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A BRST Primer*

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ABSTRACT

We develop a Hamiltonian formulation of the BRST method for quantizing constrained systems. The rigid rotor is studied in detail and the similarity of this simple quantum system to a gauge theory is explicitly demonstrated. The system is quantized as a gauge-theory and then the similarity between BRST and Gupta-Bleuler approach is displayed. We also apply our formalism to true gauge theories. Both Abelian and non-Abelian gauge theories are studied in detail. Finally, the Hamiltonian treatment of the relativistic and the spinning relativistic particle is presented.

1. INTRODUCTION

Ever since the standard model[1] became the accepted candidate for a theory of the strong, weak and electromagnetic interactions, it has become customary to assume that the fundamental theory of everything must be a gauge theory. Unfortunately the theories one will have to deal with are considerably more complicated than QED, our prototypical example of a successful gauge theory, and the tools for dealing with the additional complexity have become more difficult to understand. For example, any serious attempt to calculate in the standard model involves Faddédev-Popov ghosts,[2] objects which appear in covariant gauges in order to guarantee the gauge invariance of the original theory . The raison d'être for these ghosts is clear. The rules for manipulating them can be derived straightforwardly within the context of Euclidean path integrals. Yet, to date, the explanation of the connection between these ghosts and the canonical Hamiltonian formalism has remained tortuous at best. Furthermore, ever since the work of Becchi-Rouet and Stora[3](BRST), it has been clear that the best way to discuss the renormalization of a non-Abelian gauge theory is not by exploiting the original gauge-invariance of the theory but rather by exploiting the existence of a considerably more mysterious BRST symmetry which involves the fictional ghosts in a fundamental way. The power of this approach is so great that nowadays, with the advent of string theories, one is tempted to elevate the existence of a BRST symmetry to the level of a fundamental principle[11–14] relegating the existence of the mundane gauge-symmetry to the level of a corollary. While proceeding in this way greatly simplifies the technical problems of dealing with a gauge theory it introduces yet one more layer of formalism between the naive physical ideas and the machinery used to implement them. It seems that the complexity of the examples which one has to understand in order to learn what a BRST symmetry is are so formidable that non-experts hesitate to make the effort. Thus, while the existence of these symmetries is widely discussed, for many people the physical content of the BRST formalism remains obscure.

It is our purpose in this paper to render the content of the BRST technique more easily accessible to the non-expert by explicitly establishing the connection between constrained systems, gauge invariance and BRST symmetry. The discussion will rely on Hamiltonian formalism throughout and, at the risk of insulting the cognoscenti, will begin at the most pedestrian level and move slowly on to a discussion of the more complicated examples. Our goal is to eliminate the mystery from the method and remove the notion of a BRST symmetry from the elevated plane of non-Abelian gauge (and string) theories and put it where it belongs, into the framework of elementary quantum mechanics.

2. THE RIGID ROTOR

We begin this paper with a discussion of the problem of quantizing a classical system, namely the problem of a particle of mass m constrained to move on a circle of radius a. We begin with this discussion to establish the equivalence between a theory with constraints and a gauge theory. Once we establish this equivalence we explain the BRST formalism for dealing with a gauge theory and prove that it is simply another way of dealing with the original constrained system. The important fact which we will be seeking to bring out is that in each case, i.e. direct quantization, the gauge theory or the BRST system, it will be crucial to impose an eigenvalue condition on the states of the theory in order to eliminate spurious degrees of freedom which are artifacts of the quantization process.

The plan of this chapter is as follows:

1. First we will define the classical problem and show that if one tries to quantize it following the simplest canonical prescription one does not obtain the correct result. By this we mean that the Heisenberg equations of motion for the quantum theory will differ sharply from the corresponding equations for the classical theory.

2. We then give a prescription for modifying the Hamiltonian obtained from the usual canonical formalism so as to obtain a correct quantization of the

constrained system. The basic trick will be to divide the Hamiltonian into two pieces, one of which commutes with the constraint and another which does not. We then follow the prescription of Dirac[4] and simply throw away the part which does not commute with the constraint and show that the correct physical theory belongs to only a subspace of the full Hilbert space which is selected by imposing an eigenvalue condition on the constraint equation.

3. Once we have quantized the constrained system and derived the condition which defines physical states we show how to rewrite the same system as a gauge theory. This is accomplished by enlarging the Hilbert space of the theory beyond that of the original problem. The purpose of this discussion is to elucidate the equivalence of the original state condition and the requirement that one restricts attention to gauge-invariant states. We prove that even though one has enlarged the number of dynamical variables in this problem by restricting attention to the set of gauge-invariant states we remove all of the spurious degrees of freedom.

4. Finally, we discuss the question of BRST quantization of the same system. We show that this amounts to enlarging the number of degrees of freedom in the original problem even further and then restricting attention to what we can call the BRST-invariant states. As part of our discussion we establish the precise correspondence between the BRST-invariant states and the physical states of the gauge theory or the states of the original problem which satisfy the constraint condition.

2.1. The Problem

Let us begin by considering the classical problem of a particle of mass m constrained to move in the $x-y$ plane on a circle of radius a. The Lagrangian for this system is

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \lambda(r-a); \qquad r = \sqrt{x^2+y^2} \tag{2.1}$$

and from it we obtain the usual Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta \mathcal{L}}{\delta \dot{x}}\right) - \frac{\delta \mathcal{L}}{\delta x} &= m\ddot{x} + \frac{\lambda x}{r} = 0 \\ \frac{d}{dt}\left(\frac{\delta \mathcal{L}}{\delta \dot{y}}\right) - \frac{\delta \mathcal{L}}{\delta y} &= m\ddot{y} + \frac{\lambda y}{r} = 0 \cdot \\ \frac{\delta \mathcal{L}}{\delta \lambda} &= r - a = 0 \end{aligned} \tag{2.2}$$

Clearly, the last equation, the one which comes from varying with respect to the Lagrange multiplier, is the one which enforces the constraint on the motion of the particle. It is this constraint which makes it hard to directly quantize the system following familiar canonical formalism. If we form the Hamiltonian in the usual way, i.e., let

$$\begin{aligned} p_x &= \frac{\delta \mathcal{L}}{\delta \dot{x}}, \qquad p_y = \frac{\delta \mathcal{L}}{\delta \dot{y}}, \\ H &= \frac{1}{2m}(p_x^2 + p_y^2) + \lambda(r - a), \end{aligned} \tag{2.3}$$

we immediately notice that something has gone wrong. Although the classical equations of motion for this Hamiltonian are consistent with the constraint $r = a$ it is obvious that if we quantize the system by following the usual canonical procedure and defining

$$[x, p_x] = [y, p_y] = i, \tag{2.4}$$

the constraint equation $\Phi = (r - a) = 0$ no longer commutes with the Hamiltonian. Therefore, one obtains the classical equations of motion neither as the Heisenberg equations of motion for the quantized system, nor as a condition which can be required on the eigenstates of the Hamiltonian. In order to arrive at a quantization procedure which yields Heisenberg equations of motion consistent with the classical constrained equations of motion one has to proceed differently.

To develop a formalism which works for constrained systems begin by examining the commutators of the constraint equation and the canonical variables p_x, p_y, x

and y. Obviously, x and y commute with $\Phi = (r-a)$ and so we need only consider the commutators

$$[r, p_x] = i\frac{x}{r} \qquad \text{and} \qquad [r, p_y] = i\frac{y}{r}. \tag{2.5}$$

It is clear from this result that the linear combination

$$p_\theta = (xp_y - yp_x) \tag{2.6}$$

commutes with r and thus Φ, whereas the orthogonal combination

$$p_r = \frac{1}{2r}(xp_x + yp_y) + (p_x x + p_y y)\frac{1}{2r} \tag{2.7}$$

does not. In fact, we see that Φ and p_r form a system of canonically conjugate variables since

$$[r, p_r] = i. \tag{2.8}$$

It is now a simple matter to rewrite the Hamiltonian (2.3) as

$$H = \frac{1}{2m}p_r^2 + \frac{1}{2mr^2}p_\theta^2 + \lambda(r-a) \tag{2.9}$$

and observe that the only term in (2.9) which fails to commute with the constraint equation $\Phi = 0$ is the term proportional to p_r^2. The solution to our problem is now clear, the correct Hamiltonian to use for quantizing the constrained system is simply obtained from the canonical one by setting to zero the coefficient of the term proportional to p_r^2. If we do this, then we arrive at the Hamiltonian

$$H = \frac{1}{2mr^2}p_\theta^2 + \lambda(r-a). \tag{2.10}$$

Since the constraint $\Phi = 0$ now commutes with H they can be simultaneously diagonalized. Since Φ is a function of the non-trivial operator r, we cannot realize

$\Phi = 0$ as an operator equation; however, since Φ and H can be simultaneously diagonalized, we can restrict attention to the subspace of eigenstates of H for which the eigenvalue condition $\Phi\,|\psi\rangle = 0$ holds. Projecting the full Hilbert space onto the subspace for which the constraint condition holds is, in the case of a gauge theory, usually referred to as projecting onto *physical states.* Since in this case this projection is onto the states for which all projected operators satisfy the desired Hamilton equations of motion, we will also use the same terminology. Clearly, if we denote the projection operator onto these states by P_Φ, then

$$\begin{aligned} P_\Phi \Phi P_\Phi &= 0, \qquad P_\Phi r P_\Phi = a P_\Phi \\ P_\Phi H P_\Phi &= \frac{1}{2ma^2} p_\theta^2 \end{aligned} \quad , \tag{2.11}$$

where p_θ is the angular momentum operator for this system.

To summarize, we see that in order to quantize a constrained system one proceeds as follows:

1. First, form the Hamiltonian from the Lagrangian following the canonical procedure.
2. Next, make a canonical transformation to a new set of variables (in this case p_r, r, θ and p_θ) chosen in such a way that one of them forms a canonically conjugate pair with the constraint equation, and the others commute with the constraint.
3. Finally, set the coefficient of the terms which contain the variable conjugate to the constraint equal to zero; thus arriving at a Hamiltonian which commutes with the constraint, allowing both the energy and the constraint conditions to be simultaneously diagonalized.

2.2. The Rotor as a Gauge Theory

By now we have completely solved our original problem. We have quantized the motion of a particle constrained to move on a circle in two dimensions and obtained the expected result, i.e. that the Hamiltonian of the physical system is proportional to the square of the two-dimensional angular momentum operator. There are, however, aspects of this quantization scheme which merit further attention.

One such aspect relates to the fact that we are now working in a large Hilbert space in which only a subspace corresponds to states of physical interest. Because of this, one has to be careful and restrict attention to operators which do not map us out of this subspace. In this case, this means never considering expectation values of p_r nor any function p_r which fails to commute with the constraint. In effect this means that a correct quantization of the constrained system not only requires that we impose a restriction on our space of states, but also that we declare an entire class of non-vanishing operators unphysical. This situation is very reminiscent of what goes on in the case of a gauge theory, where one is instructed to consider Green's functions of gauge-variant operators as unphysical. The question which arises at this point is "What, for the rotor, takes the place of the time-dependent gauge transformations of the gauge-theory?".

Clearly, if the parallel is to hold, the constraint which does not commute with unphysical operators is the operator which is to be identified with the generator of *gauge transformations*. In this case the natural definition of a gauge transformation would be a unitary operator of the form

$$U_f = e^{-if(t)\Phi} \tag{2.12}$$

where $f(t)$ is an arbitrary c-number function of the time t. Under this transformation p_r is the only canonical variable which transforms non-trivially, and it goes to

$$U_f p_r U_f^\dagger = p_r + f(t). \tag{2.13}$$

The question which arises at this juncture is "Can one write a Lagrangian which has H for its canonically constructed Hamiltonian, and which is invariant under a class of gauge transformations which include the transformation (2.13) ?" . The answer to this question is a resounding yes, but one must use a first order formalism, instead of the usual second order formalism. In order to see how this works let us spend a few moments reviewing the first order formalism and then showing how, using this formalism, we can find a gauge theory which is completely equivalent to the rigid rotor.

First Order Lagrangian Formalism

By first order Lagrangian formalism we simply refer to the fact that the Hamilton equations of motion

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned} \tag{2.14}$$

defined on the phase space $(\ldots, q_i, \ldots, p_i, \ldots)$ themselves follow from a variational principle. The Lagrangian which gives these equations is what we refer to as the first order Lagrangian. In general it is constructed as follows:

$$\mathcal{L}(q_i, p_i) = \sum_i p_i \dot{q}_i - H(q_i, p_i). \tag{2.15}$$

In general a classical trajectory is a curve in phase space $(q_i(t), p_i(t))$ and the action for this path is defined to be

$$S = \int dt\, \mathcal{L}(q_i(t), p_i(t)). \tag{2.16}$$

It then follows from the usual variational procedure that an extremal trajectory satisfies the Hamilton equations (2.14). With this in mind we now wish to exhibit a first order Lagrangian for the problem of the rigid rotor which explicitly reveals its gauge structure. The trick for accomplishing this is quite general and amounts to enlarging the number of dynamical degrees of freedom to include the Lagrange multiplier λ and its canonically conjugate momentum p_λ . Once one has introduced these extra degrees of freedom one defines the change in these variables under an arbitrary time dependent gauge transformation to be:

$$\begin{gathered} \delta p_r = f(t), \qquad \delta\lambda = -\dot{f}(t) \\ \delta r = \delta p_\theta = \delta\theta = \delta p_\lambda = 0 \end{gathered} \tag{2.17}$$

With this choice we write the first order Lagrangian

$$\mathcal{L} = p_r \dot{r} + p_\theta \dot{\theta} - \frac{p_\theta^2}{2mr^2} - \lambda(r-a), \tag{2.18}$$

where it is important to note that the term involving p_λ has been intentionally left out. It is simple to verify that under a gauge transformation of the form (2.17) the change in the Lagrangian is given by

$$\delta\mathcal{L} = \frac{d}{dt}\left(f(t)(r(t)-a)\right); \tag{2.19}$$

i.e., the Lagrangian changes by a total derivative and so the action is invariant. This exhibits the embedding of the problem of the particle constrained to move on a circle into a gauge theory defined on a phase space which is larger than the one we started with.

Recovering The Hamiltonian

Having formed the Lagrangian (2.18) we see that we can reconstruct H by essentially the reverse procedure; i.e., we simply define

$$H = \dot{r}\frac{\partial\mathcal{L}}{\partial\dot{r}} + \frac{\partial\mathcal{L}}{\partial\dot{\theta}}\dot{\theta} - \mathcal{L}. \tag{2.20}$$

The reason that no term involving p_λ appears in (2.20) is that there is no term involving $\dot{\lambda}$ appearing in the Lagrangian. Note that in this first order formalism this is not to be taken to mean that there is no canonical variable conjugate to λ, it simply means that it does not appear in this formal Legendre transformation. While we have now recovered the desired form of the Hamiltonian it is important to point out that a new feature has appeared; namely, although the Lagrangian (2.18) is invariant with respect to arbitrary time dependent gauge transformations the Hamiltonian is not (although it is invariant with respect to time independent ones). This happens because the only term appearing in (2.20) which changes under a gauge-transformation is the term in which λ multiplies the constraint $(r-a)$; hence, under a gauge transformation the change in H is

$$\delta H = -\dot{f}(t)(r(t) - a), \tag{2.21}$$

and so if $\dot{f} = 0$ so does δH. This lack of invariance with respect to time dependent gauge transformations is not a problem since we see that this extra term vanishes on the sector of physical states (i.e., those states for which $\Phi = 0$).

At this time we can turn the whole procedure around and arrive at the need for imposing the condition $\Phi = 0$ without knowing that we started out to quantize the problem of a free particle constrained to move on a circle. We may simply elevate the requirement of gauge invariance to a basic principle and require that the Hamiltonian be restricted to the dynamically stable subspace of states within which it defines a gauge invariant operator. Actually, this can be stated in a way which provides a better parallel to what is done in quantizing a gauge theory. We already remarked that Φ can be identified with the generator of time independent gauge transformations, and so the space of physical states is simply the set of states which are annihilated by Φ.

There is one final detail which must be covered at this time. Since neither the operator λ nor p_λ appears in the restriction of the Hamiltonian to the space of physical (or gauge invariant states) this operator plays no role in the physics which is going on. Hence, even after going over to the space of gauge-invariant states we have not one copy of the theory of a rigid rotor but many such copies, for example we have one copy for every eigenvalue of the operator λ or of the operator p_λ. This means that one has to go a bit further in fixing our projection. Since it is p_λ which is the gauge-invariant operator, and since we wish to be able to define all gauge invariant Green's functions, it makes sense that should define physical states by requiring that they correspond to definite eigenstates of p_λ, e.g. states for which $p_\lambda |\Psi\rangle = 0$. Once one has made this choice one has established a one to one correspondence between the states of the rigid rotor and the subspace of gauge-invariant states for the theory defined by the dynamical variables r,p_r,θ,p_θ,λ and p_λ. This was exactly the correspondence we set out to establish.

To summarize our discussion, we have shown how to explicitly rewrite the problem of quantizing a theory with a constraint as a gauge theory, and that in this theory the original constraint equation is replaced by the condition that we restrict attention to states which are invariant under time independent gauge transformations. In addition, we showed that starting from the gauge-invariant Lagrangian one trivially recovers a satisfactory gauge non-invariant Hamiltonian by blindly performing a simple Legendre transformation. We then showed that this Hamiltonian is identical to the one which we started out to construct for the original constrained problem.

A question which should arises at this point is "How come this procedure looks so much like the procedure for quantizing a theory like QED in $A_0 = 0$ gauge even though we proceded without any regard to the need for gauge fixing?" The answer to this question is that in some sense we have fixed the gauge by following our naive procedure for making a Legendre transformation. The truth is that we started from one of an infinite number of equivalent Lagrangians, and in doing so arrived at a specific definition of the operator p_r. Obviously, from all that has been said, since no function of p_r is allowed to appear in physical Greens' functions, we are free to redefine p_r it by adding to it any c-number function of t. Obviously, this will neither change the physics of the gauge invariant states, nor change commutation relations of p_r with any of the other dynamical variables. Of course, such a change is simply equivalent to using one of the gauge-transformed versions of the Lagrangian (2.18) and not performing the integration by parts. If one takes such a gauge transformed Lagrangian and blindly constructs H following our rules, we obtain the result

$$H' = \frac{p_\theta^2}{2mr^2} - (\lambda - \dot{f}(t))(r - a) \tag{2.22}$$

which agrees with the one originally obtained on the sector of physical states, but differs with it outside of this sector. Thus, while we do no formal gauge-fixing, in that we do not set the variable $\lambda = 0$ (we shall see that this variable is the direct analogue of A_0) we nevertheless have fixed the gauge, in the sense that we have fixed a specific dynamics for the gauge variant sector of the theory. Although, at first glance, this is a somewhat puzzling result, it simply points out that when one quantizes QED by assuming $A_0 = 0$ one is engaging in overkill. This is a point to which we will return in the next chapter when we discuss four-dimensional Abelian gauge theories.

2.3. The Rigid Rotor and BRST Invariance

We now have two completely tractable methods for dealing with the simple problem of a particle constrained to move in a circle. In either case we can explicitly solve the system and determine everything we want to know about the sectors of physical and unphysical states. Despite this fact we will now introduce yet one more way of dealing with this system; namely, we will rewrite the gauge theory as a quantum system which possesses a generalized gauge invariance which is referred to in the literature as a BRST, or Becchi-Rouet-Stora, symmetry. At the moment, given only what we have already said, it is not possible to provide a compelling reason for this generalization. This is because the problem at hand is too simple, and having formed the Hamiltonian according to our formal rules we end up with a complete specification of all Green's functions, even those involving gauge non-invariant operators. When, in the next chapter, we discuss the quantization of QED we will see that this is not generally the case. We will show that the equation of motion for $\vec{\nabla} \cdot \vec{A}$ is undetermined up to an arbitrary time independent function. While this lack of specificity does not affect the computation of physically interesting gauge invariant quantities it does make the computation of Green's functions for the $\vec{A}$'s ill defined. Thus, it becomes impossible to derive the familiar perturbation expansion in terms of the fields $\vec{A}$, a situation which one can live with in QED, but which would make dealing with non-Abelian gauge theories unfeasible. In the usual treatments of QED one avoids this problem by fixing the gauge, i.e. adding a gauge non-invariant term to the Lagrangian. While this allows a straightforward development of the perturbation expansion it destroys manifest gauge-invariance, thus complicating other aspects of the problem. As will become clear in this discussion and in the discussion of QED to follow, the strength of the BRST formalism is that it allows one to achieve the effects of gauge fixing without ever breaking the BRST invariance of the system. Hence, one is able to completely set up perturbation theory and at the same time one does not have to sacrifice any of the simplifications that working with an exact symmetry provides.

The trick of the BRST technique is to enlarge the Hilbert space of the gauge theory and to replace the notion of a gauge transformation which shifts operators by c-number functions, by a BRST transformation which mixes operators having different statistics. To be precise let us consider the gauge theory equivalent to the rigid rotor and replace the gauge transformation (2.17) by introducing new anti-commuting variables c and $\bar{c}$ and a commuting variable b such that

$$\begin{aligned} &\delta\lambda = -\dot{c} \qquad \delta p_r = c \\ &\delta p_\lambda = \delta\theta = \delta p_\theta = \delta r = 0 \quad . \\ &\delta c = 0 \qquad \delta\bar{c} = b \qquad \delta b = 0 \end{aligned} \tag{2.23}$$

Unlike the usual gauge transformation the BRST transformation possesses the unusual property that $\delta^2 = 0$. If we now define a BRST invariant function of the dynamical variables to be a function $f(p_r, p_\theta, p_\lambda, p_b, r, \theta, \lambda, b, c, \bar{c})$ such that $\delta f = 0$, then this set of functions breaks up naturally into two classes, those which are δ of something else (which we will refer to as trivially invariant) and those which are not. The fact that the form of the BRST transformation is exactly the same as a gauge transformation, except that the role c-number function has been replaced by the anti-commuting variable c, tells us that the first order Lagrangian (2.18) , which was constructed to be gauge invariant, is guaranteed to be BRST invariant; moreover, in general, it will not correspond to a trivially invariant function.

Gauge Fixing in the BRST Formalism

The general trick which allows us to perform *gauge fixing* is to add to the Lagrangian (2.18) a trivial BRST invariant function, i.e. something which is δ of something else but which would not be invariant with respect to an ordinary gauge transformation. For example consider

$$\mathcal{L} = p_r \dot{r} + p_\theta \dot{\theta} - \frac{1}{2r^2} p_\theta^2 - \lambda(r-a) - \delta\left(\bar{c}(\dot{\lambda} - p_r + \frac{1}{2}b)\right), \qquad (2.24)$$

where the specific form of the arbitrary gauge fixing term was chosen in order to accentuate the parallel between this case and QED quantized in a *covariant gauge*. The parallel we have in mind was already alluded to, namely; one identifies λ with the time component of the gauge field A_0 and p_r, which is the only other quantity which transforms under our gauge transformation, with $\vec{\nabla} \cdot \vec{A}$ (loosely speaking). In this case, as will become clear in a moment, the extra BRST invariant gauge fixing term plays the role of adding a term of the form $(\partial_0 A_0 - \nabla \cdot \vec{A})^2$ to the QED Lagrangian.

To see how this leads to *gauge fixed* equations of motion use the definition of δ to rewrite $\mathcal{L}$ as

$$\mathcal{L} = p_r \dot{r} + p_\theta \dot{\theta} - \frac{1}{2r^2} p_\theta^2 - \lambda(r-a) - b(\dot{\lambda} - p_r) - \frac{1}{2}b^2 + \dot{\bar{c}}\dot{c} - \bar{c}c, \qquad (2.25)$$

where we have done an integration by parts to arrive at the specific form of the terms involving time derivatives of the independent anti-commuting variables c

and $\bar{c}$. If we were to proceed classically at this point we would notice that the Euler-Lagrange equation for b is simply

$$b = -(\dot{\lambda} - p_r) \tag{2.26}$$

and substituting this result into (2.25) would look like the standard two dimensional QED gauge fixed Lagrangian, at least if we make the suggested substitutions of A_0 for λ and $\vec{\nabla} \cdot \vec{A}$ for p_r . If we look at the Heisenberg equations of motion for the operators of interest (since they coincide with the Euler-Lagrange equations) we see that the additional trivially BRST invariant term accomplishes the sort of covariant gauge fixing which we set out to achieve.

There is one point which should be noticed with regard to equation (2.26) having to do with the requirement that $\delta b = 0$; namely, that once one has invoked the equations of motion one must in principle check for consistency. In this case, using the definition (2.23) we see that $\delta b = (-\ddot{c} - c)$ which is zero by virtue of the Euler-Lagrange equation for c. Of course this is no accident, it simply reflects the fact that the term added to the Lagrangian is δ of something.

While we could first use the equation of motion for b and then s case we see that varying with respect to $\dot{\lambda}$ tells us that we must identify the operator b with the operator $-p_\lambda$, which works since we assumed that p_λ was gauge invariant. Moreover, varying with respect to $\dot{\bar{c}}$ and $\dot{c}$ tells us that the variable canonically conjugate to $\bar{c}$ is $\dot{c}$ and the variable conjugate to c is $\dot{\bar{c}}$. This result is somewhat surprising if one is used to dealing with the naive quantization of the Dirac equation wherein we find that $\Psi^\dagger$ is the variable conjugate to Ψ and it in fact this change has important implications which we will consider in a moment.

Following our, by now, standard procedure for naively forming the Hamiltonian from the Lagrangian we obtain

$$\begin{aligned} H =& p_r \dot{r} + p_\theta \dot{\theta} + p_\lambda \dot{\lambda} + \Pi_c \dot{c} + \dot{\bar{c}} \Pi_{\bar{c}} - \mathcal{L} \\ =& \frac{1}{2r^2} p_\theta^2 + \lambda(r-a) + p_r p_\lambda + \frac{1}{2} p_\lambda^2 + \dot{\bar{c}}\dot{c} + \bar{c}c \end{aligned} \tag{2.27}$$

where we have used $b = -p_\lambda$, $\Pi_c = \dot{\bar{c}}$ and $\Pi_{\bar{c}} = \dot{c}$. All of the operators except c,$\bar{c}$, $\dot{c}$ and $\dot{\bar{c}}$ are assumed to have canonical commutation relations while, following the usual rules for quantizing operators obeying fermi statistics we give these operators anti-commutation relations. The only point which we have to be careful about in writing down these anti-commutation relations is that in this case things work differently than in the case of the Dirac equation where the field $\Psi^\dagger$ is the conjugate variable to Ψ. In the case of the Dirac equation the fundamental anti-commutators are the operators $\{\Psi^\dagger_\alpha, \Psi_\beta\} = \chi\delta_{\alpha\beta}$. Since these are a set of manifestly Hermitian operators we see that the coefficient χ must be real and can be normalized to unity. However, in the case of the c's and $\bar{c}$'s we need to specify the anti-commutation relations of $\dot{c}$ with $\bar{c}$ or $\dot{\bar{c}}$ with c, and not c with $\bar{c}$. In general since c and $\bar{c}$ are independent canonical variables we must assume that

$$\{\Pi_c, \Pi_{\bar{c}}\} = \{\bar{c}, c\} = 0, \tag{2.28}$$

from which it follows that

$$\frac{d}{dt}\{\bar{c}, c\} = 0; \tag{2.29}$$

or in other words that

$$\{\dot{\bar{c}}, c\} = -\{\dot{c}, \bar{c}\}. \tag{2.30}$$

Hence, we see that there is only one non-trivial anti-commutator which we have to fix, the second anti-commutation relation being fixed by the requirement of self-consistency. To fix this coefficient uniquely it is sufficient to require that the Heisenberg equation of motion

$$[H, c] = -i\dot{c} \tag{2.31}$$

or equivalently the Euler-Lagrange equation

$$-\ddot{c} - c = 0. \tag{2.32}$$

Using $H = \dot{\bar{c}}\dot{c} + \bar{c}c$ and the fact that $c^2 = \bar{c}^2 = 0$ we obtain

$$[H, c] = -\{\dot{\bar{c}}, c\}\,\dot{c} \tag{2.33}$$

which implies that $\{\dot{\bar{c}}, c\} = i$ and so $\{\dot{c}, \bar{c}\} = -i$. Note, this reversal of sign is non-trivial and implies the existence of negative metric states in the Hilbert space of this theory; however, before we discuss this point in detail let us complete our discussion of the BRST invariance of the Hamiltonian. (As an aside for the knowledgeable we should point out that it is more customary in the literature on BRST to treat the operator c as a hermitian operator and $\bar{c}$ as an independent anti-hermitian operator. That makes the notion of generalizing the symmetry more intuitive at the expense of making the Hamiltonian formalism just a little more obscure. We have chosen to follow a different tack and preserve the parallel to the Dirac equation. To translate think of the conventional operator c as our $c + \bar{c}$, and the usual $\bar{c}$ as our $c - \bar{c}$.)

The BRST Charge Operator

At this point we have enlarged our Hilbert space and replaced the notion of gauge invariance by that of BRST invariance. This allowed us to add a term to the Lagrangian (2.25) which is not invariant with respect to ordinary gauge transformations but which is trivially BRST invariant, thus accomplishing the same goal achieved by ordinary gauge fixing without losing the symmetry which needed to make the renormalization of the theory proceed apace. While the preceding discussion has allowed us to imbed the rotor problem into this bigger Hilbert space, we have not yet discussed what replaces the condition that one projects onto gauge invariant states in order to eliminate all spurious degrees of freedom. Obviously, since we have destroyed the original gauge invariance by our *gauge fixing* prescription, we no longer know that such a projection commutes with the Hamiltonian, therefore we have no way of knowing that the restriction to states which satisfy $(r - a) = 0$ and $p_\lambda = 0$ is a time independent procedure. To establish this fact we have to turn to the study of an operator which does commute with the Hamiltonian.

The obvious candidate for such an operator is the generator of the BRST transformation (2.23), Q. Since the BRST transformation Q is nilpotent (i.e. $Q^2 = 0$) and mixes operators which satisfy Bose and Fermi statistics we have to give a definition of what we mean by the generator of such a transformation. The conventional definition of this generator is that Q is an operator whose commutators with Bose operators and anti-commutators with Fermi operators satisfy

$$\begin{aligned}[Q, p_r] &= c \\ [Q, \lambda] &= -\dot{c} \\ \{Q, \bar{c}\} &= -p_\lambda\end{aligned} \tag{2.34}$$

where all other commutators and anti-commutators vanish. It is straightforward to verify the fact that

$$Q = -ic(r-a) - i\dot{c}p_\lambda \tag{2.35}$$

satisfies this requirement. It is also straightforward to explicitly check that the commutator of Q and H vanishes. Equation (2.35) is tantalizing in that it shows that the set of states satisfying the conditions $(r-a) = 0$ and $p_\lambda = 0$ belong to the dynamically stable subspace of states $|\Psi\rangle$ satisfying $Q|\Psi\rangle = 0$, i.e. the set of BRST invariant states. This fact, and the parallel between the gauge and BRST invariance of the theory, leads one to suspect that the condition for recovering the physical states of the theory should be accomplished by projecting onto BRST invariant states. This is almost true. In order to understand what the true condition is and why we must now spend a few moments rewriting the operators c and $\bar{c}$ in terms of annihilation and creation operators.

By taking the second commutator of H with the operator c we find that c satisfies the Heisenberg equation of motion

$$-\partial_0^2 c = c \tag{2.36}$$

which means that the Heisenberg operator $c(t)$ can be written as

$$\begin{aligned} c(t) =& e^{it}B + e^{-it}D \\ \bar{c}(t) =& e^{-it}B^{\dagger} + e^{it}D^{\dagger} \end{aligned} \tag{2.37}$$

so that at time $t = 0$ we have

$$\begin{aligned} c =& B + D \\ \bar{c} =& B^{\dagger} + D^{\dagger} \\ \dot{c} =& i(B - D) \\ \dot{\bar{c}} =& -i(B^{\dagger} - D^{\dagger}) \end{aligned} \tag{2.38}$$

If we now impose the conditions $c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\dot{\bar{c}}, \dot{c}\} = 0$ and $\{\dot{\bar{c}}, c\} = i$ we obtain the equations

$$\begin{aligned} & B^2 + \{B, D\} + D^2 = 0 \\ & B^{\dagger 2} + \{B^{\dagger}, D^{\dagger}\} + D^{\dagger 2} = 0 \\ & \{B, B^{\dagger}\} + \{D, D^{\dagger}\} + \{B, D^{\dagger}\} + \{B^{\dagger}, D\} = 0 \\ & \{B, B^{\dagger}\} + \{D, D^{\dagger}\} - \{B, D^{\dagger}\} - \{B^{\dagger}, D\} = 0 \\ & \{B, B^{\dagger}\} - \{D, D^{\dagger}\} - \{B, D^{\dagger}\} + \{D, B^{\dagger}\} = -1 \end{aligned} \tag{2.39}$$

which have the solution

$$\begin{aligned} & B^2 = D^2 = B^{\dagger 2} = D^{\dagger 2} = 0 \\ & \{B, D\} = \{B^{\dagger}, D\} = \{B, D^{\dagger}\} = \{B^{\dagger}, D^{\dagger}\} = 0 \\ & \{B^{\dagger}, B\} = -\frac{1}{2} \\ & \{D^{\dagger}, D\} = \frac{1}{2} \end{aligned} \tag{2.40}$$

The important thing to notice about (2.40) is the reversal in sign between the anti-commutation relations for the operators B , $B^{\dagger}$ and D,$D^{\dagger}$. If we let $|0\rangle$ denote the state for which

$$B\,|0\rangle = D\,|0\rangle = 0 \tag{2.41}$$

have norm one, then we see that the norm of the state $B^\dagger\,|0\rangle$ is minus one-half, whereas the norm of the state $D^\dagger\,|0\rangle$ is plus one-half. Hence, we see that in quantizing the fermionic sector of the theory we necessarily encounter negative norm states. This is nothing more or less than the reflection of the negative norm states which appear in the usual quantization of QED when one adds a gauge fixing term of the form $(\partial_\mu A^\mu)^2$. The fact that these states exist is unimportant since we see that the entire system is free, so we can restrict attention to the positive metric sector of the Hilbert space. Actually, it is convenient to redefine the norm of $|0\rangle$ to be minus one, since then, the states built off $D^\dagger\,|0\rangle$ will have positive norm. This is perhaps more natural since

$$H = \frac{1}{2r^2}p_\theta^2 + \lambda(r-a) + p_r p_\lambda + p_\lambda^2 + 2(B^\dagger B + D^\dagger D) \tag{2.42}$$

and so the lowest energy state of the theory belongs to the sector built of the state $D^\dagger\,|0\rangle$ tensored with arbitrary wavefunctions involving bosonic operators.

Having made these observations we are now in a position to examine the structure of the set of states $|\Psi\rangle$ which are annihilated by the BRST generator Q. Rewriting the expression for Q in terms of the fermionic annihilation and creation operators we obtain

$$Q = -i\,[B(r-a+ip_\lambda) + D(r-a-ip_\lambda)] \tag{2.43}$$

hence, we see that although the set of states annihilated by Q contains the set of states for which $(r-a)=0$ and $p_\lambda = 0$ it contains additional states; namely, all those states for which $B\,|\Psi\rangle = D\,|\Psi\rangle = 0$. This is of course all states built by tensoring the state $|0\rangle$ with arbitrary functions of the variables $\{r, p_r, \lambda, p_\lambda, \theta, p_\theta\}$. Hence, restricting attention to the space of gauge invariant states does not recover only the states of the rigid rotor. The reason this occurs is that we are dealing with an Abelian gauge theory and in fact the Hamiltonian is not only invariant with respect to BRST transformations, but also with respect to what are referred to as anti-BRST transformations; i.e., a transformation in which the role of c and $-\bar{c}$ are interchanged. To be explicit, the operator which generates the anti-BRST transformation is

$$\bar{Q} = i\bar{c}(r-a) + i\dot{\bar{c}}p_\lambda \tag{2.44}$$

which generates the transformation

$$\begin{aligned} \bar{\delta}p_r &= -\,\bar{c} \\ \bar{\delta}\lambda &= +\,\dot{\bar{c}} \\ \bar{\delta}c &= -\,p_\lambda \end{aligned} \tag{2.45}$$

with all other commutators and anti-commutators vanishing. It is a straightforward exercise to verify by explicit computation that $[Q, H] = 0$, and so we are free to impose the dual constraints that Q and $\bar{Q}$ annihilate physical states. Since, $\bar{Q}$ is nothing but the adjoint of Q we see that the second condition amounts to stating that

$$\bar{Q} = i\left\{B^\dagger(r-a-ip_\lambda) + D^\dagger(r-a+ip_\lambda)\right\} \tag{2.46}$$

As before the states for which $(r-a) = p_\lambda = 0$ satisfy this condition, however there are no longer any other states which satisfy the condition because there are no free eigenstates of the fermionic part of the Hamiltonian which are annihilated by B, $B^\dagger$, D and $D^\dagger$ simultaneously. Hence, for the Abelian theory (because the fermions satisfy free equations of motion) we need an extra anti-BRST symmetry in order to be able to impose enough constraints to recover only the physical states of the particle constrained to move on a circle. Fortunately the existence of an anti-BRST symmetry is guaranteed for this class of theory. This happens because the BRST transformation is a symmetry which relates a set of real variables to a pair of complex variables c and $\bar{c}$, and thus we expect the complex conjugate transformation to be a symmetry too.

One final remark that we should make has to do with the significance of the statement that physical states satisfy the requirement $p_\lambda = 0$. If we had proceeded by first eliminating the b field using its equation of motion we would have identified $\dot{\lambda} - p_r$ as the conjugate momentum to the variable λ and therefore, pursuing the analogy with QED, we would interpret the state condition as the parallel statement to the QED statement that we fix the gauge $\partial^\mu A_\mu = 0$. Once again we see that although this condition is not gauge-invariant it is BRST invariant by virtue of the equations of motion for the operator c.

2.4. Summary

This completes our discussion of the rigid rotor or free particle constrained to move on a circle. We have seen three ways in which this theory can be quantized by imbedding it inside a larger Hilbert space and then imposing state conditions. In the first case we accomplished a minimal quantization and the state condition amounted to what is called by Dirac imposing a first class constraint. In the second method we imbedded the entire theory in a gauge theory and showed that restriction to the space of gauge invariant states recovered the theory of the rotor and only the theory of the rotor; however, in this method we saw that we could not add a gauge fixing term without destroying the symmetry which guaranteed that we could in a time independent way restrict attention to these, and only these, states. The final method was one where we imbedded the theory into a BRST invariant system. In this case we saw that the Hamiltonian, even with a gauge fixing term added, commuted with both a BRST and anti-BRST charge and that projecting onto the sector of BRST and anti-BRST invariant states recovered a theory which was isomorphic to the first theory. We made this discussion so detailed because it was the simplest example which exhibited all of the features necessary to a discussion of the more complicated cases of QED and Yang-Mills theories. In the sections to follow we will be able to quickly discuss each of these theories and show that the quantization of each of these systems is nothing but a notationally more complex line by line transcription of the discussion given for the rigid rotor. The point we wish to make throughout is that using BRST-invariance to quantize a theory is not something special to a non-Abelian gauge theory, rather it is simply a standard trick which can always be used to simplify the quantization of a constrained system.

3. QED: THE PROTOTYPICAL GAUGE THEORY

The purpose of this chapter is to show how the theory of quantum electrodynamics (QED) provides a more complicated parallel to the case of the rigid rotor. We will show that the correct treatment of this problem is essentially a line by line transcription of the rotor problem, except for the complexity introduced by virtue of the fact that now there are an infinite number of degrees of freedom. The only really new point to be made in this section has to do with the question of why we have to perform gauge fixing in order to set up perturbation theory.

3.1. Basics

In the case of QED we start out with a theory which is already gauge invariant, hence we do not have to figure out how to impose a constraint on the quantized theory but only have to discover what constraint follows from requiring that the physical states be invariant with respect to time independent gauge transformations. Having done this we will then show how the same theory can be quantized as a theory possessing an invariance under both BRST and anti-BRST transformations and show that by restricting attention to states which are simultaneously BRST and anti-BRST invariant we recover exactly the content of the original gauge theory, even if one works with a *gauge fixed* form of the Hamiltonian.

In what follows we will restrict attention to the case of pure QED since the problem of gauge invariant quantization of the theory already occurs at this level.

The Lagrangian of pure QED is conventionally written as

$$\begin{aligned}\mathcal{L}_{free} =& \frac{1}{2}\left(\partial_0\vec{A} - \vec{\nabla}A_0\right)^2 - \frac{1}{2}\left(\vec{\nabla}\times\vec{A}\right)^2 \\ =& \frac{1}{2}\left(\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B}\right)\end{aligned} \tag{3.1}$$

where we have explicitly separated the mixed time-space and space-space terms in order to simplify the construction of the Hamiltonian by our formal Legendre transformation. Following the naive procedure which worked for the rotor we identify the field A_0 as the analogue of the Lagrange multiplier λ since its time derivative does not appear in the Lagrangian. Also, in the usual way we identify the quantities $\delta\mathcal{L}_{free}/\partial_0\vec{A} = \vec{E}$ as the canonical variables conjugate to the fields A_i and assume for them the canonical equal time commutation relations

$$[A_i(\vec{x}), E_j(\vec{y})] = i\delta_{ij}\delta^3(\vec{x}-\vec{y}) \tag{3.2}$$

where

$$\vec{E} = \partial_0\vec{A} - \vec{\nabla}A_0 \tag{3.3}$$

and where we will adopt the usual definition

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{3.4}$$

in order to simplify writing the expressions which follow.

Applying the formula

$$\mathcal{H}_{free} = \vec{E}\cdot\partial_0\vec{A} - \mathcal{L} \tag{3.5}$$

we obtain

$$H_{free} = \int d^3x \left[\frac{1}{2}\left(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right) + \vec{\nabla}A_0\cdot\vec{E}\right] \tag{3.6}$$

which can be rewritten by integrating by parts as

$$H_{free} = \int d^3x \left[\frac{1}{2}\left(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right) - A_0\vec{\nabla}\cdot\vec{E}\right] \tag{3.7}$$

Note, that as in the case of the rotor, although the Lagrangian is invariant with respect to arbitrary time and space-dependent gauge transformations of the form

$$\begin{aligned} \delta A_0(\vec{x},t) =& \partial_0 f(\vec{x},t) \\ \delta \vec{A}(\vec{x},t) =& \vec{\nabla} f(\vec{x},t) \end{aligned} \tag{3.8}$$

the Hamiltonian (3.7) is invariant under only time-independent gauge transformations. Under a time dependent gauge transformation

$$\delta H_{free} = \int d^3x\, \partial_0 f(\vec{x},t) \vec{\nabla} \cdot \vec{E} \tag{3.9}$$

where $f(\vec{x},t)$ is an arbitrary function of $\vec{x}$ and t. Thus, in order to obtain a fully gauge invariant Hamiltonian we must project onto the set of states $|\Psi\rangle$ for which

$$\vec{\nabla} \cdot \vec{E}(\vec{x})\, |\Psi\rangle = 0 \tag{3.10}$$

From the canonical commutation relations assumed for $\vec{E}(\vec{x})$ and $\vec{A}(\vec{y})$, it follows that the operator $\vec{\nabla} \cdot \vec{E}(\vec{x})$ can be identified as the generator of time independent gauge transformations; hence, the condition that one project onto a subspace in which the Hamiltonian is invariant with respect to all gauge transformations, amounts to the condition that we project onto states which are annihilated by the generators of time-independent gauge transformations. Since these transformations commute with H this projection is time independent.

We have now completed the quantization of the free Lagrangian. In order to achieve a better understanding of the meaning of the projection onto gauge invariant states it is necessary to consider the introduction of matter fields. To do this we add a to $\mathcal{L}_{free}$ the free matter field Lagrangian and an interaction term of the form $\sum_\mu J^\mu A_\mu$, where J^μ is the conserved matter current . The Lagrangian of interacting QED then takes the form

$$\mathcal{L} = \mathcal{L}_{free} + J_0 A_0 - \vec{J} \cdot \vec{A} \tag{3.11}$$

and the Hamiltonian becomes

$$H = H_{matter} + \int d^3x \left[\frac{1}{2} \left(\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B} \right) - A_0 \left(\vec{\nabla} \cdot \vec{E} - J_0) \right) + \vec{A} \cdot \vec{J} \right] \quad (3.12)$$

As in the pure gauge theory the coefficient of the term A_0 is simply the generator of the time-independent gauge transformations, where now we include in the definition of the gauge generator a piece involving the matter fields. This is because under a gauge-transformation the matter fields transform by a phase factor of the form $e^{iqf(\vec{x},t)}$, where q is the charge of the field and $f(\vec{x}, t)$ the arbitrary gauge function. From this we see that projecting onto gauge invariant states amounts to saying that one is projecting states for which Coulomb's law holds.

From this discussion and our earlier discussion of the rotor we see that if we had not started with a gauge theory we could have reinvented one by starting with a theory involving only the canonical variables $\vec{E}$ and $\vec{A}$ and imposing Coulomb's Law as a constraint . In that case the field A_0 appears as the Lagrange multiplier of the constraint equations and the first order Lagrangian constructed following the procedure we outlined for the rotor would just be Lagrangian we began with. Fortunately, the gauge theory was given to us at the outset so we were able to skip this step.

3.2. The Need For Gauge Fixing

In the discussion of the rotor we pointed out that the virtue of BRST quantization is that it allows for gauge fixing while keeping a symmetry of the theory manifest which is equivalent to gauge invariance. We also noted that we could not really explain the need for such gauge fixing within this simplest model. In this section we will take a few minutes to rectify this situation and remind the reader of the way in which Gupta-Bleuler formalism gets around the problem. Having done this we will then discuss the BRST quantization and the reader will then be in position to compare the two approaches.

As we already pointed out at this point we have completely quantized QED and eliminated all spurious degrees of freedom. In principle we can now compute all processes involving only gauge invariant physical variables. However, there is often a big difference between *in principle* and *in practice* . In order to use the Hamiltonian projected onto gauge invariant states one has to deal with Green's functions which are not manifestly Lorentz covariant and the difficulties which this introduces are well known . One usually attempts to recast the problem in such a way that the perturbation expansion can be set up in terms of the unphysical (i.e. gauge dependent) variables $A_0, \vec{A}$. Given these Green's functions one then extracts amplitudes for gauge-invariant Green's functions by manipulating them. Unfortunately, if we do this at this juncture we find that although we have completely specified all propagators for the fields $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$ (one need only check the free field case to see this) the gauge invariance of the procedure guarantees that it will not fix the propagators for the fields $A_0, \vec{A}$. To see this all one has to do is compute the Heisenberg equations of motion for the free Hamiltonian plus a c-number external conserved current coupled to the A's by a term $J_0 A_0 - \vec{J} \cdot \vec{A}$. This yields

$$\begin{aligned} \partial_0 \vec{A} &= \left(\vec{E} + \vec{\nabla} A_0\right) \\ \partial_0 \vec{E} &= \left(-\nabla^2 \vec{A} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{J}\right) \\ \partial_0 A_0 &= 0 \end{aligned} \tag{3.13}$$

from which it follows directly that

$$\partial_0^2 \left(\vec{\nabla} \cdot \vec{A}\right) = \vec{\nabla} \cdot \vec{J} \tag{3.14}$$

Clearly, (3.14) means that for a given set of classical external sources the divergence of $\vec{A}$ is undetermined up to a function $f(\vec{x}, t)$ for which $\partial_0^2 f = 0$. By the usual arguments this translates into the fact that the $\vec{A}\vec{A}$ propagator is undetermined and so we cannot set up the usual perturbation expansion without adding something to the Lagrangian in order to fix the gauge. Of course, if we do so then we lose the fact that the generators of time independent gauge transformations commute with the Hamiltonian and therefore lose the ability to project onto gauge invariant states. In that case we lose the manifest equivalence of the formalism to the unitary form of the theory and proofs of renormalizability become more complicated.

Despite the fact that fixing the gauge complicates ones life that is precisely what one used to doing in the case of QED. It is customary to discuss the theory by adding a term like $(\partial_0 A_0 - \vec{\nabla} \cdot \vec{A})^2/2\alpha$ to the Lagrangian. In this way the conjugate variable to A_0 now appears and the canonical formalism goes through in the usual fashion, except that one discovers that due to the form of the Lorentz metric at least one the component of A_μ must correspond to a negative metric particle. In order to eliminate these states from physical processes one begins by observing that the negative metric field belongs to $\partial_0 A_o - \vec{\nabla} \cdot \vec{A}$ and using the equations of motion shows that this operator is a free field. This means one can separate this operator into annihilation and creation operators and impose the condition that physical states contain no negative norm particles. Unfortunately, while this eliminates states with negative norm there are states with zero norm which cannot be eliminated so simply. The heart of the Gupta-Bleuler formalism is the observation that two states which differ by a state of zero norm can be mapped into one another by a gauge transformation; hence, one can define the physical theory by factoring the Hilbert space of states into equivalence classes, defining all states that differ by a gauge transformation as the same state. One shows that the factor space has a positive definite metric and then one has to verify unitarity by proving that all on shell Green's functions for gauge invariant operators are independent of the coefficient of the gauge fixing term. While this somewhat cumbersome approach works for QED the analagous approach for a non-Abelian gauge theory quickly runs into frightful complications. For this reason one is very interested in developing a quantization scheme which allows effective gauge fixing, but will preserve a symmetry that vastly simplifies the discussion of renormalizability.

3.3. BRST Quantization and QED

As in the case of the rigid rotor we BRST quantize the system by enlarging the number of degrees of freedom and inventing a nil-potent symmetry which mimics the structure of the gauge symmetry. The general trick for doing this is to begin by replacing the gauge function $f(\vec{x})$ by the anti-commuting field $c(\vec{x})$, a process which partially defines the operation δ to be

$$\delta A_0 = \partial_0 c \qquad \delta \vec{A} = \vec{\nabla} c \tag{3.15}$$

from which it follows that $\delta\vec{E} = \delta\vec{B} = 0$. The transformations of all of the variables other than c and $\bar{c}$ are fixed to vanish by the condition that they are gauge invariant, thus we need only define δc and $\delta\bar{c}$ in order to fix everything. The fact that $\delta^2 = 0$ immediately tells us that $\delta c = 0$ and so the most straightforward completion of our definitions is achieved by defining $\delta\bar{c} = b$ and $\delta b = 0$. Except for the fact that all variables are now functions of space-time points this is an exact transcription of the process we followed for the case of the rigid rotor.

Following our earlier procedure we write the full Lagrangian of the expanded system as the original gauge invariant Lagrangian plus a term which is δ of an arbitrary function of the other variables. In order to accomplish the purpose of producing an effective gauge fixing term of the form $(\partial_0 A_0 - \vec{\nabla}\cdot\vec{A})^2$ we choose this term so that

$$\mathcal{L}_{BRST} = \mathcal{L} + \delta\left(\bar{c}(\partial_0 A_0 - \vec{\nabla}\cdot\vec{A} + \frac{b}{2}\right) \tag{3.16}$$

which is an obviously BRST invariant quantity. Substituting the definition of δ and using the fact that by assumption $c^2 = \bar{c}^2 = 0$ we obtain

$$\mathcal{L}_{BRST} = \mathcal{L} + \frac{1}{2}b^2 + b\left(\partial_0 A_0 - \vec{\nabla}\cdot\vec{A}\right) + \dot{\bar{c}}\dot{c} - \vec{\nabla}\bar{c}\cdot\vec{\nabla}c. \tag{3.17}$$

It follows from (3.17) , as in the case of the rotor, that if we use the equation of motion for b we wind up with exactly the gauge fixed Lagrangian used in the

Gupta-Bleuler formalism. Moreover, we see that the fields c and $\bar{c}$ are free fields whose canonical quantization can be carried out straightforwardly. Once again there are two routes open to us. We could eliminate b using its equation of motion and then canonically quantize the system in the usual way. In this event we would identify b with $\partial^\mu A_\mu$ and we would have to check that $\delta b = 0$ For the case at hand $\delta b = (\partial_0^2 c - \nabla^2 c) = 0$ which, as for the case of the rotor, follows from the Euler-Lagrange equations for the field c. The second approach we could adopt is to not use the equation of motion for b and proceed to canonically quantize the system directly. In that case we have to identify the canonical momentum of the A_0 field with b and then the discussion goes through as for the case of the rotor. Once again, in either approach, the fact that $\dot{\bar{c}}$ is the conjugate variable to c etc. implies that the operators c and $\bar{c}$ generate an indefinite metric space; however, since the field is free, this causes no problems. Simple state conditions suffice to eliminate all negative metric states from consideration. This is of course the BRST reflection of exactly the negative metric states of the Gupta-Bleuler formalism, except that here things are much simpler.

The general outline of the remainder of the discussion of the Abelian theory parallels the rotor case almost exactly. We begin by forming the Hamiltonian of the system using the definition

$$\mathcal{H}_{BRST} = \vec{E} \cdot \partial_0 \vec{A} + \Pi_{A_0} \partial_0 A_0 + 2\dot{\bar{c}}\dot{c} - \mathcal{L}_{BRST} \tag{3.18}$$

where now, since $\mathcal{L}_{BRST}$ includes a term involving $\partial_0 A_0$ we must identify $-b$ with the canonical conjugate to A_0 in exactly the same way as we identified it with p_λ in the case of the rigid rotor. The Hamiltonian obtained in this way is simply

$$\mathcal{H}_{BRST} = \frac{1}{2}\left(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right) - A_0\left(\vec{\nabla}\cdot\vec{E}\right) + \Pi_{A_0}\vec{\nabla}\cdot\vec{A} + \frac{1}{2}\Pi_{A_0}^2 + \dot{\bar{c}}\dot{c} + \vec{\nabla}\bar{c}\vec{\nabla}c \tag{3.19}$$

and it is straightforward to show that it commutes with the BRST charge

$$Q = \int d^3x \left[-ic\vec{\nabla}\cdot\vec{E} - i\dot{c}\Pi_{A_0}\right] \tag{3.20}$$

by using the canonical commutation relations supplemented by

$$\{\dot{\bar{c}}(\vec{x}), c(\vec{y})\} = i\delta^3(\vec{x}-\vec{y}) \tag{3.21}$$

We also can show that this Hamiltonian commutes with the anti-BRST charge

$$\bar{Q} = \int d^3x \left[ic\vec{\nabla}\cdot\vec{E} + i\dot{\bar{c}}\Pi_{A_0}\right] \tag{3.22}$$

In exactly the same fashion as for the rigid rotor we expand the fields c and $\bar{c}$ in terms of annihilation and creation operators and can then show that the conditions $Q\,|\Psi\rangle = \bar{Q}\,|\Psi\rangle = 0$ can only be satisfied on states which satisfy the eigenvalue conditions $\vec{\nabla}\cdot\vec{E}\,|\Psi\rangle = \Pi_{A_0}\,|\Psi\rangle = 0$. Note, that had we followed the first route, the state condition $\Pi_{A_0} = 0$ would be the familiar statement that $\partial^\mu A_\mu = 0$ when restricted to the space of physical states.

4. THE CHARGED SCALAR FIELD AND GHOSTS WHICH AREN'T FREE

In the previous discussions we chose gauge-fixing conditions which lead to free equations of motion for the ghost fields c and $\bar{c}$. In the case of more complicated Yang-Mills theories this is not possible. The question then arises as to how we are to go about choosing the sector of physical states, since we saw that for the case of the rotor and QED there was more to be done than simply stating that the states were Q and $\bar{Q}$ invariant. It has long been known that for the case of scalar QED one encounters ghosts which do not satisfy free equations of motion if one chooses to work in the so called R_ξ gauges. Hence, before going on to the discussion of Yang-Mills theories it is worth spending a few moments discussing this phenomenon in that simpler setting.

4.1. The Lagrangian

The Lagrangian of this model is conventionally written as

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B}\right) + (D_0\phi)^\dagger(D_0\phi) - (\vec{D}\phi)^\dagger\cdot(\vec{D}\phi) - \mu^2\phi^\dagger\phi \tag{4.1}$$

where ϕ is a complex scalar field and

$$\begin{aligned} D_\mu\phi &= (\partial_\mu - igA_\mu)\phi \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{4.2}$$

This Lagrangian is invariant under gauge transformations

$$\begin{aligned} \phi(\vec{x},t) &\to \phi'(\vec{x},t) = e^{-igf(\vec{x},t)}\phi(\vec{x},t) \\ A_0(\vec{x},t) &\to A_0'(\vec{x},t) = A_0(\vec{x},t) - \partial_0 f(\vec{x},t) \\ \vec{A}(\vec{x},t) &\to \vec{A}'(\vec{x},t) = A_0(\vec{x},t) - \vec{\nabla}f(\vec{x},t) \end{aligned} \tag{4.3}$$

If we now proceed to construct the Hamiltonian for this system in what is by now our standard way we define

$$\begin{aligned} \Pi_\phi &= (\partial_0 + igA_0)\,\phi^\dagger \\ \Pi_{\phi^\dagger} &= (\partial_0 - igA_0)\,\phi \\ \Pi_{\vec{A}} &= \left(\partial_0\vec{A} - \vec{\nabla}A_0\right) \\ \mathcal{H} &= \Pi_\phi\partial_0\phi + \Pi_{\phi^\dagger}\partial_0\phi^\dagger + \Pi_{\vec{A}}\partial_0\vec{A} - \mathcal{L} \end{aligned} \tag{4.4}$$

so

$$\mathcal{H} = \frac{1}{2}\left(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right) + \Pi_{\phi^\dagger}\Pi_\phi + (\vec{D}\phi)^\dagger(\vec{D}\phi) + \mu^2\left(\phi^\dagger\phi\right) + A_0\left(-\vec{\nabla}\cdot\vec{E} + g\rho\right) \tag{4.5}$$

where $\rho = j_0$ is the time component of the conserved electromagnetic current

$$j_0 = \Pi_{\phi^\dagger}\phi - \Pi_\phi\phi^\dagger \tag{4.6}$$

As before, we see that this Hamiltonian is invariant with respect to all time independent gauge transformations and the only term in $\mathcal{H}$ which changes under a time dependent gauge transformation is the term

$$A_0\left(-\vec{\nabla}\cdot\vec{E}+g\rho\right) \tag{4.7}$$

and the term inside the brackets is identifiable as the generator of the time independent gauge transformations. Thus, as we already pointed out, projecting the Hamiltonian onto the subspace in which it is fully gauge-invariant is equivalent to projecting onto the space of states for which Gauss' law holds as an eigenvalue condition.

At this point our original discussion of QED tells us that we have completely specified the quantization of the theory and obtained the correct Heisenberg equations of motion for physical variables. However, as before, we have not written things in such a way that we can completely derive a perturbation expansion for gauge non-invariant Green's functions. What we want to now look at is what happens if we study a particular form of gauge fixing which was useful for discussing the Abelian-Higgs model; i.e., the so-called R_ξ gauges.

In order to set up the perturbation expansion so as to facilitate our discussion of the R_ξ gauges it is customary to rewrite ϕ as

$$\phi=\frac{1}{\sqrt{2}}\left(\phi_1+i\phi_2\right) \tag{4.8}$$

and rewrite $\mathcal{H}$ as

$$\begin{aligned}\mathcal{H}=\frac{1}{2}\left(\vec{E}\cdot\vec{E}+\vec{B}\cdot\vec{B}\right)+\frac{1}{2}\left(\Pi_1^2+\Pi_2^2\right)+\frac{1}{2}\left(\vec{\nabla}\phi_1+g\vec{A}\phi_2\right)^2\\+\frac{1}{2}\left(\vec{\nabla}\phi_2-g\vec{A}\phi_1\right)^2+\frac{1}{2}\mu^2\left(\phi_1^2+\phi_2^2\right)+A_0\left(-\vec{\nabla}\cdot\vec{E}+g\rho\right)\end{aligned} \tag{4.9}$$

where Π_1 and Π_2 are the canonical momenta of the fields ϕ_1 and ϕ_2 respectively. In addition we have that

$$\rho=j_0=i\left(\Pi_1\phi_2-\Pi_2\phi_1\right) \tag{4.10}$$

The original motivation of the R_ξ gauges was to eliminate the cross term $\vec{A}\cdot\vec{\nabla}\phi_2$ which appears in the discussion of the Higgs model. Obviously, for the case at hand we do not face this problem and the motivation for choosing this class of gauge fixing terms does not exist. Nevertheless, since what we wish to discuss is the structure of the theory for gauge fixing terms which lead to non-trivial equations of motion for the ghost field, we will consider this class of gauges anyhow. The R_ξ gauges as introduced by Fujikawa, Lee and Sanda[6] are obtained by adding a term of the general form

$$-\frac{1}{2\xi}\left(\partial^\mu A_\mu + \xi\phi_2\right)^2 \tag{4.11}$$

to the original gauge-invariant Lagrangian. We will now show why this sort of term leads to non-trivial equations of motion for the anti-commuting fields of the BRST formalism and how that affects our discussion of the choice of physical subspace.

4.2. BRST Formalism

Following our earlier line of argument to obtain the BRST form of the theory we replace the usual gauge transformation by a nilpotent transformation on a larger Hilbert space. To arrive at the form of this new transformation we simply take the form of the c-number gauge transformation, which in these variables is

$$\delta A_\mu = -\partial_\mu f \quad \delta\phi_1 = f\phi_2 \quad \delta\phi_2 = -f\phi_1 \tag{4.12}$$

and replace the c-number gauge function f by an anti-commuting variable c. Hence, we assume that in addition to the operators $\vec{A}$, ϕ_1 ϕ_2 and their conjugate momenta $\vec{E}$, Π_1 and Π_2 we add new operators c, $\bar{c}$ and an as yet undetermined field b. The form of the BRST transformation is then fixed by requiring that

$$\delta A_0 = \partial_0 c; \quad \delta\vec{A} = \vec{\nabla}c; \quad \delta\phi_1 = gc\phi_2; \quad \delta\phi_2 = -gc\phi_1 \tag{4.13}$$

in order to reproduce the form of the c-number gauge transformation, and

$$\delta c = 0; \quad \delta\bar{c} = b; \quad \delta b = 0 \tag{4.14}$$

which is a solution of the condition that $\delta^2 = 0$.

With this notation behind us we then construct the BRST analogue of the R_ξ gauges by defining the BRST Lagrangian to be

$$\mathcal{L}_{BRST} = \mathcal{L} + \delta\left(\bar{c}(\partial_0 A_0 - \vec{\nabla}\cdot\vec{A} + \xi\phi_2 + \xi b/2)\right) \tag{4.15}$$

Using the definition of δ specified in (4.13) and (4.14), (4.15) becomes

$$\mathcal{L}_{BRST} = \mathcal{L} + \frac{1}{2}b^2 + b\left(\partial_0 A_0 - \vec{\nabla}\cdot\vec{A} + gv\xi\phi_2\right) + \dot{\bar{c}}\dot{c} - \vec{\nabla}\bar{c}\cdot\vec{\nabla}c - \xi\phi_1\bar{c}c \tag{4.16}$$

from which we see that for this class of gauges the ghost fields have non-vanishing trilinear couplings to the field ϕ_1, and so no longer satisfy free equations of motion. Nevertheless, although the Hamiltonian for the fields $\bar{c}$ and c is no longer trivial, the BRST charges take the same form as in the covariant gauges, i.e. following the prescriptions of the preceding section we find

$$Q = \int d^3x \left[ic\left(-\vec{\nabla}\cdot\vec{E} + g(\Pi_1\phi_2 - \Pi_2\phi_1)\right) - i\dot{c}\Pi_{A_0}\right] \tag{4.17}$$

and the anti-BRST is not necessary because one no longer has to worry about eigenstates $|\Psi>$ of H which satisfy $c|\Psi>= \dot{c}|\Psi>=0$.

Clearly, if we start out by quantizing the theory in terms of the free field part of $\mathcal{H}$ and then considering the effects of interactions on the classification of the Q invariant states proceeds in the same way as in the covariant gauges. The only difference is that now we see that since $\mathcal{H}$ mixes states with different numbers of ghosts we cannot simply choose one particular eigenstate of the free Hamiltonian upon which to build our space of physical states. Despite this apparently new complication the state of affairs can be shown to exactly parallel this state of affairs in the covariant theory, except that now we must copy the Gupta-Bleuler formalism and show that physical states are equivalence classes of states which are annihilated by Q. The analogue to Gupta-Bleuler lies in the fact that any two states within one equivalence class differ by a state of zero norm and if we consider the theory defined on the factor space it is clearly unitary.

For terms with an uneven number of ghosts in them, the operation δ is generated by anti-commuting with the charge Q. It therefore follows that the extra term in the BRST Lagrangian ,and thus Hamiltonian, is of the form

$$\left\{ Q, \bar{c}(\partial_0 A_0 - \vec{\nabla}\cdot\vec{A} - \xi\phi_2 + \frac{\xi}{2}b) \right\} \quad . \tag{4.18}$$

Let us now assume that we begin from any one of the eigenstates of the free part of the ghost Hamiltonian (e.g. the ground state with a filled Dirac sea) and consider the states built of this state by multiplying it by an arbitrary gauge-invariant function of the variables $\vec{E}$, $\vec{A}$, ϕ_1 and ϕ_2. As we already noted this is the generic form of a state which is annihilated by Q. It then follows that this state is mapped into another state in the same sector by all of the terms in $\mathcal{H}_{BRST}$ except the terms of the form (4.18). Since we began by assuming that we started from a state which is annihilated by Q we see that the result of applying (4.18) to such a state is a state of the form

$$\{Q,\mathcal{O}\}\,|\Psi\rangle = Q\mathcal{O}\,|\Psi\rangle \tag{4.19}$$

where $\mathcal{O}$ stands for the gauge-fixing expression. Furthermore, since all of $\mathcal{H}_{BRST}$ commutes with Q it follows trivially that all states obtained by applying any power of $\mathcal{H}_{BRST}$ to a state annihilated by Q are of the form $Q\,|\Psi'\rangle$, for some state $|\Psi'\rangle$.

The only thing which remains to be shown is that a state of this type has zero norm; however, this follows directly from the fact that the norm of a state of this type is equal to

$$\langle\Psi|\,\mathcal{O}^\dagger QQ\mathcal{O}\,|\Psi\rangle = \langle\Psi|\,\mathcal{O}^\dagger\,[Q,\{Q,\mathcal{O}\}]\,|\Psi\rangle = 0 \tag{4.20}$$

The extreme simplicity of this analysis as opposed to what one has to go through to analyze what happens in the Gupta-Bleuler formalism points up more than anything else the enormous simplifications which occur because of the fact that the BRST transformation is nilpotent.

From this line of argument it follows that if we start from any one of the subspaces of the theory defined by the usual covariant gauge fixing condition and form equivalence classes of states under the relation that two states are the same if they differ by Q of some state, then any one of this infinite number of factor spaces serves as a satisfactory subspace of physical states and generates a theory isomorphic to the physical theory obtained from directly quantizing the theory without ghosts in a Hilbert space defined by constraints. It should be apparent that the general flavor of this argument will carry over to any theory wherein the addition of a gauge-fixing term forces the BRST ghost particles to satisfy non-trivial equations of motion. This completes our discussion of Abelian gauge theories.

5. NON-ABELIAN GAUGE THEORIES

Finally, for the sake of completeness, let us turn to the case of the non-Abelian gauge theories. As we will see, the quantization of these theories differs only in minor technical details from the cases which came before.

5.1. Notation

In the sections which follow we will adopt the notation used in the book *Gauge theory of elementary particle physics* by T.P. Cheng and Ling-Fong Li. In addition, in order to simplify the details of the discussion, we will restrict attention to the case of the pure gauge theory.

In defining the pure SU(2) gauge theory one introduces a set of gauge fields A^a_μ which belong to the adjoint representation of the group. The free field Lagrangian is

$$\mathcal{L} = \frac{1}{4}\left(\vec{E}^a \cdot \vec{E}^a - \vec{B}^a \cdot \vec{B}^a\right) \tag{5.1}$$

where we have adopted the notation

$$\begin{aligned} \vec{E}_i^a &= F_{0i}^a \\ \vec{B}_i^a &= \frac{1}{2}\epsilon_{ijk}F_{jk}^a \end{aligned} \tag{5.2}$$

and where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c \tag{5.3}$$

The latin indices run over the range 1 to 3 and the Greek indices from 0 to 3 with the convention that repeated indices are summed over. As in all of our previous cases we have broken out the F_{0i}^a terms and the F_{ij}^a terms separately since we are interested in constructing the Hamiltonian. This Lagrangian is invariant with respect to the general gauge transformation

$$\begin{aligned} \delta A_0^a &= -\partial_0 f^a + g\epsilon^{abc} f^a A_\mu^c \\ \delta \vec{A}^a &= -\vec{\nabla} f^a + g\epsilon^{abc} f^b \vec{A}^c \end{aligned} \tag{5.4}$$

5.2. The Hamiltonian

Following what is by now our standard procedure we form the Hamiltonian and obtain

$$\mathcal{H} = \frac{1}{2}\left(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a\right) - A_0^a\left(\vec{\nabla} \cdot \vec{E}^a - g\epsilon^{abc}\vec{E}^b\vec{A}^c\right) \tag{5.5}$$

where we can identify the coefficient of the term linear in A_0^a as the covariant divergence of the *electric field* $\vec{E}^a$. Since this is the only term in the Hamiltonian (5.5) which is not invariant with respect to the gauge-transformation (5.4) we see that the condition which defines the *physical* states of the theory is that they must be eigenstates of the covariant divergence of the $\vec{E}^a$ field with eigenvalue zero. This is of course the direct generalization of the condition that the states satisfy Gauss' Law since in the case of the non-Abelian gauge theory the gauge field is itself charged.

The non-Abelian theory is the first place in which we see the necessity for a formal structure which provides an alternative to directly solving the state condition and eliminating the unphysical degrees of freedom, since the state condition is a non-linear equation and no closed form solution of the problem can be given. Our only alternative is to perturbatively construct the desired states. This of course means that we must work in a perturbation expansion based upon the gauge non-invariant Greens' functions since, in the absence of an exact solution, we have do not have an a-priori knowledge of the Green's functions required to define gauge invariant states. As in the case of QED this requires that we do some kind of gauge fixing in order to be able to completely specify the perturbation series.

Unlike the case of QED the Gupta-Bleuler solution, wherein one adds a gauge-fixing term to the Lagrangian and then tries to sort out the behavior of the negative norm and zero norm states, is no longer a viable alternative. This is because the ghost states no longer satisfy free equations of motion even if we use covariant gauge-fixing terms. This means that we can no longer use the statement that the negative metric states are created by free-field operators in order to eliminate them. This is where the BRST formalism really shines. The logic underlying the BRST treatment of the non-Abelian theory is a complete parallel to that of the Abelian theory of a charged scalar field in an R_ξ gauge. As in all other cases one enlarges the Hilbert space to include fermionic ghosts and shows that the resulting theory has an invariance with respect to a nil-potent BRST transformation. One then adds to the Lagrangian a gauge-fixing term of the form $\{Q, \mathcal{O}\}$, which by virtue of the nil-potency of the BRST transformation automatically commutes with Q. As in all previous cases by choosing a term of this sort we are able to get a well defined set of Feynman rules and still preserve the symmetries needed to classify states. One then shows that polynomials in the Hamiltonian only add zero norm states to states which are annihilated by the BRST charge. The procedure for carrying this argument out differs only in details from the arguments given for the rotor and Abelian electrodynamics. There is however an important new feature in this problem which we will focus on in the remainder of this section.

5.3. BRST Formalism

In order to parallel the discussions which appear in the literature we will now make what we already pointed out is a simple change of basis for the fermionic ghosts and enlarge the degrees of freedom to include a set of hermitian ghost fields c^a, their anti-hermitian anti-commuting partners $\bar{c}^a$ and auxiliary bosonic fields b^a whose specific form is to be determined after adding the gauge-fixing term to the Lagrangian. As we already pointed out this amounts to taking fermionic fields ψ^a and identifying c^a with $\psi^a + \psi^{\dagger a}$ and $\bar{c}^a$ with $\psi^a - \psi^{\dagger a}$. In terms of these additional degrees of freedom we seek a BRST transformation which is the direct generalization of the gauge-transformation (5.4) with the field c^a replacing the gauge functions f^a. Hence, we seek a transformation of the form

$$\begin{aligned} \delta A_0^a &= -\partial_0 c^a + g\epsilon^{abc} c^b A_0^c \\ \delta \vec{A}^a &= -\vec{\nabla} c^a + g\epsilon^{abc} c^b \vec{A}^c \\ \delta c^a &= \frac{1}{2}\epsilon^{abc} c^b c^c \\ \delta \bar{c}^a &= b^a; \quad \delta b^a = 0 \end{aligned} \tag{5.6}$$

where the first two equations are a direct transcription of (5.4) replacing f^a by c^a and the remaining equations are fixed by the requirement that $\delta^2 = 0$

The conventional form of the covariant gauge fixed Lagrangian is

$$\mathcal{L}_{BRST} = \mathcal{L} + \delta\left(\bar{c}^a(\partial^\mu A_\mu^a + \xi b^a/2)\right) \tag{5.7}$$

Using (5.6) and the fact that $\delta(\bar{c}A) = (\delta\bar{c})A - \bar{c}\delta A$ we can rewrite this as

$$\mathcal{L}_{BRST} = \mathcal{L} + b^a\left(\partial^\mu A_\mu^a + \xi\frac{b^a}{2}\right) - \partial^\mu \bar{c}^a\left(\partial_\mu c^a - g\epsilon^{abc} c^b A^c\right) \tag{5.8}$$

As in all previous cases we see that if we use the equation of motion for the fields b^a we find that $b^a = -\frac{1}{\xi}\partial^\mu A_\mu^a$ and so the first two gauge fixing terms combine to the sort of covariant term one would introduce in the Abelian gauge theory, and the last term has the structure of a Faddédev-Popov ghost action. Of course, as in the previous cases, the fact that $\delta^2 = 0$ now follows from the equations of motion for the fields c^a. We can, however, proceed just as we did in the other cases, and not use the equations of motion to eliminate the b^a's. Except for indices, everything in the non-Abelian theory appears to parallel the case of QED in a covariant gauge; a new feature, however, is that now the Faddédev-Popov terms for the ghost fields no longer describe free particles even in the covariant case. This is, of course, no problem since we already discussed what to do when the ghosts do not trivially decouple when we discussed the case of scalar QED in the R_ξ gauges. Except for the ϵ symbols and extra indices everything works the same way in this case. One simply focuses on the bilinear terms in all fields and imposes canonical commutation and anti-commutation relations for the fields $\vec{E}^a$, $\vec{A}^a$, $\bar{c}^a$ and c^a chosen so as to yield the correct Heisenberg equations of motion. One then constructs the BRST charge which commutes with the Hamiltonian and defines the physical states by the conditions that Q annihilates them. One then shows that starting from such a state polynomials in the gauge-fixed Hamiltonian map us into a state of the same form plus at most a state of the form $Q\,|\Psi'\rangle$; moreover, one shows that all such additional states have zero norm. This allows us to reformulate everything in a positive metric space by forming equivalence classes of such states. The details of this discussion are, except for indices, the same in this formulation of the theory as for all other cases.

5.4. BRST Charge the State Condition

There is a novel aspect to the non-Abelian problem in that it is much easier to proceed by eliminating the b^a fields first rewriting them as $\partial^\mu A^a_\mu$ and then quantizing rather that proceeding in the reverse order. This is because in this case we really need to use the equations of motion for the fields c^a in order to see that δb^a vanishes. Hence, for the sake of variety and in order to simplify our discussion we will assume that first we set $b^a = \partial^\mu A^a_\mu$ and then rewrite $\mathcal{L}$. One then quantizes canonically and finds the BRST charge charge Q which generates the BRST transformation according to the rule

$$\begin{aligned} \delta A_\mu^a = \left[Q, A_\mu^a\right]; \quad \delta c^a = \{Q, c^a\} \\ \delta \bar{c}^a = \{Q, \bar{c}^a\}; \quad \delta b^a = [Q, b^a] \end{aligned} \tag{5.9}$$

to have the form

$$\begin{aligned} Q = \int d^3x \Big[& i\left(\partial_0 c^a + g\epsilon^{abc} c^b A_0^c\right)\left(\partial^\mu A_\mu^a\right) + i\left(-\vec{\nabla}\cdot\vec{E}^a + g\epsilon^{abc}\vec{E}^b\cdot\vec{A}^c\right) c^a \\ & + \frac{1}{2}\epsilon^{abc}\partial_0\bar{c}^a c^b c^c \Big] \end{aligned} \tag{5.10}$$

Obviously, as in all previous cases, the fact that the covariant gauge gives us an action involving a term $\partial_0\bar{c}\partial_0 c$ forces a negative metric formulation on the theory by requiring that $\partial_0\bar{c}^a$ and c^a are the canonically conjugate quantities satisfying non-vanishing anti-commutation relations. Insofar as the state conditions are concerned, one sees that if a state is constructed so that $\partial^\mu A_\mu^a = 0$, $-\vec{\nabla}\cdot\vec{E}^a + g\epsilon^{abc}\vec{E}^b\cdot\vec{A}^c = 0$ and so that $\epsilon^{abc}\partial_0\bar{c}^a c^b c^c$ annihilates the state then that state is annihilated by the operator Q. In other words, a state of this sort is simply a state which satisfies the original requirement of being a gauge-invariant state multiplied by any color singlet combination of the fields $\bar{c}^a$ and c^a. Obviously such a state can be constructed iteratively in perturbation theory by starting from a state with no c's and $\bar{c}$'s and requiring to lowest order in g that $\vec{\nabla}\cdot\vec{E}^a = 0$. The remainder of the argument that applying the Hamiltonian to such a state only produces a state of the same sort plus at most a zero norm state which is the Q of something goes through as before.

This concludes our discussion of BRST symmetry for Abelian and non-Abelian gauge theories. The remainder of this paper will be devoted to the problem of the relativistic particle and spinning relativistic particle because of the interest these problems have for people working on string theories.

6. THE RELATIVISTIC PARTICLE

In reality there are two reasons for concluding this paper with a discussion of the relativistic and spinning relativistic particle.[5,13,14] The first reason is that these examples demonstrate the way in which one can treat a system involving what Dirac called second class constraints. A second class constraint, as we shall see, arises when the classical equations of motion imply relations among the operators which are not consistent with their supposed canonical commutation relations. The second reason is that these examples exhibit manifest reparametrization invariance, and that is why string theorists consider them paradigms for the way BRST invariance works in more complicated cases. The relativistic particle and the spinning relativistic particle are simple quantum mechanical examples which share this property with the more difficult example of a relativistic string theory. While one can treat these examples using the BRST formalism which we have already developed without making reference to this geometrical invariance, proceeding in this way leads to a formalism in which the relationship between the BRST invariance and the underlying reparametrization symmetry is somewhat obscure. We will show in this section how one can carry out the entire BRST program starting from the reparametrization invariance. The final result will of course be equivalent to the one we obtained before.

6.1. Basics

The classical relativistic particle is defined by an action of the form

$$I = \int dt\, m \sqrt{\dot{x}_0^2 - \dot{\vec{x}} \cdot \dot{\vec{x}}} \tag{6.1}$$

where x_0 and $\vec{x}$ are the coordinates of the particle in some reference frame and t specifies a parametrization of the curve. If we now make the usual Legendre transformations

$$
\begin{aligned}
p_0 &= \frac{m\dot{x}_0}{\sqrt{\dot{x}_0^2 - \dot{\vec{x}}\cdot\dot{\vec{x}}}} \\
\vec{p} &= \frac{-m\dot{\vec{x}}}{\sqrt{\dot{x}_0^2 - \dot{\vec{x}}\cdot\dot{\vec{x}}}}
\end{aligned}
\tag{6.2}
$$

we see that even though we have assumed that x_0 and $\vec{x}$ form four independent degress of freedom their canonically conjugate momenta must satisfy the constraint

$$
{p_0}^2 - \vec{p}\cdot\vec{p} = m^2 \tag{6.3}
$$

This constraint means that beginning from this form of the action it is clearly impossible to quantize the theory in the usual fashion giving all the operators canonical commutation relations.

Dirac, in his famous paper on the subject, gave a prescription for dealing with the problem which consisted of modifying the canonical commutation relations to be consistent with the constraint. Since the spirit of this paper is to enlarge our Hilbert space in order to make the formalism as simple as possible, we will take another tack. Basically we note that the reason for the constraint among the degrees of freedom is the existence of the square root in the action. This form of action was required in order that the action be reparametrization invariant, There is, however, another way to achieve the same goal, that is to introduce a metric, e, into the problem. To be specific one can adopt an action having the form

$$
\int dt\, \frac{(\dot{x}_0^2 - \dot{\vec{x}}\cdot\dot{\vec{x}})}{2e} + \frac{1}{2}m^2 e \tag{6.4}
$$

In this case, the usual manipulations yield

$$
\begin{aligned}
p_0 &= \dot{x}_0/e \\
\vec{p} &= -\dot{\vec{x}}/e
\end{aligned}
\tag{6.5}
$$

Since the time derivative of e does not appear in the Lagrangian the Hamiltonian becomes

$$\begin{aligned} H &= p_0\dot{x}_0 + \vec{p}\cdot\dot{\vec{x}} - L \\ &= \frac{e}{2}(p_0^2 - \vec{p}\cdot\vec{p} - m^2) \end{aligned} \tag{6.6}$$

At this point we can follow our familiar procedure and quantize this system by rewriting the Lagrangian in first order formalism. Following the steps outlined in section 2 we can construct the first order Lagrangian using equations (2.15) and (6.6)

$$L = p_0\dot{x}_0 + \vec{p}\cdot\dot{\vec{x}} - \frac{1}{2}e(p_0^2 - \vec{p}\cdot\vec{p} - m^2) \tag{6.7}$$

From this equation we see that, in the first order formalism, the metric e plays the role of the Lagrange multiplier; variation of the action with respect to e producing the constraint equation (6.3).

Having written the first order Lagrangian we are ready to quantize the system. As we will see the quantization of point particle is identical to the rigid rotor. Therefore in this section we will outline the procedure and the reader can fill in the necessary steps or read off the corresponding formulae from section 2. As before, the Lagrangian has a local symmetry that is generated by the constraint (6.3). This local symmetry will enable us to explicitly reveal the gauge structure of our Lagrangian. Under an arbitrary gauge transformation we have

$$\begin{aligned} &\delta x_0 = \xi p_0; \quad \delta\vec{x} = -\xi\vec{p} \\ &\delta p_0 = 0; \quad \delta\vec{p} = 0; \quad \delta e = \dot{\xi} \end{aligned} \tag{6.8}$$

where ξ is the gauge parameter. It is a simple exercise to check that the Lagrangian is indeed invariant under this time dependent gauge transformation. Once again the Hamiltonian is not gauge invariant. The variation of the Hamiltonian under a gauge transformation is given by

$$\frac{\dot{\xi}}{2}({p_0}^2 - \vec{p}\cdot\vec{p} - m^2)$$

Since the variation of the Hamiltonian is proportional to the constraint, the lack of invariance is not a problem. Here, as in all our previous cases, in order to obtain a gauge invariant Hamiltonian we must restrict attention to the sector of states for which $p_0^2 - \vec{p}\cdot\vec{p} = m^2$ Thus, for this theory, the gauge invariant Hamiltonian is identically zero and all the dynamics is in the constraint condition. Thus, we see that once again, we are dealing with a trivial theory and we could stop at this point: however, since this obviously has the form of a gauge theory it is amusing to carry out a BRST treatment of the problem

6.2. BRST Formalism

The BRST formalism for this sort of problem begins the same way as in all other cases. We replace the gauge function ξ by two anticommuting c-number variables c and $\bar{c}$ and a commuting variable b such that

$$\begin{aligned} &dx_0 = cp_0; \quad d\vec{x} = -c\vec{p} \\ &dp_0 = 0; \quad d\vec{p} = 0; \quad de = \dot{c} \\ &dc = 0 \\ &d\bar{c} = b; \quad db = 0 \end{aligned} \tag{6.9}$$

where the two first conditions just rewrite the gauge transformation. The next conditions come from requiring that $d^2 = 0$. As in the case of QED and the rotor it is possible to find a second nilpotent invariance $\bar{d}$ such that $d\bar{d} + \bar{d}d = 0$; namely,

$$\begin{aligned} &\bar{d}x_0 = \bar{c}p_0; \quad \bar{d}\vec{x} = -\bar{c}\vec{p} \\ &\bar{d}p_0 = 0; \quad \bar{d}\vec{p} = 0; \quad \bar{d}e = \dot{\bar{c}} \\ &\bar{d}\bar{c} = 0; \quad \bar{d}c = b; \quad \bar{d}b = 0 \end{aligned} \tag{6.10}$$

Once again the two first conditions are obvious rewritings of the gauge transformation condition. The remaining conditions follow from the nilpotency of $\bar{d}$ and the requirement that it anticommutes with d

6.3. Gauge Fixing

The general trick which we have use to allow us to fix the gauge is to add to the Lagrangian a function that is trivially BRST invariant; in general, this is achieved by adding to the Lagrangian a function which is δ of something else. For the point particle we can choose to add $\delta\bar{c}(\dot{e} + b/2)$. The reader should compare this gauge fixing term to the one we used for the rotor equation (2.24). If we adopt this tack then the full Lagrangian has the form

$$L = p_0\dot{x}_0 + \vec{p}\cdot\dot{\vec{x}} - \frac{1}{2}e({p_0}^2 - \vec{p}\cdot\vec{p} + m^2) + b\dot{e} + b^2/2 - \dot{\bar{c}}\dot{c} \tag{6.11}$$

To carry out the BRST formalism we would follow the same steps as in chapter 2. Once again the independent canonical variables c and $\bar{c}$ must satisfy the commutation relations

$$\begin{aligned} \{\dot{\bar{c}}, c\} &= i \\ \{\dot{c}, \bar{c}\} &= -i \end{aligned} \tag{6.12}$$

As before, the reversal of the sign in equation (6.12) implies the existence of negative metric states in the full Hilbert space including the ghosts.

In the BRST formalism the notion of gauge invariance is replaced by BRST invariance. To conclude the quantization of the point particle we write down the nilpotent BRST charge. The charge we are looking for must satisfy the following nontrivial commutation and anti-commutation relations,

$$[Q, x_0] = cp_0; \quad [Q, \vec{x}] = -c\vec{p}; \quad \{Q, \bar{c}\} = b \tag{6.13}$$

It is straightforward to verify that the operator $Q = \frac{i}{2}c({p_0}^2 - \vec{p}\cdot\vec{p}) + i\dot{c}b$ satisfies this requirement.

The rest of the quantization proceeds exactly as for the rotor. Since this was analysed very carefully in section 2 we will not repeat it here. Instead we will conclude by studying the reparametrization invariance of the system, which is a new a feature of our Lagrangian.

6.4. Reparametrization Invariance

In addition to the invariance (6.13) the Lagragian (6.7) exhibits another symmetry, namely under local reparametrizations. It is easy to verify that, in fact, each term in (6.7) is invariant with respect to the transformation

$$\begin{aligned} \delta_r x_0 &= \epsilon \dot{x}_0; \quad \delta_r \vec{x} = \epsilon \dot{\vec{x}} \\ \delta_r p_0 &= \epsilon \dot{p}_0; \quad \delta_r \vec{p} = \epsilon \dot{\vec{p}} \\ \delta_r e &= \epsilon \dot{e} + \dot{\epsilon} e \end{aligned} \tag{6.14}$$

where ϵ is the gauge parameter. We see that x_μ and p_μ transform under 1-dimensional diffeomorphisms as scalars, while e transforms as a density. It is also obvious that (6.14) is equivalent to (6.8) with $\xi = \epsilon e$ by the equations of motion $\dot{p}_\mu = 0$ and (6.5). However, both of these symmetries leave (6.7) invariant even without invoking the equations of motion, and so one might suspect that they are not equivalent off shell. It turns out[8] that there is a linear combination of the two symmetries which is trivial: define $\delta_t = \delta_r - \delta$ and $\xi = \epsilon e$. Then

$$\begin{aligned} \delta_t x_0 &= \delta_r x_0 - \delta x_0 = \epsilon \frac{\delta S}{\delta p_0} \\ \delta_t \vec{x} &= \delta_r \vec{x} - \delta \vec{x} = \epsilon \frac{\delta S}{\delta \vec{p}} \\ \delta_t p_0 &= \delta_r p_0 - \delta p_0 = -\epsilon \frac{\delta S}{\delta x_0} \\ \delta_t \vec{p} &= \delta_r \vec{p} - \delta \vec{p} = -\epsilon \frac{\delta S}{\delta \vec{x}} \\ \delta_t e &= 0 \quad . \end{aligned} \tag{6.15}$$

The action does not vary:

$$\begin{aligned} \delta_t S &= \frac{\delta S}{\delta x_0} \delta_t x_0 + \frac{\delta S}{\delta \vec{x}} \delta_t \vec{x} + \frac{\delta S}{\delta p_0} \delta_t p_0 + \frac{\delta S}{\delta \vec{p}} \delta_t \vec{p} \\ &= \epsilon \left(\frac{\delta S}{\delta x_0} \frac{\delta S}{\delta p_0} + \frac{\delta S}{\delta \vec{x}} \frac{\delta S}{\delta \vec{p}} - \frac{\delta S}{\delta p_0} \frac{\delta S}{\delta x_0} - \frac{\delta S}{\delta \vec{p}} \frac{\delta S}{\delta \vec{x}} \right) \\ &= 0 \quad . \end{aligned} \tag{6.16}$$

The symmetry generated by δ_t has no physical significance, and exists for any action written in first order form. Its triviality will be the key to relating the

BRST formalism derived from the reparametrizations (6.14) to the one discussed in then previous sections. However, there is one proviso: in general, disentangling two different BRST symmetries is easy only if they anticommute. This in turn requires the underlying gauge symmetries to commute. Let us check that for δ and δ_r. We obtain

$$[\delta, \delta_r]x = \epsilon\dot{\xi}p - \delta_r\xi p \tag{6.17}$$

The commutator will be zero if and only if $\delta_r = \epsilon\dot{\xi}$. This requirement is not entirely unexpected: the gauge symmetry (6.8) is valid even for a finite parameter ξ. This function must be a 1-dimensional scalar, so that under reparametrizations $\delta_r\xi = \epsilon\dot{\xi}$. It is now easy to verify that indeed $[\delta, \delta_r] = 0$. We now have all the ingredients we need to show the equivalence of the two symmetries of (6.7) . In the next section we will study the associated BRST invariances and how they are related.

6.5. More BRST Symmetries

First, recall the BRST transformations constructed from (6.8) , namely

$$\begin{aligned} &dx_0 = cp_0; \quad d\vec{x} = -c\vec{p} \\ &dp_0 = 0; \quad d\vec{p} = 0; \quad de = \dot{c} \\ &dc = 0 \\ &d\bar{c} = b; \quad db = 0 \ . \end{aligned} \tag{6.18}$$

The technique to derive the BRST transformation associated with (6.14) is by now completely standard, and we obtain

$$\begin{aligned} &d_r x_0 = \alpha\dot{x}_0; \quad d_r\vec{x} = \alpha\dot{\vec{x}} \\ &d_r p_0 = \alpha\dot{p}_0; \quad d_r\vec{p} = \alpha\dot{\vec{p}} \\ &d_r e = \alpha\dot{e} + \dot{\alpha}e \quad d_r c = \alpha\dot{c} \\ &d_r\alpha = \alpha\dot{\alpha} \\ &d_r\bar{\alpha} = g; \quad d_r g = 0 \ . \end{aligned} \tag{6.19}$$

Now we demand $\{d, d_r\} = 0$. This condition is solved by

$$\begin{aligned} d\alpha &= 0 \\ dg &= 0; \quad d\bar{\alpha} = 0 \\ d_r b &= \alpha \dot{b}; \quad d_r \bar{c} = \alpha \dot{\bar{c}} \ . \end{aligned} \tag{6.20}$$

We may now combine the two BRST symmetries into one by defining $\Delta = d_r - d$. Obviously , we have $\Delta^2 = 0$. We now change variables:

$$c = \alpha e - \beta$$

and note that Δ can be decomposed into two new BRST transformations:

$$\Delta = d_\alpha + \delta_\beta \ ,$$

with

$$\begin{aligned} d_\alpha x_0 &= \alpha \frac{\delta S}{\delta p_0}; \quad d_\alpha \vec{x} = \alpha \frac{\delta S}{\delta \vec{p}} \\ d_\alpha p_0 &= -\alpha \frac{\delta S}{\delta x_0}; \quad d_\alpha \vec{p} = -\alpha \frac{\delta S}{\delta \vec{x}} \\ d_\alpha \alpha &= \alpha \dot{\alpha} \quad , \end{aligned} \tag{6.21}$$

corresponding to the gauge invariance δ_t, and

$$\begin{aligned} d_\beta x_0 &= \beta p_0; \quad d_\beta \vec{x} = -\beta \vec{p} \\ d_\beta p_0 &= 0; \quad d_\beta \vec{p} = 0 \\ d_\beta e &= \dot{\beta} \ , \end{aligned} \tag{6.22}$$

which is an exact replica of the BRST transformation d. Since $\{d_\alpha, d_\beta\} = 0$, and because we convinced ourselves that d_α is free of any physical content, we have the option of reducing the space of states we are working in by imposing the condition that none of our states depend on the coordinate α. The problem we are left with is now manifestly equivalent to the BRST formalism discussed in sections (6.2) and (6.3).

7. RELATIVISTIC SPINNING PARTICLE

Having discussed the case of the simple relativistic particle we really have exhibited all of the interesting new features which one encounters in carrying out a BRST quantization of any constrained quantum mechanical system. However, for the sake of completeness we conclude with the treatment of the spinning relativistic particle,[5] since this is a supersymmetric quantum mechanical example which parallels the introduction of fermions into a supersymmetric string theory. The only slightly new feature encountered in dealing with the relativistic spinning particle is the fact that because the problem is supersymmetric we encounter an additional constraint when we quantize the theory. This new feature will force us to introduce commuting ghosts in addition to the anti-commuting ghosts we encountered in the previous sections.

7.1. BRST Quantization

To simplify things we restrict attention to the case of the massless spinning particle. The action for this can be obtained from equation (6.4) by introducing a superpartner χ to the variable e and a superpartner ψ for the variable x. The supersymmetric Lagrangian for the spinning point particle is given by

$$L = \frac{1}{2e}(\dot{x}_0^2 - \dot{\vec{x}} \cdot \dot{\vec{x}}) + \frac{i}{2}\psi_0\dot{\psi}_0 - \frac{i}{2}\vec{\psi} \cdot \dot{\vec{\psi}} + +i\frac{\chi}{e}\dot{x}_0\psi_0 + i\frac{\chi}{e}\dot{\vec{x}} \cdot \vec{\psi} \tag{7.1}$$

As in the case of the point particle we want to rewrite this in the first order formalism.

$$L = p_0\dot{x}_0 + \vec{p} \cdot \dot{\vec{x}} + \frac{i}{2}\psi_0\dot{\psi}_0 - \frac{i}{2}\vec{\psi} \cdot \dot{\vec{\psi}} - \frac{e}{2}({p_0}^2 - \vec{p} \cdot \vec{p}) + i\chi p_0\psi_0 - i\chi\vec{p} \cdot \vec{\psi} \tag{7.2}$$

Examining (7.2) we see that χ, the superpartner of e, also plays the role of a Lagrange multiplier. Variation of the Lagrangian with respect to χ yields the constraint equation

$$p_0\psi_0 - \vec{p}\cdot\vec{\psi} = 0 \tag{7.3}$$

Once again, this constraint corresponds to a local symmetry, with respect to which the Lagrangian is invariant.

$$\begin{aligned} &\delta_\chi x_0 = \eta\psi_0; \quad \delta_\chi \vec{x} = -\eta\vec{\psi} \\ &\delta_\chi \psi_0 = i\eta p_0; \quad \delta_\chi \vec{\psi} = i\eta\vec{p} \\ &\delta_\chi p_0 = 0; \quad \delta_\chi \vec{p} = 0; \quad \delta_\chi e = 2\eta\chi; \quad \delta_\chi \chi = i\dot{\eta} \end{aligned} \tag{7.4}$$

where η is a fermionic gauge parameter. In addition to this constraint, variation of the Lagrangian with respect to the Lagrange multiplier e gives another constraint. The local symmetry corresponding to that constraint has the form

$$\delta_e x_0 = \xi p_0; \quad \delta_e \vec{x} = -\xi\vec{p}; \quad \delta_e p_0 = 0; \quad \delta_e \vec{p} = 0; \quad \delta_e e = \dot{\xi} \tag{7.5}$$

As in all previous cases we see that although the Lagrangian is invariant with respect to these symmetries the Hamiltonian is not. One easily verifies that in order to obtain a gauge invariant Hamiltonian one must restrict attention to *gauge invariant states*, i.e. states which are annihilated by the constraint.

As usual we create a BRST charge corresponding to the symmetries (7.4) and (7.5) , and write it $d = d_e + d_\chi$. The quantization parallels the discussion given in all previous examples. Corresponding to the invariance (7.4) we introduce commuting *bosonic* c-number variables γ and $\bar{\gamma}$ and a c-number anticommuting variable β . Similarly, corresponding to the symmetry (7.5) we introduce c-number anticommuting variables c and $\bar{c}$ and a commuting variable b. With these conventions the

BRST transformations take the form

$$
\begin{aligned}
&dx_0 = cp_0 + \gamma\psi_0 : \quad d\vec{x} = -c\vec{p} - \gamma\vec{\psi} \\
&dp_0 = 0; \quad d\vec{p} = 0 \\
&d\psi_0 = i\gamma p_0; \quad d\vec{\psi} = i\gamma\vec{p} \\
&de = \dot{c} + 2\gamma\chi; \quad d\chi = i\dot{\gamma} \\
&dc = -i\gamma^2; \quad d\gamma = 0 \\
&d\bar{c} = b; \quad db = 0 \\
&d\bar{\gamma} = \beta; \quad d\beta = 0
\end{aligned}
\tag{7.6}
$$

It is easy to check that this set of equations guarantees that $d^2 = 0$ but not that $d_e^2 = d_\chi^2 = 0$ separately. The corresponding identity for the gauge symmetries reads

$$[\delta_{\chi,1}, \delta_{\chi,2}] = \delta_e$$

where the parameter ξ of δ_e is expressed in terms of the fermionic gauge parameters as $\xi = 2i\eta_1\eta_2$. Since the two gauge symmetries cannot be written in terms of two algebras that close separately, it is no surprise that we cannot decompose the corresponding BRST symmetries into two nilpotent ones which anticommute. However, we would like to carry out the program of BRST quantization as in all the previous examples, and therefore it would be quite convenient if we could find a trick that decouples the two symmetries d_e and d_χ. This trick consists of making the bosonic ghost nilpotent, i.e. declaring $\gamma^2 = 0$. One may think of γ then as a pair of fermionic fields. An inspection of (7.6) now shows that if we impose this condition we may separate d into two BRST symmetries d_e and d_χ which satisfy $d_e^2 = d_\chi^2 = \{d_e, d_\chi\} = 0$. One can now fix the gauge by introducing into the Lagrangian a term of the form $\delta_e\delta_\chi f$, where f is some function of the basic fields. Choosing one such function one then carries out the canonical quantization procedure in the usual way and checks that the BRST charges commute with the Hamiltonian, and that the condition Q_e and Q_χ annihilate a state enforce the original constraint equations. We leave the details of this procedure to the reader and conclude with an outline of what has to be done to show that the reparametrizations are incorporated properly in this problem.

7.2. Reparametrization Invariance

Just as we did in the case of the bosonic partlcle, we now ask for a more geometrical way of understanding the gauge symmetries we encountered. Of course, the invariances we wish to discuss now are the well known superreparametrizations. Let us concentrate first on the bosonic part of these transformations, the reparametrizations:

$$\begin{aligned} &\delta_r x_0 = \epsilon \dot{x}_0; \quad \delta_r \vec{x} = \epsilon \dot{\vec{x}} \\ &\delta_r p_0 = \epsilon \dot{p}_0; \quad \delta_r \vec{p} = \epsilon \dot{\vec{p}} \\ &\delta_r \psi_0 = \epsilon \dot{\psi}_0; \quad \delta_r \vec{\psi} = \epsilon \dot{\vec{\psi}} \\ &\delta_r e = \dot{e}\epsilon + \dot{\epsilon} e; \quad \delta_r \chi = \dot{\chi}\epsilon + \dot{\epsilon}\chi \quad . \end{aligned} \tag{7.7}$$

The technique we apply here to show the equivalence of (7.7) and (7.5) is the same we used for the bosonic particle. Define $\delta_1 = \delta_e$ for $\xi = \epsilon e$ and $\delta_2 = \delta_\chi$ for $\eta = \epsilon\chi$. Then it is not hard to see that $\delta_t = \delta_r - \delta_1 + \delta_2$ is a trivial symmetry:

$$\begin{aligned} &\delta_t x_0 = \epsilon \frac{\delta S}{\delta p_0}; \quad \delta_t \vec{x} = \epsilon \frac{\delta S}{\delta \vec{p}} \\ &\delta_t p_0 = -\epsilon \frac{\delta S}{\delta x_0}; \quad \delta_t \vec{p} = -\epsilon \frac{\delta S}{\delta \vec{x}} \\ &\delta_t \psi_0 = -i\epsilon \frac{\delta S}{\delta \psi_0}; \quad \delta_t \vec{\psi} = i\epsilon \frac{\delta S}{\delta \vec{\psi}} \\ &\delta_t e = 0; \quad \delta_t \chi = 0 \quad . \end{aligned} \tag{7.8}$$

Quite obviously a symmetry of the form (7.8) will leave any action with bosonic and fermionic degrees of freedom invariant. The BRST transformation associated with (7.7) and (7.4) can be written as

$$\Delta = d_\chi + d_\tau \tag{7.9}$$

and after the change of variables

$$c = \beta - \alpha e; \quad \gamma = \kappa + i\alpha\chi \tag{7.10}$$

we may decompose this symmetry as follows:

$$\Delta = d_\alpha + d_{\beta,\kappa} \quad , \tag{7.11}$$

where $d_{\beta,\kappa}$ is an exact copy of the transformation d of (7.6) with c, γ replaced by β, κ and d_α represents the BRST transformation associated with the trivial gauge symmetry (7.8) , namely

$$\begin{aligned} d_\alpha x_0 &= \alpha\frac{\delta S}{\delta p_0}; \quad d_\alpha \vec{x} = \alpha\frac{\delta S}{\delta \vec{p}} \\ d_\alpha p_0 &= -\alpha\frac{\delta S}{\delta x_0}; \quad d_\alpha \vec{p} = -\alpha\frac{\delta S}{\delta \vec{x}} \\ d_\alpha \psi_0 &= -i\alpha\frac{\delta S}{\delta \psi_0}; \quad d_\alpha \vec{\psi} = i\alpha\frac{\delta S}{\delta \vec{\psi}} \\ d_\alpha \alpha &= \alpha\dot{\alpha} \quad . \end{aligned} \tag{7.12}$$

Both d_α and $d_{\beta,\kappa}$ are nilpotent and they anticommute:

$$d_\alpha^2 = d_{\beta,\kappa}^2 = \{d_\alpha, d_{\beta,\kappa}\} = 0 \quad . \tag{7.13}$$

It is now clear how to proceed in order to establish the equivalence of two symmetries that are related by the equations motion: an appropriate change of variables in the ghost sector will yield a trivial BRST transformation, which we called d_α, and the remainder is then equivalent to the BRST invariance we started with, even though its form may be quite different.

The careful reader may have felt a little uneasy when we combined (7.4) and (7.7) into one BRST symmetry, since the associated fermionic gauge symmetry does not represent a closed algebra. Only up to the equations and together with the reparametrizations does it close. At the heart of this problem lies the fact that we did not bother to write the spinning particle in 1-dimesional superspace.

7.3. Spinning Particle in Superspace

We will conclude in this section with a presentation of the spinning particle in superspace, focusing mainly on the gauge symmetry, i.e. the superreparametrizations. We will see that they are related by the equations of motion to the symmetries we studied thus far, and it will be left to the reader to spell out in detail the BRST decomposition into trivial pieces and the symmetry d described by (7.6). The supercovariant derivative we will use is $\mathbf{d} = \partial_\theta - i\theta\partial_t$, and it acts on the superfields:

$$\begin{aligned} X &= x - i\theta\sqrt{e}\phi; \\ P &= \psi + \theta\sqrt{e}p; \\ G &= \frac{1}{\sqrt{e}} + i\theta\frac{\chi}{e} \quad . \end{aligned} \tag{7.14}$$

These superfields are judiciously chosen so that the component expansion of the superfield action[9]

$$S = \int dt d\theta \left[iG\mathbf{d}X\mathbf{d}P - \frac{1}{2}P\mathbf{d}P \right] \tag{7.15}$$

has a simple form:

$$S = \int dt[\dot{x}p - \frac{1}{2}ep^2 + i\phi\dot{\psi} - \frac{i}{2}\psi\dot{\psi} + i\chi\phi p] \quad . \tag{7.16}$$

The superreparametrizations may be written in superspace as

$$\begin{aligned} \delta X &= [V\partial_t + \frac{i}{2}(\mathbf{d}V)\mathbf{d}]X; \\ \delta P &= [V\partial_t + \frac{i}{2}(\mathbf{d}V)\mathbf{d}]P; \\ \delta G &= [V\partial_t + \frac{i}{2}(\mathbf{d}V)\mathbf{d} - \frac{1}{2}\partial_t V]G \quad , \end{aligned} \tag{7.17}$$

and they read in component language, with $V = \epsilon + 2\theta\frac{\eta}{\sqrt{e}}$,

$$\begin{aligned} \delta x &= \epsilon\dot{x} + \eta\phi; \quad \delta p = \epsilon\dot{p} + \frac{\eta}{e}(\dot{\psi} - \chi p) \\ \delta\psi &= \epsilon\psi + i\eta p; \quad \delta\phi = \epsilon\dot{\phi} + i\frac{\eta}{e}(\dot{x} + i\chi\phi) \quad . \end{aligned} \tag{7.18}$$

If we now use the equation of motion $\phi = \psi$, we can integrate out ϕ and arrive at the action (7.2) . The symmetries (7.18) are easily recognized to be equivalent to (7.4) and (7.5) up to terms that vanish by the equations of motion. Therefore, the method of quantization we described in section (7.1) really emodies all the symmetries present for the spinning string.

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