

SPLITTING OF THE P STATES IN HEAVY
QUARKONIUM SYSTEMS AND THE NATURE
OF THE SPIN-DEPENDENT POTENTIAL*

CLAUDIO O. DIB AND FREDERICK J. GILMAN

*Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305*

and

PAULA J. FRANZINI

Theory Division, CERN, CH-1211 Geneva 23, Switzerland

ABSTRACT

We consider the behavior of the splittings between the $L = 1$ states in heavy quarkonium systems. With standard assumptions, the ratio of mass splittings, $R_x = (M_2 - M_1)/(M_1 - M_0)$, is shown to grow slowly as we go from the lowest P states to their radial excitations, a behavior which is in accord with other theoretical calculations and is consistent with experiment. This behavior is the opposite of what is expected from a naive argument which we present. We show why the naive argument fails, and how the phenomenological division of the potential into components which correspond to exchange of a Lorentz scalar and vector can be made so as to obtain other behaviors.

Submitted to *Physical Review D*

* Work supported by the Department of Energy, contract DE-AC03-76SF00515.

The bound states of heavy quarks and antiquarks have proven to be a key element in our understanding of strong interactions by giving us a subset of all hadrons whose spectroscopy we can calculate quantitatively. The insight thereby gained can then also be used to understand at least semi-quantitatively hadronic systems involving light quarks and, if only by process of elimination, to highlight those states which can not be interpreted as quark-antiquark bound states and which then must have their origin in more exotic quark and/or gluon combinations.^[1] We consider here a limited but interesting aspect of the spectroscopy of heavy quark-antiquark systems, related to the character of the spin-dependent forces. We examine the behavior of the level splittings between the $L = 1$ states as we go from the lowest P states to their radial excitations.

The general form for the spin-dependent potential in a system composed of a heavy quark and a heavy antiquark has been shown to be:^[2]

$$\begin{aligned}
V_{SD}(r) = & \left(\frac{\vec{S}_1 \cdot \vec{L}}{2m_1^2} + \frac{\vec{S}_2 \cdot \vec{L}}{2m_2^2} \right) \left(\frac{-dV(r)}{rdr} + 2\frac{dV_2(r)}{rdr} \right) \\
& + \frac{(\vec{S}_1 + \vec{S}_2) \cdot \vec{L}}{m_1 m_2} \frac{dV_2(r)}{rdr} \\
& + \frac{1}{6m_1 m_2} \left(6\vec{S}_1 \cdot \hat{r} \vec{S}_2 \cdot \hat{r} - 2\vec{S}_1 \cdot \vec{S}_2 \right) V_3(r) \\
& + \frac{2}{3m_1 m_2} \vec{S}_1 \cdot \vec{S}_2 V_4(r),
\end{aligned} \tag{1}$$

where $V(r)$ is the usual spin-independent potential and V_2 , V_3 , and V_4 spin-dependent ones. These extra terms originate in expectation values of color electric and magnetic fields which are different than those that enter in the spin-independent potential and in principle are new objects which are not simply related to $V(r)$.

Nevertheless, it provides some physical insight to consider the non-relativistic reduction of the four-fermion interaction arising from the exchange of vector and

scalar fields between quark and antiquark. In momentum space this is represented by an interaction:

$$L_{int} = \tilde{s}(q^2) \bar{u}u\bar{v}v + \tilde{v}(q^2) \bar{u}\gamma_\mu u\bar{v}\gamma^\mu v. \quad (2)$$

If we do an expansion in powers of v^2/c^2 , the static limit is the spin-independent potential $v(r)+s(r)$, while the spin-dependent terms give the Breit-Fermi potential, which in configuration space for the equal mass case is:

$$\begin{aligned} V_{SD}(r) = & \frac{\vec{S} \cdot \vec{L}}{2m^2} \left(-\frac{dv(r) + ds(r)}{rdr} + 4\frac{dv(r)}{rdr} \right) \\ & + \frac{1}{12m^2} \left(6\vec{S} \cdot \hat{r}\vec{S} \cdot \hat{r} - 2\vec{S} \cdot \vec{S} \right) \left(\frac{dv(r)}{rdr} - \frac{d^2v(r)}{d^2r} \right) \\ & + \frac{1}{6m^2} \left(2\vec{S} \cdot \vec{S} - 3 \right) \nabla^2 v(r). \end{aligned} \quad (3)$$

The term $-(dv(r) + ds(r))/rdr$ in the first line is due to the familiar Thomas precession, and it is followed by usual spin-orbit, tensor (on the second line), and spin-spin (on the third line) interactions, each with a coefficient related to $v(r)$ or $s(r)$. We can compare this to the more general spin-dependent potential in Eq. (1) in the equal mass case, rewriting it a bit in the form

$$\begin{aligned} V_{SD}(r) = & \frac{\vec{S} \cdot \vec{L}}{2m^2} \left(\frac{-dV(r)}{rdr} + 4\frac{dV_2(r)}{rdr} \right) \\ & + \frac{1}{12m^2} \left(6\vec{S} \cdot \hat{r}\vec{S} \cdot \hat{r} - 2\vec{S} \cdot \vec{S} \right) V_3(r) \\ & + \frac{1}{6m^2} \left(2\vec{S} \cdot \vec{S} - 3 \right) V_4(r). \end{aligned} \quad (4)$$

Comparing what is in Eq. (3) to the generic decomposition in Eq. (4) involving V_2 , V_3 , and V_4 we see first that the spin-independent potential V is given by the sum of the vector and scalar potentials, $v + s$. Second, the spin dependent potentials V_2 , V_3 , and V_4 are all expressible in terms of derivatives of only the vector part of

the potential, v . Hence, if v is related to gluon exchange and its associated $1/r$ behavior, then the potentials V_2 , V_3 , and V_4 are all short range in character.

This encourages the following standard division: the scalar term is long range and associated with quark confinement, while the vector term is short range and associated with one-gluon exchange. The associated physical picture^[3] has confinement due to a color flux tube that connects the quark and antiquark; as they rotate around each other the flux tube rotates along with them. Consequently there are no spin-dependent forces generated from this part of the potential, aside from the Thomas term which comes in with a minus sign and is generated from the spin rotation associated with Lorentz transforming from the center-of-mass to the quark or antiquark rest frame. It is from the Coulomb-like piece that one obtains the spin-dependent terms that we are long accustomed to in atomic physics: a spin-orbit interaction (minus the piece due to Thomas precession), a tensor interaction, and a spin-spin interaction. This description has been very successful with respect to predicting and interpreting the data on heavy quarkonium systems and it is within its context that we shall work in this paper.

Let us focus on the splittings of the 3P_J states, *i.e.*, the quark-antiquark states with $L = 1$ and $J = 0, 1, \text{ or } 2$. The spin-orbit and tensor terms cause this splitting, and within the picture of vector and scalar exchanges their contributions to the 3P_J state masses are^[4]

$$M({}^3P_2) = \bar{M} + a - 2b/5 \quad (5a)$$

$$M({}^3P_1) = \bar{M} - a + 2b \quad (5b)$$

$$M({}^3P_0) = \bar{M} - 2a - 4b, \quad (5c)$$

where the matrix elements a and b are defined as

$$a = \frac{1}{2m^2} \left\langle -\frac{ds}{rdr} + 3\frac{dv}{rdr} \right\rangle, \quad (6a)$$

$$b = \frac{1}{12m^2} \left\langle \frac{dv}{rdr} - \frac{d^2v}{dr^2} \right\rangle. \quad (6b)$$

We can summarize the relative values of the matrix elements in terms of one number by forming the ratio

$$R_\chi = \frac{M(^3P_2) - M(^3P_1)}{M(^3P_1) - M(^3P_0)} = \frac{2a - \frac{12}{5}b}{a + 6b}. \quad (7)$$

If only the Coulomb-like vector part of the potential, $v(r)$, is present, $R_\chi = 0.8$. As the strength of the scalar term, $s(r)$, is increased, there is more cancellation between the two terms on the right-hand-side of Eq. (6a) and the matrix element a decreases, as then does R_χ .

This brings us to the question at issue in this paper: How should the ratio R_χ behave as we go from one set of P states to another? A naive argument goes as follows. The 2P states are characterized by an average radius which is larger than that of the 1P states (and similarly for the 3P compared to the 2P, etc.). Therefore these states should “live” more in the confining, Lorentz scalar part of the potential and the value of the ratio R_χ should be smaller for the 2P (sometimes called χ') as compared to the 1P (χ) states of a given quarkonium system. This argument can be cast in a more quantitative form by noting that if $v(r) \propto 1/r$, then all the terms in Eq. (6) that involve v are proportional to $\langle 1/r^3 \rangle$, while if $s(r) \propto r$ the term in Eq. (6a) involving s is proportional to $\langle 1/r \rangle$ and the whole discussion boils down to the assertion that naively $\langle 1/r \rangle / \langle 1/r^3 \rangle$ should increase as we go from the 1P to 2P states.

This argument is wrong! It disagrees with evidence provided by the bottomonium system where the theoretical predictions^[5-10] almost all would have $R_\chi < R_{\chi'}$. Furthermore, the most recent experimental results^[11] also tend in the same direction, with $R_\chi = 0.67 \pm 0.06$ and $R_{\chi'} = 0.69 \pm 0.05$.

How does the naive argument go wrong? It is a somewhat subtle point. Even though it is true that $\langle r^2 \rangle$ is bigger for the χ' than the χ states, it does not

necessarily follow that $\langle 1/r \rangle / \langle 1/r^3 \rangle$ is bigger. The latter statement depends on the potential. With Coulomb wave functions, it does indeed grow, as $\langle 1/r \rangle / \langle 1/r^3 \rangle = 6r_B^2$ for the χ states and $9r_B^2$ for the χ' states (r_B being the Bohr radius). But with a three dimensional harmonic oscillator potential this same quantity is a factor of 9/13 smaller for the χ' states. The wave function does spread out in space, $\langle r \rangle$ and $\langle r^2 \rangle$ increase for the χ' states, but the ratio $\langle 1/r \rangle / \langle 1/r^3 \rangle$ does not. By numerical calculation it appears that more generally $\langle 1/r \rangle / \langle 1/r^3 \rangle$ increases with radial excitation when the potential is of the form $V(r) \propto -r^\alpha$ with $\alpha < 0$, and conversely, it decreases when the potential is of the form $V(r) \propto r^\alpha$ with $\alpha > 0$.

The more interesting question now is what happens in the physical case when the potential has both ‘‘Coulomb’’ (one gluon exchange) and confining (linear) components. From the above discussion we expect that the ratio of mass splittings will tend to decrease when the wave functions are mostly sensitive to the one gluon exchange part of the potential and will increase when they are sensitive to the confining part. This is indeed the case, as is seen by studying the situation in various regimes with the Cornell potential,^[12]

$$V(r) = \frac{-\beta}{r} + kr = \frac{-0.52}{r} + \frac{r}{(2.34 \text{ GeV}^{-1})^2}, \quad (8)$$

with the two coefficients having been adjusted to fit the charmonium spectrum, although the model does a quite adequate job in describing bottomonium as well. The Schrodinger equation can be put in dimensionless form by using the variables $\rho = \mu\beta r = r/r_B$ and $K = k/(\beta^3\mu^2)$, where μ is the reduced mass and $r_B = 1/\beta\mu$ is the Bohr radius of the corresponding purely Coulomb potential problem.

Figure 1 shows the ratio of mass splittings

$$R_\chi = \frac{M(^3P_2) - M(^3P_1)}{M(^3P_1) - M(^3P_0)}$$

for the 1P and 2P levels of the Cornell potential as a function of the scaled variable K . The critical assumption, even if it is the most straightforward one, has been

made that the $-\beta/r$ piece of the potential is a Lorentz four-vector and the kr piece a Lorentz scalar *for all values of r* . As is to be expected from our previous discussion, $R_\chi < R_{\chi'}$ when K is large and the confining part of the potential plays a dominant role even for the lowest bound lying states. Indeed, it is only for very small values of K that we get an inversion of this behavior and $R_\chi > R_{\chi'}$. Inspection of Figure 2 indicates that this happens for $K < 1/34$.

The charmonium and bottomonium systems correspond to values of K indicated by the arrows in Figure 1 (corresponding to $m_c = 1.84$ GeV and $m_b = 5.17$ GeV) and are well within the region where $R_\chi < R_{\chi'}$. The values of R_χ that correspond to these two situations are shown in Table 1.

State	R_χ (experiment ^[11])	R_χ (theory)
χ_c	0.48 ± 0.01	0.538
χ'_c	—	0.588
χ_b	0.67 ± 0.06	0.717
χ'_b	0.69 ± 0.05	0.727

Table 1. R_χ for charmonium and bottomonium computed using the Cornell potential.

The agreement with experiment is quite good for both charmonium and bottomonium considering that nothing about spin-dependent effects was used as an input in the choice of parameters. For charmonium, however, the absolute magnitude of the χ_c splittings is about a factor of two smaller than experiment.

For quark masses above ~ 13 GeV with the Cornell potential, which corresponds to $K = 1/34$, the situation will change and R_χ will be smaller for the χ' instead of the χ states. But even with very high mass quarks, for sufficiently high radial excitations which “live” primarily in the confining part of the potential we

expect the situation to revert back again to larger values of R_χ as we go up in principal quantum number. This is seen in Table 2, where we have again used the Cornell potential with the same assumptions except we consider the case of toponium with $m_t = 50$ GeV. The corresponding value of K is indicated by the arrow in Figure 2.

State	1P	2P	3P	4P	5P	6P	7P	8P	9P
R_χ	0.79763	0.79689	0.79656	0.79645	0.79642	0.79644	0.79647	0.79651	0.79656

Table 2. R_χ for toponium computed with the Cornell potential and $m_t = 50$ GeV.

The values of R_χ do indeed decrease until we get to the 5P state, and then increase again, but we need to look at four or five significant figures to see the effect! The situation is completely dominated by the Coulomb piece of the potential and all values of R_χ are very close to 0.8. Moreover, the absolute magnitudes of the mass splittings are on the order of tens of MeV, making them hard to measure to the needed accuracy.

A number of calculations of the splittings of the P states have been done, each with its own prescription for the Lorentz structure of the potential. As can be seen from Table 3, they have varying success in accounting for the data, although several are in quite good accord with experiment.

Model	$R_{\chi_c} = 0.48 \pm 0.02$	$R_{\chi_b} = 0.67 \pm 0.06$	$R_{\chi'_b} = 0.69 \pm 0.05$
Beavis ^[5]	0.97	0.96	1.0
Buchmuller ^[6]	0.61	0.76	0.75
McClary and Byers ^[7]	0.35	0.45	0.48
Moxhay and Rosner ^[8]	0.42	0.42	0.42
Gupta <i>et al.</i> ^[9]	0.50	0.64	0.67
Olsson and Suchyta ^[10]	0.55	0.78	0.79

Table 3. Various theoretical predictions for R_χ

This brings us to the more general question of the division of the potential into pieces with different Lorentz transformation properties. When a calculation of the potential is made from first principles, for example, with lattice gauge theory techniques, then this question will have a definitive answer. Until that time we are left in the somewhat unsatisfactory state of having to make a prescription that is in accord with theory and experiment as it is known at any given time. When we did this for the Cornell potential above, it was “natural” to make the Coulomb piece arise from exchange of a Lorentz four-vector and the confining linear piece correspond to exchange of a Lorentz scalar. But is this so natural at all values of r ? It surely does not follow from any known physics that the $1/r$ behavior from one-gluon exchange at short distance should be continued to all r and, moreover, its Lorentz character maintained in what is, after all, a phenomenological potential.

In order to see how sensitive the splittings of the P states are to our assumptions and to see if, in particular, we could have come to opposite conclusions about the behavior of R_χ with a plausible input, we have done the following computation.

We start from the Richardson potential,^[13]

$$V(r) = \frac{8\pi}{33 - 2n_f} \Lambda \left(\Lambda r - \frac{f(\Lambda r)}{\Lambda r} \right), \quad (9)$$

with

$$f(t) = 1 - 4 \int_1^{\infty} \frac{dq}{q} \frac{e^{-qt}}{\ln^2(q^2 - 1) + \pi^2}. \quad (10)$$

In momentum space it can be written in a more transparent form as one term:

$$\tilde{V}(q^2) = -\frac{4}{3} \frac{12\pi}{33 - 2n_f} \frac{1}{q^2 \ln(1 + q^2/\Lambda^2)}, \quad (11)$$

where we take $\Lambda = 0.398$ GeV and n_f is the number of light quarks at the relevant momentum scale for renormalization (taken equal to three). There is no obvious way to separate things into vector and scalar pieces, even though the Richardson potential was designed to have the correct behavior at both short and long distances. For our purposes we make the arbitrary division that

$$v(r) = V_{\text{Richardson}}(r) \left[e^{r^2/a^2} \right] \quad (12a)$$

and

$$s(r) = V_{\text{Richardson}}(r) \left[1 - e^{r^2/a^2} \right], \quad (12b)$$

where a is a distance scale which provides the smooth division between a vector character at short distances and scalar character at large distances. By construction,

$$V_{\text{Richardson}} = v(r) + s(r).$$

In Figures 3 and 4 we have plotted R_χ for the charmonium and bottomonium systems, respectively, as a function of the distance scale, a , using Eqs. (5), (6), and (12) and masses $m_c = 1.49$ GeV and $m_b = 4.88$ GeV appropriate to the Richardson potential.

These two Figures make it clear that it is possible to invert the usual ordering in which $R_\chi < R_{\chi'}$ by choosing the distance parameter a large enough, *i.e.*, making the potential vector in character out to fairly large distances. The price is in getting

values of both R_χ and $R_{\chi'}$ which are not far from 0.8, the value associated with a Coulomb potential, which is in disagreement with the data for both charmonium and bottomonium. A value of a around 0.5 fermi is needed to fit charmonium and about half of that for bottomonium, with the absolute magnitudes of the mass splittings coming out in good agreement as well.

We do not propose this as a serious method to calculate the splittings of the P states in onium systems. Aside from being arbitrary, it does not give quantitative agreement using a single value of a with the known data. However, it does show us that other assumptions about the Lorentz character of the heavy quark potential are possible and that one can obtain different behaviors for R_χ depending on the assumptions one makes. Turning it around, the behavior of R_χ as we go from the lowest P states to their radial excitations is sensitive to the Lorentz character of the exchanges between heavy quarks as it depends on their distance and can be used as a tool to understand this more detailed and more quantitative feature of the potential.

FIGURE CAPTIONS

- 1) The ratio R_χ for the Cornell potential as a function of the scaled, dimensionless variable K for the χ and χ' states, respectively. The arrows indicate the values of K corresponding to charmonium and bottomonium with $m_c = 1.84$ GeV and $m_b = 5.17$ GeV, respectively.
- 2) The ratio R_χ for the Cornell potential for small values of the scaled, dimensionless variable K for the χ and χ' states, respectively. The arrow indicates the value of K corresponding to toponium with $m_t = 50$ GeV.
- 3) The ratio R_χ for the Richardson potential with the division into scalar and vector parts as in Eq. (12) as a function of the distance a for the χ (solid line) and χ' (dashed line) states of charmonium, respectively.
- 4) The ratio R_χ for the Richardson potential with the division into scalar and vector parts as in Eq. (12) as a function of the distance a for the χ (solid line) and χ' (dashed line) states of bottomonium, respectively.

REFERENCES

1. J. L. Rosner, rapporteur talk at Hadron 87, Tsukuba, Japan, April 16 - 18, 1987 and University of Chicago preprint EFI-87-26-CHICAGO, 1987 (unpublished) gives an overview of our present understanding of hadron spectroscopy.
2. E. Eichten and F. Feinberg, Phys. Rev. D23, 2724 (1981); D. Gromes, Z. Phys. C26, 401 (1984). See, E. Eichten, *Proceedings of the 11th SLAC Summer Institute on Particle Physics*, SLAC Report No. 267, edited by P. M. McDonough (SLAC, Stanford, 1983), p. 497.
3. W. Buchmuller, Phys. Lett. 112B, 479 (1982). See also J. L. Rosner, J. de Phys. 46, C2, Supp. 2 (1985); R. D. Pisarski and J. D. Stack, Fermilab preprint FERMILAB-PUB-86/122-T, 1986 (unpublished), and M. G. Olsson

- and C. J. Suchyta, University of Wisconsin preprint MAD/PH/327, 1987 (unpublished) and Phys. Rev. D35, 1738 (1987).
4. J. L. Rosner, *Proceedings of the 1985 Symposium on Lepton and Photon Interactions at High Energies*, edited by M. Konuma and K. Takahashi (Kyoto University, Kyoto, 1986), p. 448.
 5. D. Beavis *et al.*, Phys. Rev. D20, 2345 (1981)
 6. W. Buchmuller, Ref. 3.
 7. R. McClary and N. Byers, Phys. Rev. D28, 1632 (1983).
 8. P. Moxhay and J. L. Rosner, Phys. Rev. D28, 1132 (1983).
 9. R. Gupta *et al.*, Phys. Rev. D34, 201 (1986).
 10. M. G. Olsson and C. J. Suchyta, Ref. 3.
 11. J. Lee-Franzini, invited talk at the 1987 SLAC Summer Institute, August 10 - 21, 1987 (unpublished).
 12. E. Eichten *et al.*, Phys. Rev. D17, 3090 (1978) and D21, 203 (1980).
 13. J. L. Richardson, Phys. Lett 82B, 272 (1979).

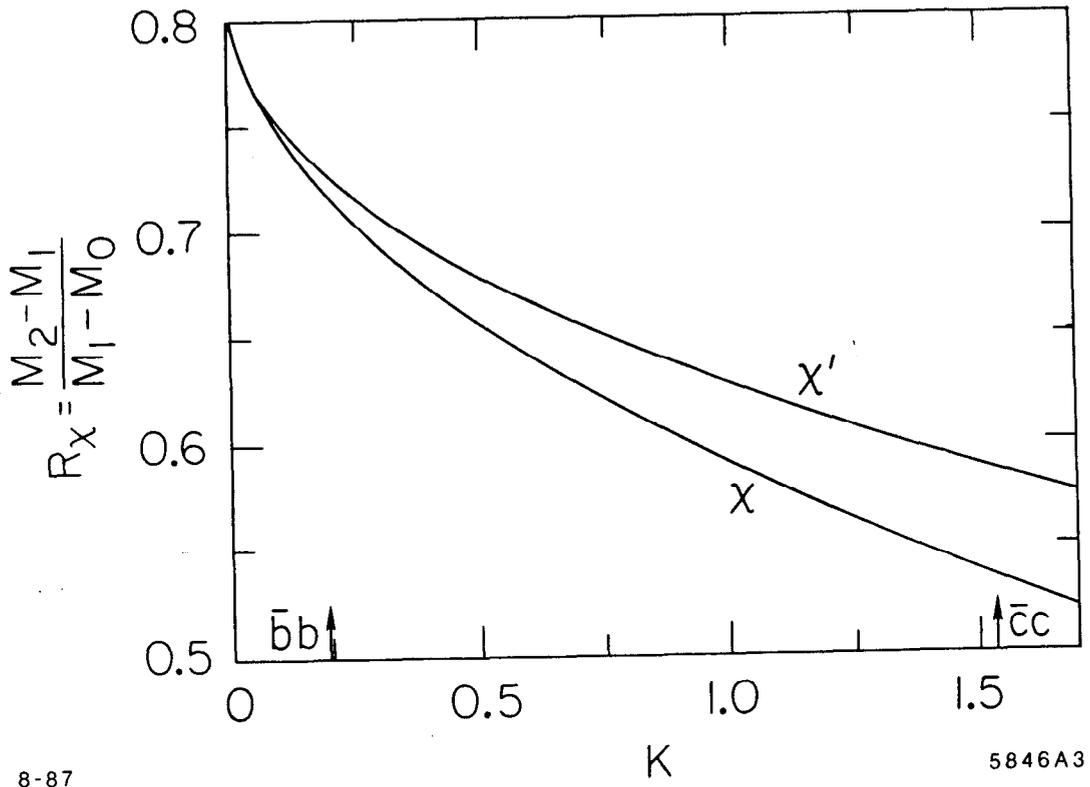


Fig. 1

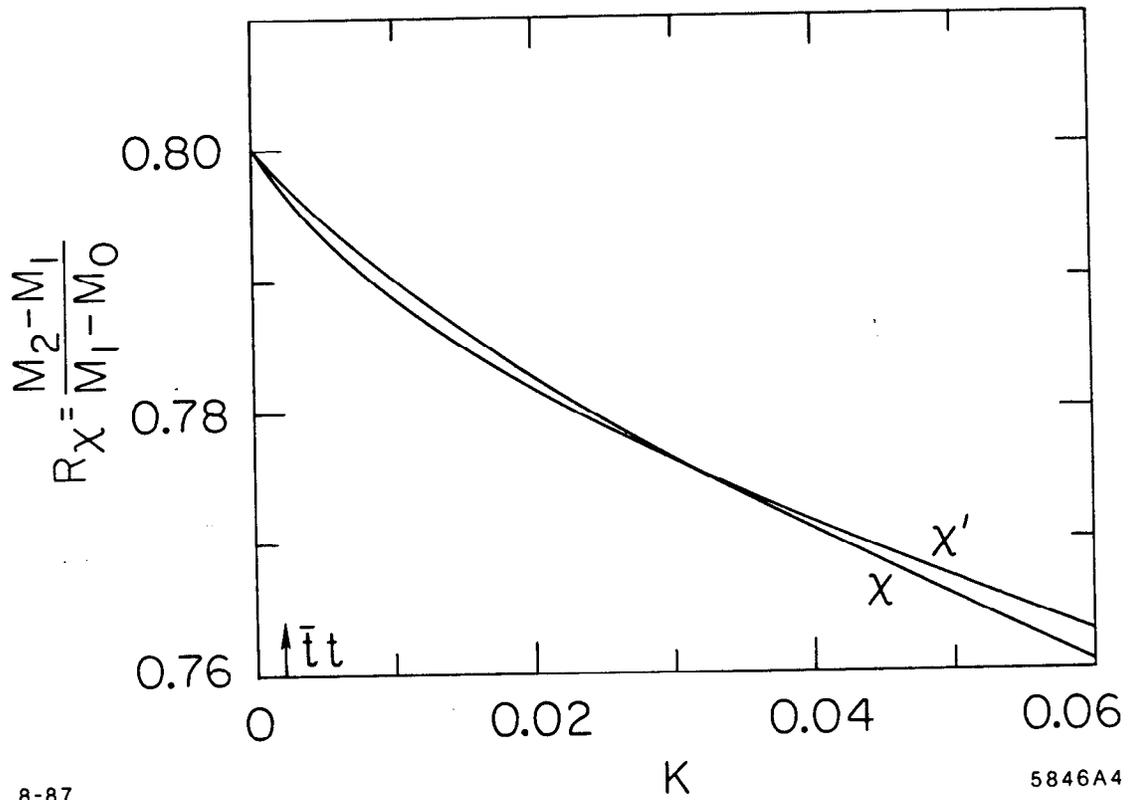
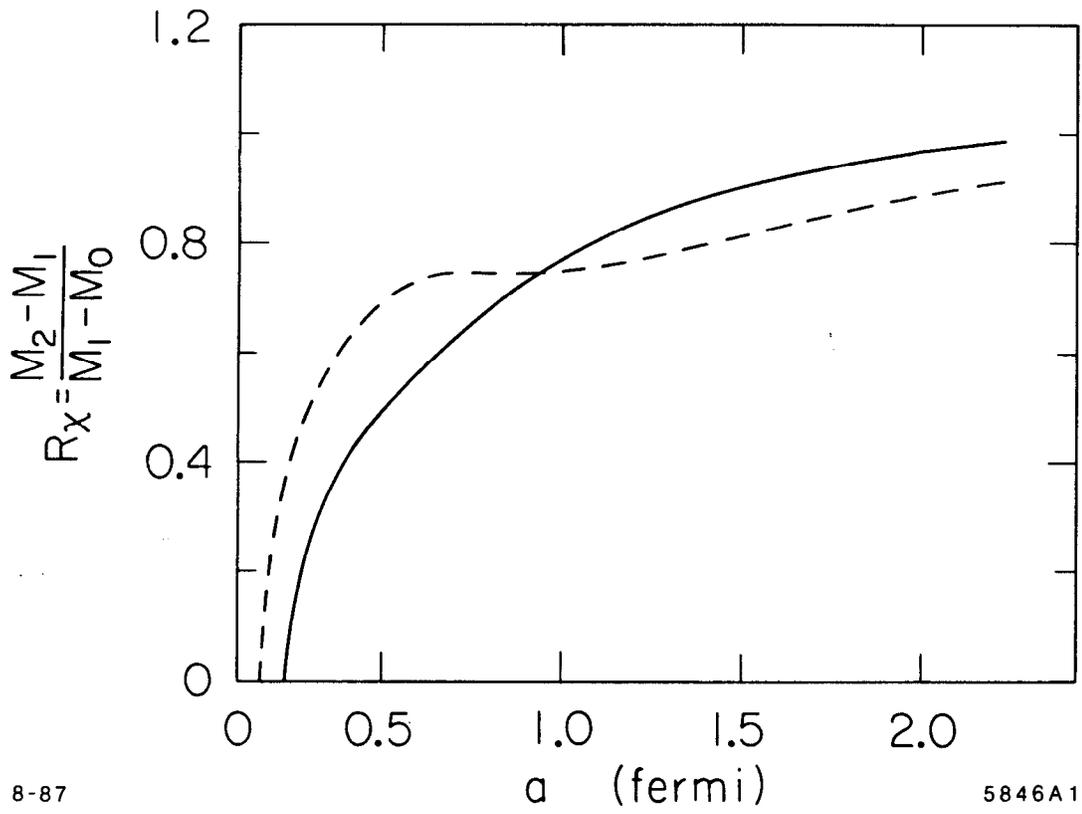


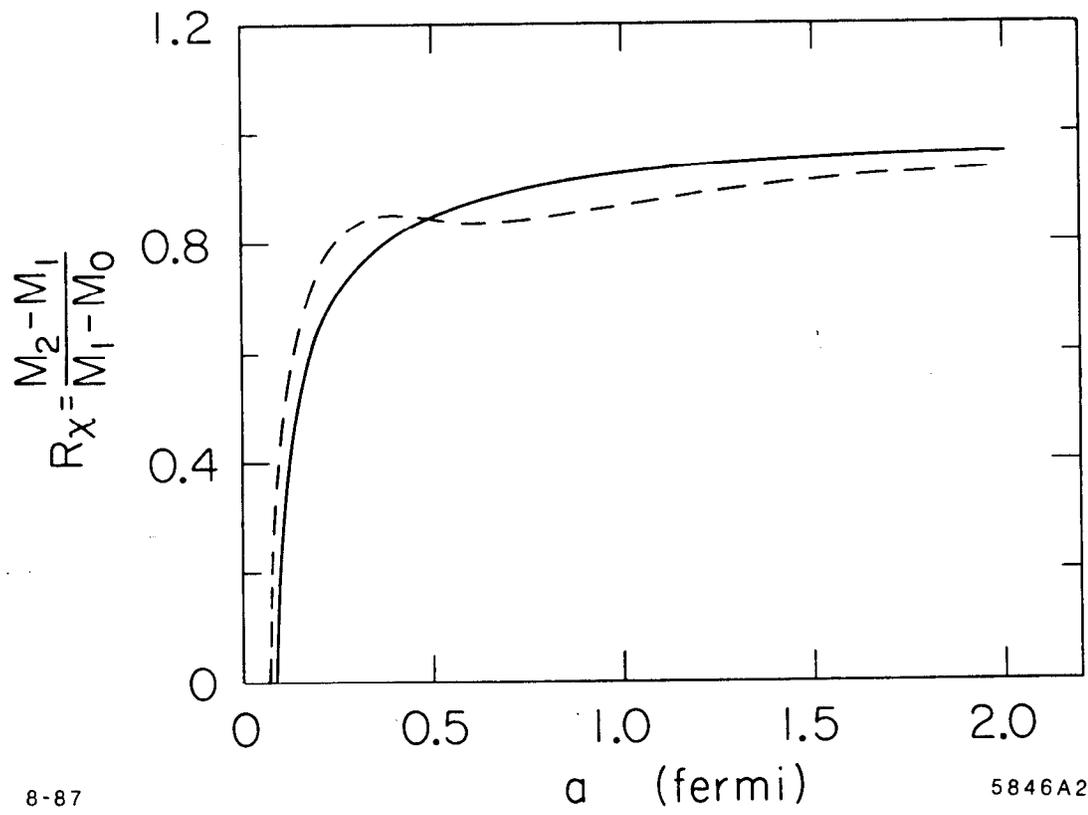
Fig. 2



8-87

5846A1

Fig. 3



8-87

5846A2

Fig. 4