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QUANTIZATION ON THE LIGHT-CONE: RESPONSE TO A COMMENT BY C. R. HAGEN*

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ABSTRACT

We show that if carefully defined, light-cone commutation relations are not inconsistent with the charge and Lorentz invariance of the vacuum. We demonstrate this for charge invariance with an explicit example.

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The method of discretized light-cone quantization[1] (DLCQ) has recently been applied to a series of field theories in one-space and one-time dimension, including Yukawa theory[1] (charged fermions coupled to scalar bosons), quantum electrodynamics,[2] and quantum chromodynamics (for arbitrary N_{color}).[3] By diagonalizing the Hamiltonian defined at a given light-cone time on a discrete momentum Fock basis, one obtains not only the charge zero spectrum but also the bound state wavefunctions of each field theory for arbitrary coupling constant and fermion mass. The numerical results agree with previous calculations of the spectrum where they are available.

The Comment by C. R. Hagen[5] raises several issues concerning the consistency of field theory quantized on the light cone. These should not be confused with gauge-fixing problems, such as the choice of light-cone gauge $A^+ = 0$. A recent discussion of such problems is given in ref. 4. Indeed Hagen's remarks are directed to the application of DLCQ to Yukawa theory where there are no gauge fields.

Hagen's first remark[5] [see his equation (1)] concerns a possible unwanted term in the Poincare algebra. Our choice of periodic boundary conditions trivially eliminates this term. The remainder of his comment reiterates claims made in his earlier papers that the light-cone vacuum is neither charge nor Lorentz invariant. Although these issues are not directly related to our paper, they should be evaluated on their merit, independently of their relevance to our work. In fact, we shall show, by explicit counterexample that Hagen's central "theorem" on which his main objections rely is false. In particular, we will investigate the claim that a charge-invariant light-cone vacuum cannot be defined;[6] the discussion of Lorentz invariance is similar.

Hagen's theorem assumes a conserved current, the existence of a spectral decomposition for vacuum expectation values of this current, and standard light-cone equal-x^+ commutation relations for scalar fields. To avoid any possible ambiguity or controversy arising from light-cone quantization techniques we will employ non-interacting scalar fields of mass m quantized at equal time. Although quantized conventionally, these fields will be shown to satisfy all the requirements necessary to test Hagen's theorem.

For simplicity, we work in the relevant two dimensions throughout. In equal-time quantization, commutators are fixed at $x^0 = 0$, but because in this example the fields are free, they are known for all x. In particular, $\langle[\phi(x), \phi(0)]\rangle$ is known at $x^+ \equiv x^0 + x^1 = 0$, where

$$\langle[\phi(x), \phi(0)]\rangle_{x^+=0} \equiv i\Delta(x; m^2)_{x^+=0} = \frac{-i}{4}\epsilon(x^-). \qquad (1)$$

This will be demonstrated below. ($\epsilon(x)$ is the antisymmetric step function.)

Now allow for a conserved current by making $\phi(x)$ complex, with $\frac{1}{\sqrt{2}}\phi_1(x)$ and $\frac{1}{\sqrt{2}}\phi_2(x)$ the real and imaginary parts. The commutator of ϕ and ϕ^* is as in Eq.(1) . The current-current commutator, with

$$j^\mu(x) =: \phi_2(x)\partial^\mu\phi_1(x) - \phi_1(x)\partial^\mu\phi_2(x): \tag{2}$$

may be expressed in a general spectral form,

$$\langle[j^\mu(x), j^\nu(0)]\rangle = \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)\int_0^\infty dM^2\sigma(M^2)i\Delta(x; M^2) \tag{3}$$

where, for free fields,

$$\sigma(M^2) = \Theta\left(1 - \frac{4m^2}{M^2}\right)\frac{\left(1 - \frac{4m^2}{M^2}\right)^{\frac{1}{2}}}{2\pi M^2}. \tag{4}$$

The assumptions of Hagen's proof[6] are therefore satisfied.

The charge operator on the light-cone may be computed, also unambiguously:

$$\begin{aligned} Q &\equiv \int \frac{dx^-}{2} j^+(x) = \int \frac{dx^-}{2}\left[j^0(x) + j^1(x)\right] \\ &= i\int dk^1 \left[a_1^\dagger(k^1)a_2(k^1) - a_2^\dagger(k^1)a_1(k^1)\right], \end{aligned} \tag{5}$$

where $a_i(k^1)$ and $a_i^\dagger(k^1)$ are the *usual* equal-time creation and annihilation operators. The charge defined in equation Eq.(5) is thus explicitly identical to the charge Q defined conventionally at equal time,

$$Q \equiv \int dx^1 j^0(x), \tag{6}$$

as current conservation guarantees. It is obvious that

$$Q\,|0\rangle = 0 \tag{7}$$

and that, since $\partial_0 Q = 0$ and $\partial_1 Q = 0$,

$$\partial_+ Q = 0, \tag{8}$$

in conflict with Hagen's published claim that a charge-invariant light-cone vacuum does not exist. In this case the light-cone and ordinary vacuum are identical. We re-emphasize that although quantized at equal time, free charged scalar fields satisfy all the assumptions enumerated in Hagen's theorem, and yet contradict his result. There is therefore no reason to give his work any further attention when considering light-cone physics.

To uncover the source of Hagen's error, it is necessary to examine his calculation in detail. From Eq.(3), for a general field theory,

$$\langle [j^+(x), j^+(0)] \rangle = -4\partial_-^2 \int_0^\infty dM^2 \sigma(M^2) i\Delta(x; M^2). \tag{9}$$

The commutator $\Delta(x; M^2)$ is singular on the light cone, so the evaluation of Eq.(9) and related commutators at $x^+ = 0$ may depend on the procedure employed. Hagen formally shows $\langle [\partial_+ Q, j^+(0)] \rangle$ to be non-zero by first differentiating Eq.(9) with respect to x^+ and performing one x^- derivative, then setting x^+ identically to zero and replacing $\Delta(x; M^2)$ with the standard equal-x^+ commutator. Finally, he performs the second x^- derivative and integrates over x^-.

Several facts about Hagen's procedure indicate its speciousness. First, it gives zero for $\langle [Q(x^+ = 0), j^+(0)] \rangle$, a possibly finite number for $\langle [\partial_+ Q(x^+ = 0), j^+(0)] \rangle$, and divergences for higher derivatives in x^+. However, as Q is conserved, it should have no x^+ dependence, and there is no good reason to set x^+ to zero at an intermediate step. Had it been left arbitrary, Hagen would have obtained the conventional result that $\langle [Q(x^+), j^+(0)] \rangle$ is zero for *all* x^+. Higher derivatives in x^+ computed at this stage would then trivially give zero for all x^+. Also, it can be seen from Eq.(9) that these quantities all involve an integral over a total derivative in x^-. That Hagen doesn't obtain zero for this, and that his result depends on a particular sequence of integration and differentiation suggests that he's not working with well-defined integrals. That this is the case may be seen by carefully defining the commutator in Eq.(9). The formal representation in two dimensions,

$$i\Delta(x; M^2) = \int \frac{d^2k}{2\pi} \delta(k^2 - M^2) \epsilon(k^0) exp\{-ik \cdot x\}, \tag{10}$$

is, in light-cone variables,

$$i\Delta(x;M^2) = \lim_{\eta\to 0^+} \int_0^\infty \frac{dk^+}{2\pi k^+}\Big[exp\frac{-i}{2}\{k^+(x^- - i\eta) + \frac{M^2}{k^+}(x^+ - i\eta)\} - exp\frac{i}{2}\{k^+(x^- + i\eta) + \frac{M^2}{k^+}(x^+ + i\eta)\}\Big]. \tag{11}$$

Convergence factors are explicitly and necessarily included to ensure that the integral exists, and that manipulations such as the rearrangement of integration and differentiation make sense.

Near the light cone Eq.(11) reduces to

$$i\Delta(x;M^2)_{x^2\sim 0} = \frac{-i}{4}\left[\epsilon(x^+) + \epsilon(x^-)\right] + O(x^2)$$

or equivalently,

$$= \frac{-i}{2}\epsilon(x^0)\Theta(x^2) + O(x^2). \tag{12}$$

The damping prescription in Eq.(11) uniquely defines $\epsilon(y)$ and $\Theta(y)$ in Eq.(12) to be zero and one-half, respectively, when y is identically zero. It therefore reproduces the standard equal-x^+ commutator

$$i\Delta(x;M^2)_{x^+=0} = \frac{-i}{4}\epsilon(x^-). \tag{13}$$

Substituting Eq.(11) into Eq.(9) yields, after appropriate differentiation and integration,

$$\left\langle\left[\partial_+ Q(x^+), j^+(0)\right]\right\rangle = \lim_{\eta\to 0^+}\frac{-i}{4}\int_0^\infty dM^2 M^2\sigma(M^2)\times \int_{-\infty}^{\infty} dk^+\delta(k^+)exp\{\frac{-i}{2}\frac{M^2}{k^+}x^+ - \frac{M^2\eta}{2\mid k^+\mid}\}. \tag{14}$$

(If desired, a limiting procedure could also be introduced to explicitly define $\delta(k^+)$, with the same conclusion.) As long as all the steps leading to this were well-defined, that is, as long as η was kept infinitesimal but not zero, then Eq.(14) and all higher derivatives in x^+ are unambiguously zero independently of x^+. This is consistent with an invariant vacuum as illustrated in our initial free scalar field example. Hagen's result is obtained if both x^+ and η are set to zero identically, but then his intermediate manipulations are evidently suspect.

Finally, if one insists on employing the commutator at equal-x^+ explicitly, Eq.(11) with x^+ at zero has been shown to provide an adequate definition of the scalar light-cone commutator, and it produces consistent results as long as η is taken to zero only at the end.[7]

Computations involving singular quantities in a field theory are always potentially ambiguous. In general, one must define such quantities by a prescription, when one exists, consistent with the desired properties of that theory. That at least one prescription does exist for the light-cone commutators in question consistent with the charge and Lorentz invariance of the vacuum has been demonstrated above. It seems peculiar to insist instead on another which is ill-defined and which pretends to render inconsistent not only light-cone quantization but even standard free scalar field theory.

References and Footnotes

1. H. C. Pauli and S. J. Brodsky, Phys. Rev. **D32**, 1993 (1985); Phys. Rev. **D32**, 2001 (1985).
2. T. Eller, H. C. Pauli and S. J. Brodsky, Phys. Rev. **D35**, 1493 (1987).
3. K. Hornbostel, in preparation.
4. Light cone quantization and the various pitfalls are discussed in detail in the literature. See for example P. J. Steinhardt in Annals of Physics 128, 425-447 (1980) and references therein. Noncovariant gauges are reviewed by G. Leibbrandt, University of Guelph preprint (1987).
5. C. R. Hagen, preceding Comment.
6. C. R. Hagen, Phys. Lett. **90B** (1980) 405;
   C. R. Hagen, Phys. Rev. **D24** (1981) 2605
7. Alternatively, the equal-x^+ commutator may be defined in analogy to the Johnson-Low (equal-time) prescription (Cf. K. Johnson and F. E. Low. Prog. Theor. Phys. Suppl. **37-38** (1966) 74), by subtracting the x^+-ordered product at $-\eta$ from that at $+\eta$ as $\eta \to 0^+$. This also produces the standard commutator, Eq.(1), and in this case it is the Feynman $i\epsilon$ prescription in the propagator which serves to regulate the small k^+ region in Eq.(14).