{1}{2}\right)\cdot\left[A_{00}\cdot 1 + A_{20}\cdot 5^{\frac{1}{2}}\left(\frac{3}{2}g_\epsilon(z) - \frac{1}{2}\right)\right], \tag{A30}$$

$$A_{2\pm2}(\epsilon) = \int_0^{\sin\Phi_m} \frac{dz}{\sin\Phi_m} \cdot \left(\frac{15}{8}\right)^{\frac{1}{2}} \left(1 - \frac{z^2}{\sin^2\Phi_m}\right) \cdot \left\{A_{2\pm2} \cdot \left(\frac{15}{8}\right)^{\frac{1}{2}} [1 - g_\epsilon(z)]\right\}. \quad \text{(A31)}$$

Thus, from equation (A29), the total density (integrated over all angles), $f_s(s)ds$, is given by

$$\begin{aligned} \frac{f_s(s)}{s} ds &= \frac{f_g(t)}{t} dt \cdot A_{00}(\epsilon) \\ &= \frac{f_g(t)}{t} dt \left\{ 1 + A_{20} \cdot 5^{\frac{1}{2}} \epsilon \left[1 + 6\epsilon \left(1 - \frac{2\epsilon + 1}{4(\epsilon + 1)\sin\Phi_m} \ln \frac{1 + \sin\Phi_m}{1 - \sin\Phi_m} \right) \right] \right\}. \end{aligned} \quad \text{(A32)}$$

This expression has the limiting forms

$$\frac{f_s(s)}{s} ds \to \frac{f_g(t)}{t} dt \left(1 + 5^{\frac{1}{2}} A_{20} \frac{v_\oplus^2}{t^2}\right) = \frac{f_g(t)}{t} dt \left(1 + \frac{a_2}{16} \frac{v_\oplus^2}{v_\odot^2}\right) \qquad t \gg v_\oplus, \quad \text{(A33)}$$

$$\frac{f_s(s)}{s} ds \to \frac{f_g(t)}{t} dt \left(1 + \frac{1}{5^{\frac{1}{2}}} A_{20}\right) = \frac{f_g(t)}{t} dt \left(1 + \frac{a_2}{80} \frac{t^2}{v_\odot^2}\right) \qquad t \ll v_\oplus, \quad \text{(A34)}$$

From these equations it is clear that the errors generated by ignoring the over (or under) density due to the quadrupole anisotropy are small compared to the limit (2.4).

The variable $\epsilon = v_\oplus^2/t^2$ was originally introduced to parameterize the energy of the incident WIMPs (considered a variable) in order to describe their distribution at the position of the Earth (considered fixed). But one might just as well adopt the viewpoint that $\epsilon \equiv v^2/t_0^2$ parameterizes the Sun's gravitational potential, v^2, at various distances, in order to describe the distribution of WIMPs of some fixed energy. From this perspective, equations (A32) through (A34) show that an observer moving toward the Sun will see the overdensity of WIMPs (due to their quadrupole moment) increase linearly with the Sun's gravitational potential. As the gravitational potential reaches $\sim 2/5$ of the WIMP's energy, the overdensity will level off. This is shown in Figure 3, where the A_{20} term of equation (A32) is graphed. Note that no overdensity arises from the A_{22} term.

The capture rate can be affected not only by the overall density, but also by the spin-2 behavior of the distribution at the Earth. (The spin-2 anisotropy in f_g does, in fact, also induce a spin-1 anisotropy in f_s. However, as discussed below, this does not affect capture.) The $A_{2m}(\epsilon)$ can be evaluated using equations (A30) and (A31) but, unfortunately, the resulting analytic expressions are complicated and not very illuminating. I present the results graphically in Figure 3, and give here the analytic expressions for their limiting behavior,

$$A_{20}(\epsilon) = (1+2\epsilon)A_{20} \qquad A_{2\pm2}(\epsilon) = (1-\frac{1}{2}\epsilon)A_{2\pm2} \qquad \epsilon \ll 1, \tag{A35}$$

$$A_{20}(\epsilon) = -\frac{4}{7}A_{20} \qquad A_{2\pm2}(\epsilon) = \frac{4}{7}A_{2\pm2} \qquad \epsilon \gg 1. \tag{A36}$$

As WIMPs of a given energy move closer to the Sun, their quadrupole (2,0) coefficient initially increases linearly with their *kinetic* energy. However, when the potential and kinetic energies are of the same order, the coefficient declines and reverses sign, before levelling off. The last part of this behavior is easily understood: When the potential energy is large, the WIMPs are likely to be in the process of "turning around", and hence to be travelling at $\sim 90°$ to their original direction. Thus, a predominantly polar distribution becomes equatorial, and vice versa. The $m = \pm 2$ coefficients behave completely differently. They decay right from the beginning, but eventually stabilize at $\sim$ half their initial values.

I turn now to the analysis of the effect of spin-2 anisotropies in f_s on the capture rate. The first question to be addressed is, how does the motion of the Earth affect the capture of a single "shell" in velocity space (represented by speed s)? If the Earth were standing still, the capture rate would, by equations (1.1) and (2.23), be proportional to

$$\frac{f_s(s)}{s}\left(1-\frac{s^2}{v_c^2}\right) \rightarrow \int_0^{\pi} d\cos\delta \int_0^{2\pi} d\phi \frac{f_s(s,\delta,\phi)}{s}\left(1-\frac{s^2}{v_c^2}\right) = \left\langle \frac{1}{s} \right\rangle - \frac{\langle s \rangle}{v_c^2}, \tag{A37}$$

where

$$\langle h(s,\delta,\phi)\rangle \equiv \int_0^{\pi} d\cos\delta \int_0^{2\pi} d\phi f_s(s,\delta,\phi) h(s,\delta,\phi). \tag{A38}$$

To ask how capture is affected by the Earth moving with respect to the shell, is to ask how these averages are affected by a Galilean transformation:

$$\left\langle \frac{1}{s} \right\rangle - \frac{\langle s\rangle}{v_c^2} \rightarrow \left\langle \frac{1}{\mid \mathbf{s} - \mathbf{v}_{\oplus} \mid} \right\rangle - \frac{\langle \mid \mathbf{s} - \mathbf{v}_{\oplus} \mid \rangle}{v_c^2}. \tag{A39}$$

The Earth velocity vector is orthogonal to both the Earth-Sun axis and the axis of the ecliptic. Its coordinates are

$$\mathbf{v}_{\oplus} = \left(v_{\oplus}, \frac{\pi}{2}, \frac{\pi}{2} \right), \tag{A40}$$

and thus

$$\begin{aligned} \mid \mathbf{s} - \mathbf{v}_{\oplus} \mid^{-1} = s^{-1} \Bigg\{ 1 &+ i\left(\frac{2\pi}{3}\right)^{\frac{1}{2}} \left[Y_{11}(\delta,\phi) + Y_{1-1}(\delta,\phi) \right] \frac{v_{\oplus}}{s} \\ &- \left[\left(\frac{\pi}{5}\right)^{\frac{1}{2}} Y_{20}(\delta,\phi) + \left(\frac{3\pi}{10}\right)^{\frac{1}{2}} \left(Y_{22}(\delta,\phi) + Y_{2-2}(\delta,\phi) \right) \right] \frac{v_{\oplus}^2}{s^2} + \ldots \Bigg\}, \end{aligned} \tag{A41}$$

$$\begin{aligned} \mid \mathbf{s} - \mathbf{v}_{\oplus} \mid = s \Bigg\{ 1 &- i\left(\frac{2\pi}{3}\right)^{\frac{1}{2}} \left[Y_{11}(\delta,\phi) + Y_{1-1}(\delta,\phi) \right] \frac{v_{\oplus}}{s} \\ &+ \left[\frac{1}{3} + \left(\frac{\pi}{45}\right)^{\frac{1}{2}} Y_{20}(\delta,\phi) + \left(\frac{\pi}{30}\right)^{\frac{1}{2}} \left(Y_{22}(\delta,\phi) + Y_{2-2}(\delta,\phi) \right) \right] \frac{v_{\oplus}^2}{s^2} + \ldots \Bigg\}, \end{aligned} \tag{A42}$$

From these and equation (A11), one may evaluate

$$\left\langle \frac{1}{\mid \mathbf{s} - \mathbf{v}_{\oplus} \mid} \right\rangle = \frac{f_s^0(s)}{s} \left\{ 1 - \left(\frac{1}{20}\right)^{\frac{1}{2}} \frac{v_{\oplus}^2}{s^2} A_{20}(\epsilon) - \left(\frac{3}{40}\right)^{\frac{1}{2}} \frac{v_{\oplus}^2}{s^2} \left[A_{22}(\epsilon) + A_{2-2}(\epsilon) \right] \right\}, \tag{A43}$$

$$\langle \mid \mathbf{s} - \mathbf{v}_{\oplus} \mid \rangle = f_s^0(s) s \left\{ 1 + \frac{1}{3} \frac{v_{\oplus}^2}{s^2} + \left(\frac{1}{180}\right)^{\frac{1}{2}} \frac{v_{\oplus}^2}{s^2} A_{20}(\epsilon) + \left(\frac{1}{120}\right)^{\frac{1}{2}} \frac{v_{\oplus}^2}{s^2} \left[A_{22}(\epsilon) + A_{2-2}(\epsilon) \right] \right\}. \tag{A44}$$

The second term in equation (A44) is not a consequence of the spin-2 behavior of the distribution and has already been taken into account in the treatment of the isotropic case in Section II. Using equations (A43), (A44), (A25), (A37), (A39), and (2.14), the fractional error generated by ignoring anisotropies in f_s is found to be

$$\frac{a_2}{160}\frac{v_\oplus^2}{v_\odot^2}\left(\frac{s^2-2v_\oplus^2}{s^2}\right)\left\{3\frac{A_{22}(\epsilon)}{A_{22}}-\frac{A_{20}(\epsilon)}{A_{20}}\right\}\frac{1+\frac{1}{3}\frac{s^2}{v_c^2}}{1-\frac{s^2}{v_c^2}}. \tag{A45}$$

The quantity in braces can be evaluated in the low and high ϵ limits using equations (A35) and (A36), and is 2 and 16/7 respectively. Consequently, the largest errors in the total capture rate are in the region $v_\oplus \ll v_c \ll v_\odot$, where they are

$$\sim \frac{7}{240}a_2\frac{v_\oplus^2}{v_\odot^2}. \tag{A46}$$

These errors are thus very small, even smaller than those generated by ignoring the overdensity [eqns. (A33) and (A34)].

As I mentioned above, the spin-2 anisotropy in f_g induces a spin-1 anisotropy in f_s. However, because the spin-2 anisotropy has no $m=\pm 1$ components, the induced spin-1 anisotropy must be proportional to Y_{10}. This may be seen from equation (A15), which follows from azimuthal symmetry. Since there are no Y_{10} terms in equations (A41) and (A42), this anisotropy does not affect capture.

In some circumstances, one may be interested in the capture rate averaged over the year of the Earth's orbit, as opposed to the $\sim$ 200 million years of the Sun's orbit. For example, if the evaporation or annihilation rate of WIMPs in the Earth's core were high compared to the Sun's orbit frequency, it would be inappropriate to take an average over such long time scales. In this case, the WIMP distribution could still be regarded as symmetric about the axis of the ecliptic and so could be written,

$$f_g(t)dt\left[1+a_1'\frac{t}{v_\odot}P_1(\cos\theta')+a_2'\frac{t^2}{v_\odot^2}P_2(\cos\theta')+\ldots\right]\frac{d\cos\theta'}{2}, \tag{A47}$$

where

$$P_1(x)=x, \qquad P_2(x)=\frac{3}{2}x^2-\frac{1}{2}, \tag{A48}$$

and the a_i' are of order unity or less. Equation (A47) is similar to equation (2.15), except that it contains an extra (dipole) term and the coefficients are $\sim$ 8 times larger. The larger coefficients reflect the absence of the accidental symmetry discussed in Section II. The spin-2 term may be analyzed exactly as in this appendix and, as I show below, the dipole term does not affect capture. This means that the errors generated by ignoring these anisotropies are $\sim$ 8 times larger than those in expressions (A33) and (A46). While these errors are still small compared to one, they are no longer small compared to the limit (2.4).

That the dipole term does not affect capture may be seen as follows: In the coordinates of this appendix, the dipole is proportional to $[Y_{11}(\theta,\phi)-Y_{1-1}(\theta,\phi)]$. By equation (A15), this gives rise to anisotropies in f_s which are proportional to $[Y_{11}(\delta,\phi) - Y_{1-1}(\delta,\phi)]$ and $[Y_{21}(\delta,\phi) + Y_{2-1}(\delta,\phi)]$. These anisotropies are orthogonal to all the terms in equations (A41) and (A42), which implies that they do not affect capture.

APPENDIX B: High Cut-Off-Velocity Behavior of η

In this appendix, I derive equation (2.30),

$$\eta = 1 - \frac{7}{3}\xi^2 \qquad\qquad v_c \gg v_\odot, \tag{B1}$$

where

$$\xi \equiv \frac{v_\oplus}{v_c}. \tag{B2}$$

I also argue that, in the interval between the range of validity of this equation, and the range of validity of equation (2.28),

$$\eta = 1 - \frac{10}{3}\xi^2 + \frac{79}{15}\xi^4 \qquad\qquad (2^{\frac{1}{2}}+1)v_\oplus < v_c \ll v_\odot, \tag{B3}$$

η is well represented by the latter functional form.

To facilitate the discussion, I introduce the following functions. Let $C_g(\xi)$, $C_s(\xi)$, and $C_e(\xi)$ be respectively the capture rates of the Earth when it is in free space, when it is one astronomical unit from the Sun but at rest in the Sun's frame, and when it is moving in its actual orbit:

$$C_j(\xi) \equiv \sigma n v_e^2 \int_0^{v_c} du \frac{f_j(u)}{u}\Big(1 - \frac{u^2}{v_c^2}\Big) = \sigma n v_e^2 \int_0^{v_c} du \frac{f_j(u)}{u}\Big(1 - \frac{u^2}{v_\oplus^2}\xi^2\Big). \qquad \text{(B4)}$$

And define the functions $\nu(\xi)$ and $\rho(\xi)$ to be

$$\nu(\xi) \equiv \frac{\sigma n v_e^2 \int_0^{v_c} ds f_s(s)/s}{C_s(\xi)}, \qquad \rho(\xi) \equiv \frac{\sigma n v_e^2 f_s(v_c)}{C_s(\xi)}. \qquad \text{(B5)}$$

Using equations (B4), (2.2), and (2.3), and the spherical harmonic expansion of $1/\mid \mathbf{s} - \mathbf{v}_\oplus \mid$, one may verify the identity,

$$C_g(0) \equiv C_s(0) \equiv C_e(0). \qquad \text{(B6)}$$

The function ν has the asymptotic behavior

$$\nu(\xi) = 1 + B\frac{v_\odot^2}{v_c^2}, \qquad v_c \gg v_\odot \qquad \text{(B7)}$$

where

$$B \equiv \frac{\int_0^\infty ds f_s(s) s/v_\odot}{\int_0^\infty ds f_s(s) v_\odot/s} \qquad \text{(B8)}$$

is a number of order 1. From equation (2.18), $\nu(\xi)$ may be evaluated in the opposite limit,

$$\nu(\xi) = 2 \qquad v_\oplus \ll v_c \ll v_\odot. \qquad \text{(B9)}$$

The function $\rho(\xi)$ may also be evaluated in the limits. Clearly, $\rho(0) = 0$, and

$$\rho(\xi) = 4 \qquad v_\oplus \ll v_c \ll v_\odot. \qquad \text{(B10)}$$

For concreteness, I give the following evaluations for the simple case of an isotropic

Maxwell-Boltzmann distribution:

$$B = 1, \quad \nu\Big(\frac{v_\oplus}{v_\odot}\Big) = e{-}1 \sim 1.7, \quad \rho\Big(\frac{v_\oplus}{v_\odot}\Big) = 2 \qquad [\text{Maxwell} - \text{Boltzmann}\ \bar{v}^2 = \frac{3}{2}v_\odot^2]. \tag{B11}$$

Using these definitions and relations, $C_s(\xi)$ may be evaluated:

$$\begin{aligned}
C_g(\xi) &\equiv \sigma n v_e^2 \int_0^{v_c} dt \frac{f_g(t)}{t}\Big(1 - \frac{t^2}{v_\oplus^2}\xi^2\Big) \\
&= \sigma n v_e^2 \int_{2^{\frac{1}{2}}v_\oplus}^{(v_c^2+2v_\oplus^2)^{\frac{1}{2}}} ds \frac{f_s(s)}{s}\Big[1 - \Big(\frac{s^2 - 2v_\oplus^2}{v_\oplus^2}\Big)\xi^2\Big] \\
&= \sigma n v_e^2 \Bigg\{ \int_{2^{\frac{1}{2}}v_\oplus}^{v_c} ds \frac{f_s(s)}{s}\Big(1 - \frac{s^2}{v_\oplus^2}\xi^2\Big) + 2\xi^2 \int_{2^{\frac{1}{2}}v_\oplus}^{v_c} ds \frac{f_s(s)}{s} \\
&\qquad + \int_{v_c}^{(v_c^2+2v_\oplus^2)^{\frac{1}{2}}} ds \frac{f_s(s)}{s}\Big(\frac{v_c^2 - s^2 + 2v_\oplus^2}{v_c^2}\Big)\Bigg\} \\
&\simeq \big[1 + 2\xi^2\nu(\xi) + \xi^4\rho(\xi)\big]\, C_s(\xi).
\end{aligned} \tag{B12}$$

By equation (B10), the last term may be dropped. Then

$$C_s(\xi) = \big[1 - 2\xi^2\nu(\xi)\big]C_g(\xi). \tag{B13}$$

Now consider a single shell in *velocity space* of WIMPs in the Galactic halo. Recall that this shell is isotropic and therefore induces an isotropic shell in the neighborhood of the Earth. If the Earth were standing still, the capture rate due to this shell would be

$$\frac{dC_s(s;\xi)}{ds} ds = \sigma n v_e^2 \frac{f_s(s)}{s}\Big(1 - \frac{s^2}{v_\oplus^2}\xi^2\Big) ds. \tag{B14}$$

As seen from the Earth frame, different parts of the shell will be moving with different velocities

$$u(s,\theta) = \left(s^2 + v_\oplus^2 - 2sv_\oplus \cos\theta\right)^{\frac{1}{2}}, \tag{B15}$$

where θ measures the angle from the Earth's direction. Therefore, the actual capture rate of this shell by the Earth is

$$\frac{dC_e(s;\xi)}{ds}ds = \sigma n v_e^2 f_s(s) ds \int_0^\pi \frac{d\cos\theta}{2}\frac{1}{u}\left(1 - \frac{u^2}{v_\oplus^2}\xi^2\right)$$
$$= \sigma n v_e^2 f_s(s) ds \int_0^\pi \frac{1}{2}\frac{d\cos\theta}{\left(s^2 + v_\oplus^2 - 2sv_\oplus\cos\theta\right)^{\frac{1}{2}}}\left(1 - \frac{s^2 + v_\oplus^2 - 2v_\oplus s\cos\theta}{v_\oplus^2}\xi^2\right), \tag{B16}$$

which may be evaluated (as in Appendix A),

$$\frac{dC_e(s;\xi)}{ds}ds = \sigma n v_e^2 \frac{f_s(s)}{s} ds \left[1 - \frac{s^2\left(1 + \frac{v_\oplus^2}{3s^2}\right)}{v_c^2}\right] \qquad s < v_c - v_\oplus. \tag{B17}$$

As indicated, equation (B17) is only valid for $s < v_c - v_\oplus$. All the WIMPs in the range $v_c - v_\oplus < s < v_c$ are eligible for capture in the Sun frame, but not all are eligible in the Earth frame. Furthermore, some of the WIMPs in the range $v_c < s < v_c + v_\oplus$ are eligible for capture in the Earth frame, but none are eligible in the Sun frame. The total capture rate from these latter two regions is (after one integration)

$$\sigma n v_e^2 \int_0^\pi \frac{d\cos\theta}{2}\frac{f(v_c)}{v_c^2}\left[v_c - v_\oplus - \left(v_c^2 + v_\oplus^2 - 2v_c v_\oplus\cos\theta\right)^{\frac{1}{2}}\right]^2 = \frac{4}{3}C_s(\xi)\rho(\xi)\left[\xi^2 + \mathcal{O}\left(\xi^4\right)\right], \tag{B18}$$

where I have made the approximation

$$\frac{f(s)}{s^2} = \frac{f(v_c)}{v_c^2}\left[1 + \mathcal{O}\left(\xi^2\right)\right] \qquad v_c - v_\oplus < s < v_c + v_\oplus. \tag{B19}$$

Then, by integrating equation (B17) over its range of validity and adding expression (B18), one obtains,

$$C_e(\xi) = C_s(\xi)\left\{1+\frac{1}{3}[-\nu(\xi)+\rho(\xi)]\xi^2\right\} = C_g(\xi)\left\{1-\left[\frac{7}{3}\nu(\xi)-\frac{1}{3}\rho(\xi)\right]\xi^2\right\}, \quad \text{(B20)}$$

or

$$\eta = 1 - \left[\frac{7}{3}\nu(\xi) - \frac{1}{3}\rho(\xi)\right]\xi^2 \qquad v_c \gg v_\oplus, \qquad \text{(B21)}$$

It is important to emphasize that equation (B21) is valid in the entire range given. In particular, in the range $v_\oplus \ll v_c \ll v_\odot$ where $\nu = 2$ and $\rho = 4$, it reduces to equation (B3). In the range $v_c \gg v_\odot$, where $\nu = 1$ and $\rho = 0$, it reduces to equation (B1). To get an idea how equation (B21) behaves in the intermediate region, assume that, for velocities below the characteristic velocity, f_g can be written

$$f_g(t) = \kappa\left(1 - a\frac{t^2}{v_\odot^2}\right) \qquad v_\oplus \ll t \lesssim v_\odot \qquad \text{(B22)}$$

where a is a parameter of order unity. Then, to lowest order, equation (B18) becomes

$$\eta = 1 - \left(\frac{10}{3} + \frac{a}{9}\frac{v_c^2}{v_\odot^2}\right)\xi^2 = 1 - \frac{10}{3}\xi^2 - \frac{a}{9}\frac{v_\oplus^2}{v_\odot^2} \qquad v_\oplus \ll v_c \lesssim v_\odot \qquad \text{(B23)}$$

From this it is apparent that if η is approximated by equation (B3) in the region below $v_\odot$, and if an arbitrary smooth transition to equation (B1) is made in the region above $v_\odot$, then the fractional errors in η will be within the limit (2.4). For example, consider the concrete case given in equation (B11),

$$\eta = 1 - \left(\frac{7}{3}e - 3\right)\xi^2 \simeq 1 - \left(\frac{10}{3} + .01\right)\xi^2. \qquad \text{(B24)}$$

This is extremely close to equation (B3).

FIGURE CAPTIONS

1. Capture rates in units of $10^{17}\mathrm{s}^{-1}$ for Dirac neutrinos of various masses (in GeV). Direct capture is shown before (solid line) and after (dashed line) correction for the Earth's motion in the potential well of the Sun. Direct capture plus indirect capture from the lower range (see text) is shown by a dotted line, and direct plus all indirect capture is shown by a dotdash line. The Galactic WIMP distribution and first Earth model of Paper II are assumed. Note the change scale at ~ 43 GeV.

2. Capture rates in units of $10^{17}\mathrm{s}^{-1}$ for Dirac neutrinos of various masses (in GeV). Direct capture is shown by a dotdash line. Indirect capture from the lower range (see text) is shown by a solid line. The sum is shown by a dotted line. The Galactic WIMP distribution and first Earth model of Paper II are assumed.

3. Propagation of anisotropies in the Sun's gravitational potential shown as functions of the $\log_{10}$ of the square of the ratio of the Earth's velocity to the WIMP's velocity-at-infinity. The $l = 2, m = 0$ (solid line) and $l = 2, m = \pm 2$ (dotted line) components are shown as fractions of their values at infinity. The "overdensity" in the $l = m = 0$ component [that is, the correction to eqn. (2.2)] due to an $l = 2, m = 0$ anisotropy at infinity is shown as a dotdash line. The relative normalization for these last two components is given in the text.

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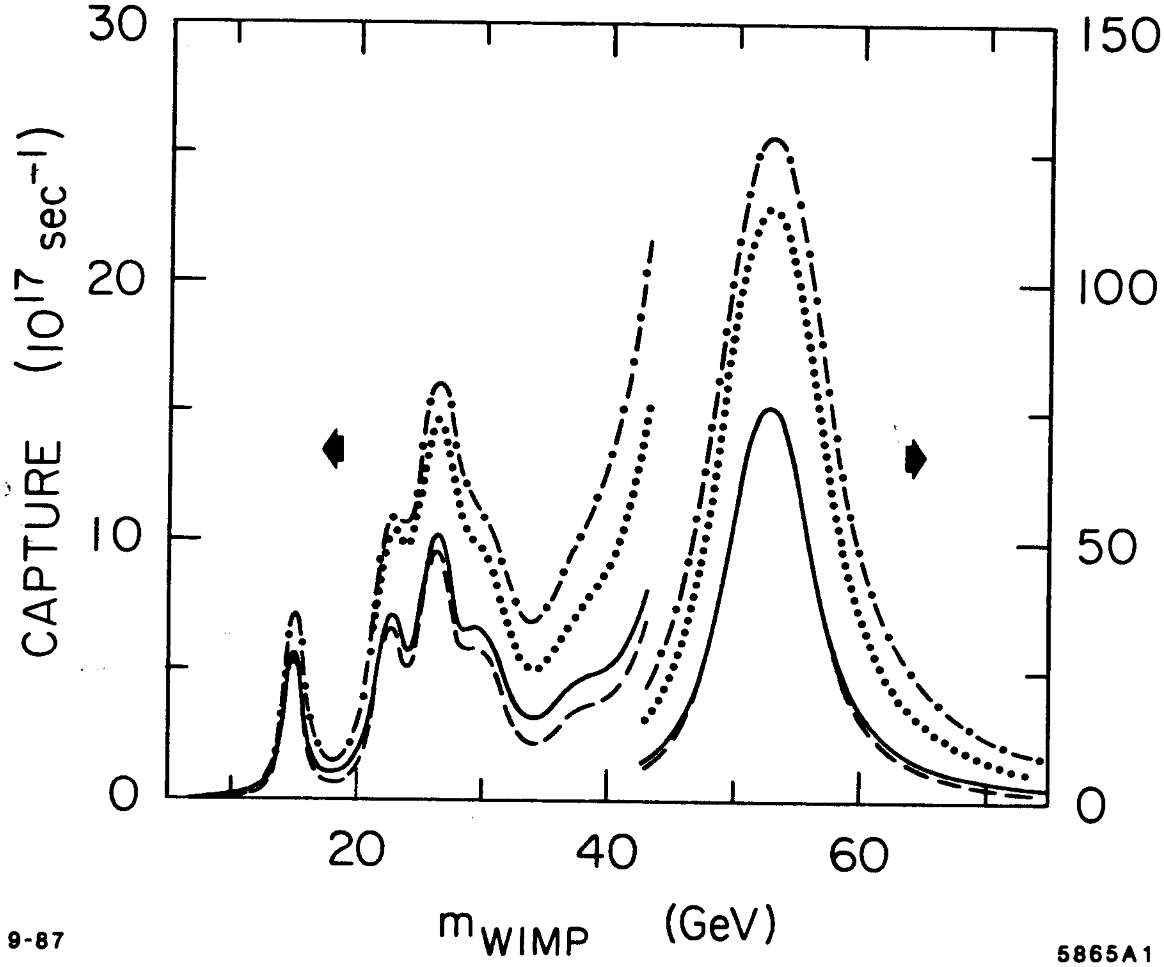
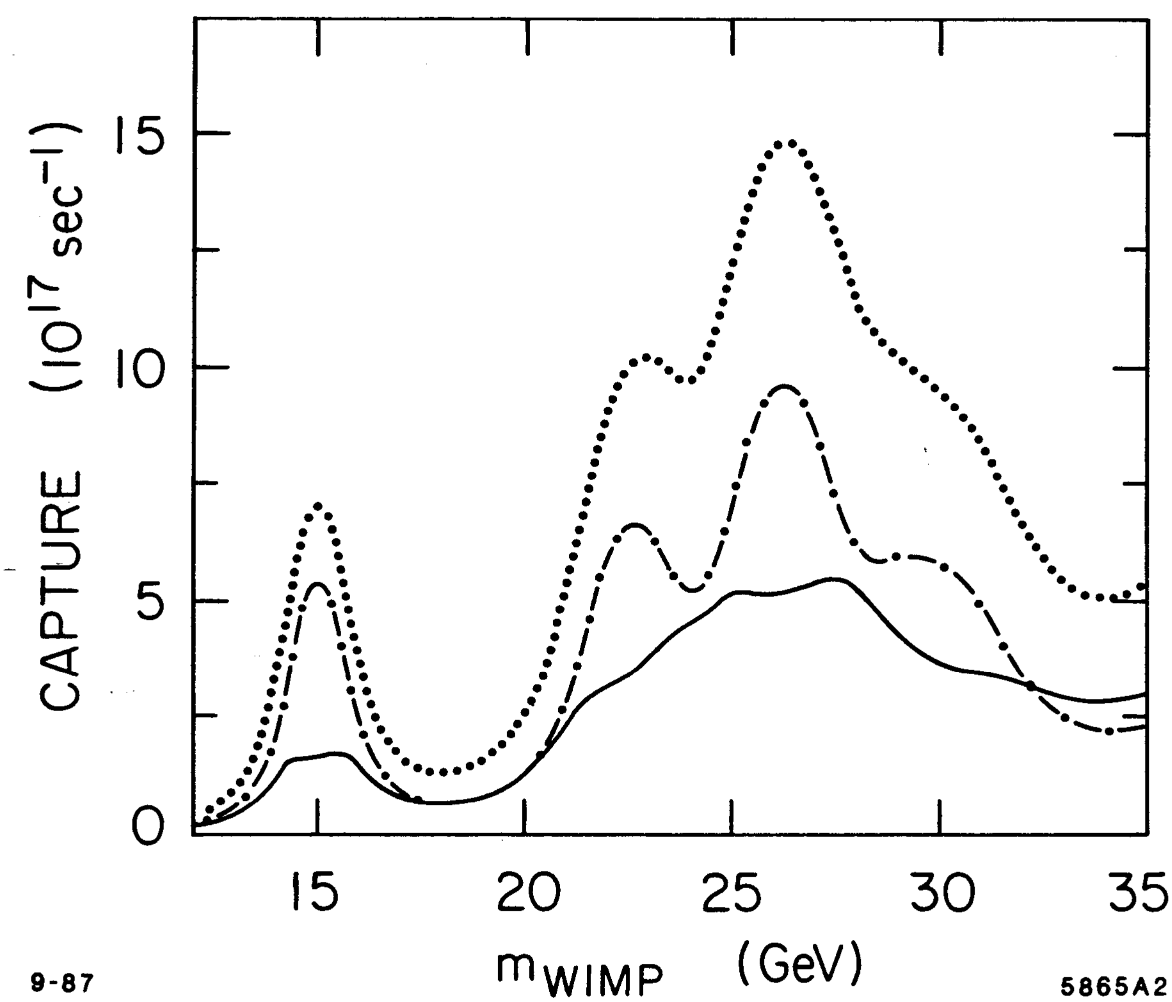

30
20
10
0
150
100
50
0
CAPTURE (10^17 sec^-1)
20
40
60
m_WIMP (GeV)
9-87
5865A1

Fig. 1



CAPTURE $(10^{17}\,\mathrm{sec}^{-1})$
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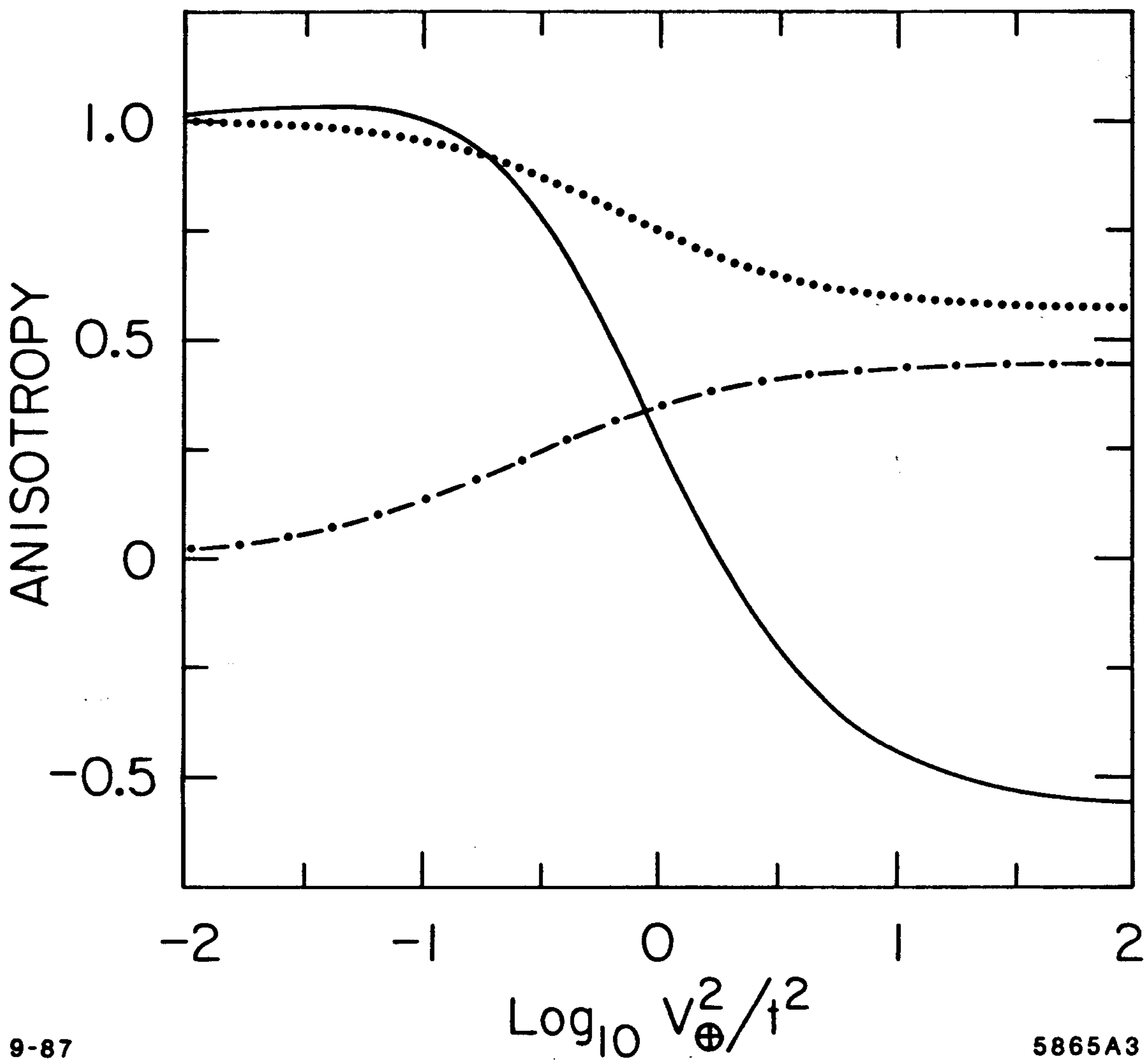
10
15
15
20
25
30
35
m_{WIMP} (GeV)
9-87
5865A2

Fig. 2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0.5
0
−0.5
−2
−1
0
1
2
$\mathrm{Log}_{10}\, V_{\oplus}^2/t^2$
9-87
5865A3

Fig. 3