SLAC – PUB – 4391
August 1987
(T/E/A)

QUANTUM BEAMSTRAHLUNG FROM GAUSSIAN BUNCHES*

PISIN CHEN

Stanford Linear Accelerator Center,
Stanford University, Stanford, CA 94305

ABSTRACT

The method of Baier and Katkov is applied to calculate the correction terms to the Sokolov–Ternov radiation formula due to the variation of the magnetic field strength along the trajectory of a radiating particle. We carry the calculation up to the second order in the power expansion of $\dot{B}\tau/B$, where τ is the formation time of radiation. The expression is then used to estimate the quantum beamstrahlung average energy loss from e^+e^- bunches with gaussian distribution in bunch currents. We show that the effect of the field variation is to reduce the average energy loss from previous calculations based on the Sokolov–Ternov formula or its equivalent. Due to the limitation of our method, only an upper bound of the reduction is obtained.

1. INTRODUCTION

For future e^+e^- linear colliders, radiation induced by beam-beam collision is expected to be very strong [1]. This radiation, called beamstrahlung, would cause substantial loss of energy and degredation on energy resolution. Due to these concerns, the study of the subject has been intensive during recent years. Rigorously speaking, the problem is very complex in the sense that the e^+e^- bunches would be continuously deformed during collision. A complete analytic treatment would be formidable if not impossible. Fortunately, it occurs that in a large range of beam parameters the bunches would only be slightly deformed. It is therefore a reasonable approximation to assume no bunch deformation in a calculation as a first attempt.

Himel and Sigrest [2] estimated the average beamstrahlung energy loss in a conceptual 5 TeV+5 TeV collider, with number of particles in each bunch $N = 1.2 \times 10^8$, and beam size $\sigma_r = 2.5$ Å, $\sigma_z = 0.4\ \mu$m. The calculation assumes uniform particle distribution within a cylinder bunch, where the radius is $R = 2\sigma_r$ and the length is $L = 2\sqrt{3}\,\sigma_z$. By assuming no disruption, each particle would execute a linear trajectory at a fixed impact parameter with respect to the oncoming bunch. An approximate radiation power spectrum based on the well-known Sokolov–Ternov formula for uniform magnetic fields [3] was then used to obtain an average fractional energy loss of $\langle\epsilon\rangle = 14.5\%$ [4].

Recently Blankenbecler and Drell [5] studied this problem with a different approach. They sum over individual potential scatterings of a test charge traversing through the oncoming bunch in the target's rest frame. This is an Eikonal type aproximation but retaining one more order in the expansion of the phase. The result agrees reasonably well with Himel and Siegrest. Soon after Bell and Bell [6] showed that the two approaches are equivalent to the extent that the spin flip contribution to the radiation was omitted in the Blankenbeckler–Drell calculation, which is minor.

On the other hand, there have also been efforts to calculate beamstrahlung from gaussian bunches. Noble developed a computer simulation code for beamstrahlung with negligible disruption [7]. During the collision, at each time step the Sokolov–Ternov radiation probability is invoked based on the local field strength. The result for the same beam parameters turns out to be very close to the calculations on an equivalent cylinder bunch. Yokoya independently developed a computer code which is capable of simulating both beamstrahlung and disruption effects [8]. Again the Sokolov–Ternov formula was used in the code.

* Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.

Presented at the Workshop on New Developments in Particle Acceleration Techniques
Orsay, France, June 29—July 4, 1987

As is well-known, the Sokolov–Ternov formula was derived by assuming a uniform field, whereas in the problem of beamstrahlung the field is both spacially finite and inhomogeneous. In the calculations invoking uniform cylindrical charge distribution, the radiating particles do experience constant field along the trajectories inside the target bunch if disruption is neglected. But to enter and to exit the target, the particles would encounter abrupt changes of field strength. In the case of gaussian bunches the field strength is continuously changing during the traverse of the radiating particle even if disruption is neglected. The smooth variation in the latter case can be considered as a smearing of the abrupt discontinuity in the former case. One natural question therefore arises as to how important this "slope" effect would be in the context of modifying the average fractional energy loss in beamstrahlung.

In this paper we present a calculation based on a method developed by Baier and Katkov [9,10], which enables one to calculate radiation intensity in inhomogeneous fields. The homogeneity of a field along particle's trajectory can be tested by the condition

$$\frac{\dot{B}\tau}{B} \ll 1 \quad , \tag{1}$$

where $\dot{B}$ characterizes the change of the field within the radiation formation time τ. While saving only the zeroth order of $\dot{B}\tau/B$ in the radiation formula, Baier and Katkov reproduce the well-known expressions in both classical and quantum regimes. Our task is to retain higher terms in the expansion of $\dot{B}\tau/B$, and apply to our specific problem.

In Sec. 2 we review briefly the Baier–Katkov method. We then derive the extra term for the radiation formula up to the order $(\dot{B}\tau/B)^2$ for head-tail symmetric inhomogeneous fields in Sec. 3. Our result shows a reduction from the leading Sokolov–Ternov contribution in the quantum regime and no effect in the clasical regime. A physical argument is given to explain these facts. The expression is then applied to the specific numerical example of Himel and Siegrest in Sec. 4. Due to the limitation of our perturbative approach to the Baier–Katkov method, however, we are only able to estimate the upper bound of the radiation reduction due to the "slope" effect, which gives a lower bound of average fractional energy loss $\langle\epsilon\rangle \gtrsim 10.2\%$ for the Himel–Siegrest parameters.

2. BAIER–KATKOV METHOD

Our starting point is the Baier–Katkov method of radiation calculation [9,10]. A similar method had been used earlier by Schwinger [11]. The method is based on the realization that when the radiating particle is ultrarelativistic, its radiation in a magnetic field is a quasi-classical problem. By that we mean the motion of an electron becomes more and more "classical" as its energy increases that it makes sense to describe the particle by its trajectory. The radiation is therefore viewed as induced by the bending of the trajectory. The only role that quantum physics plays is the noncommutativity between the electron field and the photon field, and the conservation of initial and final energies in a discrete manner. The general expression of radiation intensity (in the Coulomb gauge) is

$$I = \alpha \int \frac{d^3k}{(2\pi)^2} \langle i| \int dt_1 \int dt_2\, e^{i\omega(t_1-t_2)} M^* \;(t_2)\, M\,(t_1)\, |f\rangle \quad , \tag{2}$$

where $\alpha = 1/137$ is the fine structure constant, $(\omega, \vec{k})$ the four-momentum of the photon, $\langle i|$, $\langle f|$ the initial and final states of electron, respectively, and M the transition matrix. To the accuracy of the order of $1/\gamma$, Baier and Katkov show that the phase factor from M^*M

$$e^{i\vec{k}\cdot\vec{r}(t_2)} e^{i\vec{k}\cdot\vec{r}(t_1)} = \exp\left\{ i \left[\omega\tau + \frac{\mathcal{H}}{\mathcal{H}-\omega} \left(\vec{k}\cdot(\vec{r}(t_2) - \vec{r}(t_1)) - \omega\tau \right) \right] \right\} \quad , \tag{3}$$

where $\tau \equiv t_2 - t_1$ and $t = t_1 + t_2$, commutes with both the Hamiltonian $\mathcal{H}$ and the electron momentum $\vec{p}$. After summing over the spins of the final electron and polarizations of the photon, and averaging over the initial electron spins, the radiation intensity can be written as

$$\frac{dI}{dt} = \frac{\alpha}{(2\pi)^2} \int d^3k \int_{-\infty}^{\infty} d\tau\, G(\vec{v}(t_1), \vec{v}(t_2)) \exp\left\{ i \left[\omega\tau + \frac{\mathcal{E}}{\mathcal{E}'} \left(\vec{k} \cdot (\vec{r}(t_2) - \vec{r}(t_1)) - \omega\tau \right) \right] \right\} \quad , \tag{4}$$

where $\mathcal{E}$ and $\mathcal{E}'$ are the initial and final energies of the electron and

$$\begin{aligned} G(\vec{v}(t_1), \vec{v}(t_2)) = \frac{1}{4} \Bigg[& \left(1 + \frac{\mathcal{E}}{\mathcal{E}'}\right)^2 (\vec{v}(t_2) \cdot \vec{v}(t_1) - 1) \\ & + \left(\frac{\omega}{\mathcal{E}'}\right)^2 \left(\vec{v}(t_2) \cdot \vec{v}(t_1) - 1 + \frac{2}{\gamma^2} \right) \Bigg] \quad . \end{aligned} \tag{5}$$

From now on we will simplify the notations by designating $\vec{v}_1$ and $\vec{v}_2$ for $\vec{v}(t_1)$ and $\vec{v}(t_2)$, respectively. Similar notations apply for $\vec{r}(t)$. It is observed that the dominant contribution of the τ integration in Eq. (4) comes from the value at $\dot{v}\tau \sim 1/\gamma$. This corresponds to the situation where the electron position vector has swept through an angle $1/\gamma$, or correspondingly the outcoming photon lies within an open cone of angle $1/\gamma$. We shall call this period of time the radiation formation time τ, and the corresponding distance of travel by the electron the radiation formation length, ℓ_R. Since $1/\gamma \ll 1$ we can Taylor expand $\vec{v}_2$ and $\vec{r}_2$ in terms of $\vec{v}_1$ and $\vec{r}_1$:

$$\begin{aligned} \vec{v}_1 \cdot \vec{v}_2 &= \vec{v}_1 \cdot \left[\vec{v}_1 + \dot{\vec{v}}_1\tau + \frac{1}{2}\ddot{\vec{v}}_1\tau^2 + \frac{1}{6}\dddot{\vec{v}}_1\tau^3 + \ldots \right] \quad , \\ \vec{k} \cdot (\vec{r}_2 - \vec{r}_1) &= \vec{k} \cdot \left[\vec{v}_1\tau + \frac{1}{2}\dot{\vec{v}}_1\tau^2 + \frac{1}{6}\ddot{\vec{v}}_1\tau^3 + \ldots \right] \quad . \end{aligned} \tag{6}$$

In their paper [9] Baier and Katkov truncated the expansion at $\ddot{\vec{v}}_1\tau^2$, thus the assumption was

$$\frac{(1/6)|\dddot{\vec{v}}_1|\tau^3}{(1/2)|\ddot{\vec{v}}_1|\tau^2} \ll 1 \quad . \tag{7}$$

Since $B \propto \dot{v}$ in a magnetic field, and $\vec{v}^2 =$ constant, we have $\vec{v} \cdot \dot{\vec{v}} = 0$. Taking time derivatives successively, we have

$$\vec{v} \cdot \ddot{\vec{v}} = -\dot{\vec{v}} \cdot \dot{\vec{v}} \quad , \quad \vec{v} \cdot \dddot{\vec{v}} = -3\dot{\vec{v}} \cdot \ddot{\vec{v}} \quad , \text{etc.} \tag{8}$$

Using these relations the assumption can be translated into $\dot{B}\tau/B \ll 1$, as is in Eq. (1). Now we define a dimensionless, Lorentz invariant parameter Υ:

$$\Upsilon = \gamma \cdot \frac{B}{B_c} = \frac{2}{3}\frac{\omega_c}{\mathcal{E}} = \frac{\dot{v}\gamma^3}{\mathcal{E}} \quad , \tag{9}$$

where $B_c = m^2c^3/e\hbar \simeq 4.4 \times 10^{13}$ Gauss is the Schwinger critical field strength, and ω_c is the critical frequency in classical sychrotron radiation. The radiation intensity for electrons in an inhomogenous field satisfying Eq. (1) can then be obtained in terms of Υ:

$$\frac{dI_0}{dt} = \begin{cases} \frac{2}{3}\alpha m^2\Upsilon^2 \left(1 - \frac{55\sqrt{3}}{16}\Upsilon + 48\Upsilon^2 + \ldots\right) \quad , & \Upsilon \ll 1 \quad . \\ \frac{32}{3^5\pi}\Gamma\left(\frac{2}{3}\right) \alpha m^2 (3\Upsilon)^{2/3} + \ldots \quad , & \Upsilon \gg 1 \quad . \end{cases} \tag{10}$$

In the above equation the expression for $\Upsilon \ll 1$ is the well-known formula for classical synchrotron radiation, including the quantum correction first derived by Schwinger [11], and independently by Sokolov, Klepikov and Ternov [12], and higher terms in Υ. The expression for $\Upsilon \gg 1$ corresponds to the synchrotron radiation in the extreme quantum limit studied by many people, but we will from now on simply call it Sokolov–Ternov formula [3]. The fact that Baier and Katkov reproduce these formulas in a straightforward manner suggests the power of this method.

3. RADIATION FROM INHOMOGENEOUS FIELDS

Consider a magnetic field that points to the direction transverse to the axis where an electron enters, and its strength that varies along the axis. Let $t = 0$ when the electron passes the geometric center of the field. We are interested in the case where the field variation is such that $B(t)$ is an even function in t, which is also called head-tail symmetric. Since from Lorentz force $\dot{v} \propto B(t)$, we see that $\ddot{v} \propto \dot{B}(t)$ is an odd function in t. Therefore, in the study of radiation from a head-tail symmetric inhomogeneous magnetic field, the terms linear in $\ddot{v}$ would vanish when integrating over t. This means the leading correction term is of the order $\ddot{v}^2$. We should thus retain the Taylor expansion in the integrand G up to the term $\vec{v}_1 \cdot \ddddot{\vec{v}}_1 \tau^4$ where the recurrence relation

$$\vec{v} \cdot \ddddot{\vec{v}} = -3\ddot{\vec{v}} \cdot \ddot{\vec{v}} - 4\dot{\vec{v}} \cdot \dddot{\vec{v}} \quad , \tag{11}$$

which is obtained from one more derivative on Eq. (8), links the term with $\ddot{\vec{v}} \cdot \ddot{\vec{v}}$ and $\dot{\vec{v}} \cdot \dddot{\vec{v}}$ where both are even functions in time.

As for the phase, retaining terms up to $\ddot{\vec{v}} \cdot \ddot{\vec{v}}$ we have

$$\exp\left\{i\left[\omega\tau + \frac{\mathcal{E}}{\mathcal{E}'}\left(\vec{k} \cdot (\vec{r}_2 - \vec{r}_1) - \omega\tau\right)\right]\right\} = \exp\left\{i\left(\Phi_0 + \Phi_1\right)\right\} \quad , \tag{12}$$

where

$$\Phi_0 = u\mathcal{E}\tau\left[1 - \hat{n} \cdot \vec{v} - \frac{1}{2}\hat{n} \cdot \dot{\vec{v}}\tau + \frac{1}{6}\dot{\vec{v}} \cdot \dot{\vec{v}}\tau^2\right] \quad ,$$

and $u \equiv \omega/\mathcal{E}'$, $\hat{n} \equiv \vec{k}/\omega$, is the phase angle that gives rise to Eq. (10) in the previous section, and

$$\Phi_1 = u\mathcal{E}\tau\left[\frac{1}{8}\frac{\dot{\vec{v}} \cdot \ddot{\vec{v}}}{\dot{v}^3}\dot{v}^3\tau^3 + \frac{1}{120}\left(3\frac{\ddot{\vec{v}} \cdot \ddot{\vec{v}}}{\dot{v}^4} + 4\frac{\dot{\vec{v}} \cdot \dddot{\vec{v}}}{\dot{v}^4}\right)\dot{v}^4\tau^4\right]$$

is the additional phase that we retain. Notice that in Φ_1 and the last term in Φ_0 we had made the approximation of replacing $\hat{n}$ by $\vec{v}$.

We further assume that $\Phi_1 \ll 1$, which is usually satisfied if only $u \not\gg 1$, or the final energy of the electron $\mathcal{E}' \gg m$. This does not introduce extra assumption since the Baier–Katkov method has already assumed relativistic electron before and after emitting the photon. Therefore we make the following approximation:

$$\exp\left\{-i\left(\Phi_0 + \Phi_1\right)\right\} \simeq \left(1 - i\Phi_1\right)\exp\left\{-i\Phi_0\right\} \quad . \tag{13}$$

Retaining terms to the same order in the integrand G, and combining with Eq. (13), we find the integrand to be

$$G = G_0 + G_1 + G_2 \quad , \tag{14}$$

where

$$G_0 = -\frac{1}{\gamma^2}(1+u) - \frac{1}{2}\left(1 + u + \frac{u^2}{2}\right)\dot{v}^2\tau^2$$

is the part that reproduces the Sokolov–Ternov formula, $G_1 \propto \dot{B}\tau/B$ is an odd function in time and would give zero contribution for head-tail symmetric fields, and G_2 is

$$\begin{aligned} G_2 = &-\left(1 + u + \frac{u^2}{2}\right)\left(\frac{1}{8}\frac{\dot{B}^2}{B^2} + \frac{1}{6}\frac{\ddot{B}}{B^3}\right)\dot{v}^4\tau^4 \\ &+ i\frac{u\mathcal{E}}{120}\left(\frac{1+u}{\gamma^2}\right)\left(3\frac{\dot{B}^2}{B^5} + 4\frac{\ddot{B}}{B^4}\right)\dot{v}^5\tau^5 \\ &+ i\frac{u\mathcal{E}}{120}\left(1 + u + \frac{u^2}{2}\right)\left(9\frac{\dot{B}^2}{B^5} + 2\frac{\ddot{B}}{B^4}\right)\dot{v}^7\tau^7 \quad . \end{aligned} \tag{15}$$

In the above expression the vector products $\dot{\vec{v}} \cdot \ddot{\vec{v}}$ and $\dot{\vec{v}} \cdot \dddot{\vec{v}}$ have been replaced by $B\dot{B}$ and $B\ddot{B}$.

This is because the only components that $\ddot{\vec{v}}$ and $\dddot{\vec{v}}$ contribute are proportional to $\vec{v}\times\dot{\vec{B}}$ and $\vec{v}\times\ddot{\vec{B}}$, respectively.

Following the mathematical techniques used by Baier and Katkov [9], we introduce angles θ and φ, where θ is the angle between the unit vector $\hat{n}$ of photon propagation and the plane $(\vec{v},\dot{\vec{v}})$, and φ is the angle between the projection of $\hat{n}$ on $(\vec{v},\dot{\vec{v}})$ and $\vec{v}$, i.e.,

$$\hat{n}\cdot\vec{v} = v\cos\varphi\sin\theta \quad , \quad \hat{n}\cdot\dot{\vec{v}} = \dot{v}\sin\varphi\cos\theta \quad . \tag{16}$$

Taking into account the fact that up to terms of highest order in $1/\gamma^2$ the principal contribution comes from small θ and φ, and by shifting the origin of τ to $\tau+\varphi/\dot{v}$, the phase can be written as

$$\begin{aligned}\Phi_0 &= u\mathcal{E}\left[(1-\hat{n}\cdot\vec{v})\,\tau - \frac{1}{2}\,\vec{n}\cdot\dot{\vec{v}}\,\tau^2 + \frac{1}{6}\,\dot{\vec{v}}^2\tau^3\right] \\ &= \frac{u\mathcal{E}\mu^{3/2}}{2\dot{v}}\left(x+\frac{1}{3}\,x^3+y+\frac{1}{3}\,y^3\right) \quad ,\end{aligned} \tag{17}$$

where

$$\mu \equiv 1 - v^2\cos^2\theta \simeq \frac{1}{\gamma^2}+\theta^2 \quad ,$$

and

$$x = \frac{1}{\sqrt{\mu}}\varphi \quad \text{and} \quad y = \frac{1}{\sqrt{\mu}}\dot{v}\tau \quad .$$

With the definition of Υ in Eq. (9) the coefficients in the phase can be symbolized by

$$b = \frac{3}{2}\eta = \frac{\alpha}{2\Upsilon}\left(\gamma^2\mu\right)^{3/2} \quad .$$

The radiation intensity associated with head-tail symmetric inhomogeneous field is then

$$\frac{dI_2}{dt} = \frac{4\alpha}{(2\pi)^2}\int k^2 dk d\sin\theta\frac{\mu}{\dot{v}}\int_0^\infty dx\int_0^\infty dy\; G_2\exp\left\{-ib\left(x+\frac{1}{3}\,x^3+y+\frac{1}{3}\,y^3\right)\right\} \quad . \tag{18}$$

Recall that $u=\omega/\mathcal{E}'=\omega/(\mathcal{E}-\omega)$, and $k^2dk=\omega^2d\omega$, we find that

$$k^2dk = \frac{\mathcal{E}^3u^2du}{(1+u)^4} \quad . \tag{19}$$

The intergrations over x and y give Bessel functions of fractional order $K_{1/3}(\eta)$ and $K_{2/3}(\eta)$. For the evaluation of the integral over u it is convenient to introduce the representation [13]

$$\frac{1}{(1+u)^m} = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma(-s)\Gamma(m+s)}{\Gamma(m)}u^s ds \quad , \tag{20}$$

where $1-m<\lambda<0$. After this transformation the integration over u turns the Bessel functions into gamma functions, multiplied by a factor $(\gamma^2\mu)^{-3(s+n)/2}$ among other things. We can then carry out integration over $\sin\theta\approx\theta$ by the following formula [13]:

$$\int_{-\infty}^{\infty}(\gamma^2\mu)^{-3(s+n)/2}\gamma d\theta = \int_{-\infty}^{\infty}(1+\gamma^2\theta^2)^{3(s+n)/2}\gamma d\theta = \sqrt{\pi}\frac{\Gamma(3s/2+(3n-1)/2)}{\Gamma(3s/2+3n/2)} \quad . \tag{21}$$

All integrations in Eq. (18) are straightforward, though tedious. The result before carrying out the final integration over s is

$$\frac{dI_2}{dt} = \frac{\alpha\gamma^2}{\pi^{3/2}}\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} ds\, 2^s(3\Upsilon)^s \frac{\Gamma(3s/2+1)}{\Gamma(3s/2+3/2)}\left[\frac{\Gamma(s+4)}{\Gamma(4)}+\frac{\Gamma(s+2)}{\Gamma(2)}\right]\frac{\Gamma(-s)}{\Gamma(s+3)}$$

$$\times\Bigg\{-\left(\frac{9}{2}\frac{\dot{B}^2}{B^2}+6\frac{\ddot{B}}{B}\right)\Bigg[\Gamma\left(\frac{s}{2}+\frac{11}{6}\right)\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)\Gamma\left(\frac{s}{2}+\frac{7}{6}\right)$$

$$-\frac{1}{2}\frac{\Gamma(s+3)}{\Gamma(s+2)}\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)\Gamma\left(\frac{s}{2}+\frac{7}{6}\right)\Gamma\left(\frac{s}{2}+\frac{5}{6}\right)\Gamma\left(\frac{s}{2}+\frac{3}{6}\right)$$

$$+\Gamma\left(\frac{s}{2}+\frac{13}{6}\right)\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)\Gamma\left(\frac{s}{2}+\frac{5}{6}\right)\Bigg] \tag{22}$$

$$+\left(\frac{81}{40}\frac{\dot{B}^2}{B^2}+\frac{9}{20}\frac{\ddot{B}}{B}\right)\Bigg[\frac{256}{3}\Gamma\left(\frac{s}{2}+\frac{11}{6}\right)\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)\Gamma\left(\frac{s}{2}+\frac{7}{6}\right)$$

$$+\frac{224}{3}\Gamma\left(\frac{s}{2}+\frac{13}{6}\right)\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)\Gamma\left(\frac{s}{2}+\frac{5}{6}\right)$$

$$-\frac{128}{s+3}\Gamma\left(\frac{s}{2}+\frac{15}{6}\right)\Gamma\left(\frac{s}{2}+\frac{13}{6}\right)\Gamma\left(\frac{s}{2}+\frac{11}{6}\right)\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)$$

$$-\frac{216}{9}(s+2)\Gamma\left(\frac{s}{2}+\frac{9}{6}\right)\Gamma\left(\frac{s}{2}+\frac{7}{6}\right)\Gamma\left(\frac{s}{2}+\frac{5}{6}\right)\Gamma\left(\frac{s}{2}+\frac{3}{6}\right)\Bigg]\Bigg\} .$$

where $-1 < \lambda < 0$. The above expression includes only contributions from the $\dot{v}^4\tau^4$ and $\dot{v}^7\tau^7$ terms in Eq. (15) because it can be shown that the contribution from the $\dot{v}^5\tau^5$ term is significantly smaller, and thus negligible.

The integral over s can be evaluated by closing the contour of integration either to the right for $\Upsilon \ll 1$, or to the left for $\Upsilon \gg 1$. For $\Upsilon \ll 1$, we have

$$\frac{dI_2}{dt} = 0 \quad , \quad \Upsilon \ll 1 \quad , \tag{23}$$

identically. For $\Upsilon \gg 1$ we have, to the leading order in Υ,

$$\frac{dI_2}{dt} = -\frac{\alpha\gamma^2}{\sqrt{\pi}}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{6}\right)\frac{41}{54}\left(\frac{61}{60}\frac{\dot{B}^2}{B^2}-\frac{13}{90}\frac{\ddot{B}}{B}\right)(6\Upsilon)^{-2/3} \quad , \quad \Upsilon \gg 1 \quad . \tag{24}$$

This result is valid for any head-tail symmetric inhomogeneous magnetic field which satisfies the assumptions given previously.

Now we apply Eq. (24) to the field from a relativistic gaussian bunch with standard deviation σ_z:

$$B(t) = B_0 e^{-2t^2/\sigma_z^2} \quad , \tag{25}$$

where the time of flight of the test electron traversing the oncoming bunch is $t = z/2$. Then we get

$$\frac{dI_2}{dt} = -\frac{32\,\Gamma(2/3)}{3^5}\alpha m^2(3\Upsilon)^{2/3}\left\{\frac{\Gamma(1/6)}{360\sqrt{\pi}}F^2\left[12997\left(\frac{2t}{\sigma_z}\right)^2 - 1804\right]\right\} \quad , \quad \Upsilon \gg 1 \quad , \tag{26}$$

where $F \equiv \ell_R/\sigma_z$ is the formation length parameter associated with $\ell_R(\omega)$ in the quantum limit for photon frequency $\omega = \mathcal{E}$:

$$\ell_R(\omega = \mathcal{E}) = \frac{\lambda\!\!\!^{-}_c \gamma}{\Upsilon}\left(\frac{\omega_c}{\mathcal{E}}\right)^{1/3} = \left(\frac{3}{2}\right)^{1/3} \frac{\lambda\!\!\!^{-}_c \gamma}{\Upsilon^{2/3}} \quad . \tag{27}$$

Combining Eqs. (23) and (26) with Eq. (10), we obtain

$$\frac{dI}{dt} = \frac{dI_0}{dt} + \frac{dI_2}{dt} = \begin{cases} \frac{2}{3}\alpha m^2 \Upsilon^2 \left(1 + \frac{55\sqrt{3}}{16}\Upsilon + 48\Upsilon^2 + \ldots\right) \quad , & \Upsilon \ll 1 \quad , \\ \frac{32\,\Gamma(2/3)}{3^5}\alpha m^2 (3\Upsilon)^{2/3} \left\{1 - \frac{\Gamma(1/6)}{360\sqrt{\pi}} F^2 \left[12997\left(\frac{2t}{\sigma_z}\right)^2 - 1804\right]\right\} \quad , & \Upsilon \gg 1 \quad . \end{cases} \tag{28}$$

Our result can be appreciated by the following physical arguments: Consider the differential radiation intensity $P(\omega)$ where

$$\frac{dI}{dt} = \int_0^{\mathcal{E}} P(\omega) d\omega \quad . \tag{29}$$

In the classical limit $P(\omega)$ in the case of a uniform field scales as

$$P(\omega) \sim \begin{cases} \omega^{1/3} \quad , & \omega \lesssim \omega_c \quad , \\ \frac{\omega}{\omega_c} e^{-\omega/\omega_c} \quad , & \omega \gtrsim \omega_c \quad , \end{cases} \tag{30}$$

as shown by the solid curve in Fig. 1. As is introduced in Eq. (9), classical limit $\Upsilon \ll 1$ corresponds to the situation $\omega_c \ll \mathcal{E}$, meaning the typical frequency of radiated photons is much less than the kinetic energy of the radiating particles. Thus the entire spectrum of Eq. (30) is observable. On the contrary, the extreme quantum limit $\Upsilon \gg 1$ corresponds to $\mathcal{E} \ll \omega_c$, therefore the spectrum beyond the electron energy is kinematically forbidden, and the observable spectrum scales roughly as $\omega^{1/3}$.

In the case of nonuniform fields the spectrum differs from that of uniform fields. In the classical limit the problem has been studied by Coïsson [14], and independently by Bagrov, Fedosov and Ternov [15]. It is found that for a short magnet which is comparable in length with ℓ_R, the radiation spectrum is modified in such a way that the low-frequency regime is suppressed in favor of high frequencies beyond ω_c. The total intensity, however, remains the same. The prediction was confirmed by Bossart et al. [16] with observations in SPS at CERN. We can extrapolate this fact by suggesting that when the magnet length $L^* \ll \ell_R$, the spectrum would be a constant independent of ω up to a maximum frequency $\omega^* \sim \omega_c(\ell_R/L^*)$ (see the dashed curve in Fig. 1). Our result for the classical limit shows that the total intensity dI/dt is the same for uniform and gaussian fields. This is a confirmation of the previous studies.

The situation for short magnets is different in the quantum limit. Again, spectrum beyond $\mathcal{E}$ is energetically forbidden. But now that the low frequency regime is suppressed, the overall intesnity is reduced. This explains why our dI_2/dt is opposite in sign from dI_0/dt. From Eq. (28) it can be seen that when $\ell_R \ll \sigma_z$, or when the bunch is very long, $dI_2/dt \to 0$, and we have vanishing correction to the Sokolov–Ternov formula. A pronounced effect occurs when ℓ_R is not much smaller than σ_z.

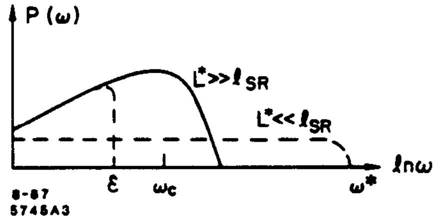

Fig. 1. Radiation spectrum in the two asymptotic limits. For long magnets, $L^* \gg \ell_R$, we have the well-known spectrum in solid curve. In the opposite limit $L^* \ll \ell_R$, the spectrum approaches a constant. In quantum limit we observe only the low frequency regime.

4. QUANTUM BEAMSTRAHLUNG

We now apply Eq. (28) to a specific example. In order to appreciate the slope effect, we choose to calculate the same set of beam parameters first discussed by Himel and Siegrest, where the Lorentz factor for 5 TeV beams is $\gamma = 1 \times 10^7$, number of particles per bunch $N = 1.2 \times 10^8$, bunch size $\sigma_z = 0.4\mu\text{m}$ and $\sigma_r = 2.5$ Å.

To focus on the longitudinal effect, we assume "cylindrical gaussian" bunches, i.e., uniform density in $r \le 2\sigma_r$ and gaussian in z. Then

$$\Upsilon(\rho, \varsigma) = \Upsilon_0 \rho e^{-\varsigma^2/2} \quad , \tag{33}$$

where $\rho = r/\sigma_r$, $\varsigma = z/\sigma_z$ are the normalized coordinates, and

$$\Upsilon_0 = \frac{r_e \lambda\!\!\!^{-}_e \gamma N}{\sqrt{2\pi}\sigma_r \sigma_z} = 5094 \gg 1 \tag{34}$$

is the reference beamstrahlung parameter corresponding to twice the field strength (i.e., $|\vec{B}| \simeq |\vec{E}|$) [17] at $(\rho, \varsigma) = (1, 0)$ in the target bunch. The formation length parameter F is also a function of ς and ρ:

$$F(\rho, \varsigma) = \left(\frac{3}{2}\right)^{1/3} \frac{\lambda\!\!\!^{-}_c \gamma}{\Upsilon^{2/3} \sigma_z} = \left[\left(\frac{3}{2}\right)^{1/3} \frac{\lambda\!\!\!^{-}_c \gamma}{\Upsilon_0^{2/3} \sigma_z}\right] \rho^{-2/3} e^{\varsigma^2/3} = F_0 \rho^{-2/3} e^{\varsigma^2/3} \quad , \tag{35}$$

where the reference formation length parameter

$$F_0 = \left(\frac{3}{2}\right)^{1/3} \frac{\lambda\!\!\!^{-}_c \gamma}{\Upsilon_0^{2/3} \sigma_z} = \frac{0.015\ \mu\text{m}}{0.4\ \mu\text{m}} = 0.0375 \quad . \tag{36}$$

Let us first calculate the average energy loss based on Sokolov–Ternov formula (i.e., dI_0/dt). Let $\epsilon = (\mathcal{E} - \mathcal{E}')/\mathcal{E}$ be the fractional energy loss of an electron having impact parameter ρ. Then the average fractional energy loss of the entire bunch is

$$\langle \epsilon \rangle = \frac{1}{\mathcal{E}} \frac{\int \int (dI_0/dt) \rho d\rho dt}{\int \rho d\rho} \tag{37}$$

for our cylindrical gaussian bunches. Replacing dt by $(\sigma_z/2) d\varsigma$, and defining

$$\Gamma_0 \equiv \frac{\lambda\!\!\!^{-}_c \gamma}{\sigma_z} \quad , \tag{38}$$

we find for the leading Sokolov–Ternov term

$$\langle\epsilon_0\rangle_\infty = \frac{16\cdot\Gamma(2/3)}{243}\cdot\frac{\alpha}{\Gamma_0}\cdot(3\Upsilon_0)^{2/3}\langle\rho\rangle^{2/3}\int_{-\infty}^{\infty} e^{-\varsigma^2/3}d\varsigma \quad , \tag{39}$$

where the mean impact parameter

$$\langle\rho\rangle \equiv \left[\frac{\int_0^2 \rho^{5/3}d\rho}{\int_0^2 \rho d\rho}\right]^{3/2} = 1.30$$

in our example. For the Himel–Siegrest parameters we have

$$\langle\epsilon_0\rangle_\infty = 15.2\% \quad , \tag{40}$$

which agrees reasonably well with previous calculations [2,4].

To include the correction term we should realize that our perturbation breaks down before dI_0/dt and dI_2/dt becomes equal in magnitude at some point $\varsigma = \varsigma_c$ from the centroid of the bunch, beyond which the total intensity would turn negative and be certianly unphysical. Since we lack the knowledge on the behavior of higher order terms, we can only estimate the upper bound of the reduction effect by extending dI_2/dt all the way to ς_c and assuming total suppression beyond that point, as shown schematically in Fig. 2. From Eq. (28) this threshold occurs at

$$\frac{\Gamma(1/6)}{360\sqrt{\pi}}\left(\frac{\ell_{R0}}{\sigma_z}\right)^2 \rho^{-4/3}e^{2\varsigma_c^2/3}\left(12997\varsigma_c^2 - 1804\right) = 1 \quad . \tag{41}$$

From this equation it is obvious that the cut-off ς_c is radial dependent. For the sake of simplicity in our discussion, we make a further approximation by evaluating ς_c at the mean impact parameter $\langle\rho\rangle = 1.30$, and we get

$$\varsigma_c = 1.49 \quad . \tag{42}$$

Thus the mean radiation loss is suppressed to

$$\langle\epsilon\rangle_{\varsigma_c} = \langle\epsilon_0\rangle_{\varsigma_c} + \langle\epsilon_2\rangle_{\varsigma_c} \quad , \tag{43}$$

where

$$\langle\epsilon_0\rangle_{\varsigma_c} = \frac{16\cdot\Gamma(2/3)}{243}\frac{\alpha}{\Gamma_0}(3\Upsilon_0)^{2/3}\left[\langle\rho\rangle^{2/3}\cdot\int_{-\varsigma_c}^{\varsigma_c} e^{-\varsigma^2/3}d\varsigma\right] \quad .$$

and

$$\langle\epsilon_2\rangle_{\varsigma_c} = -\frac{16\Gamma(2/3)}{243}\frac{\alpha}{\Gamma_0}(3\Upsilon_0)^{2/3}\left[\frac{\Gamma(1/6)}{360\sqrt{\pi}}F_0^2\langle\rho\rangle^{-2/3}\int_{-\varsigma_c}^{\varsigma_c} e^{\varsigma^2/3}(12997\varsigma^2 - 1804)d\varsigma\right] \quad .$$

Plugging in numbers we get

$$\langle\epsilon_0\rangle_{\varsigma_c} = 0.78\langle\epsilon_0\rangle_\infty = 11.8\% \tag{44}$$

and

$$\langle\epsilon_2\rangle_{\varsigma_c} = -0.11\langle\epsilon_0\rangle_\infty = -1.6\% \quad . \tag{45}$$

Thus the corrected quantum beamstrahlung average fractional energy loss is

$$\langle\epsilon\rangle \gtrsim \langle\epsilon\rangle_{\varsigma_c} = 10.2\% \quad . \tag{46}$$

This is substantially different from the previous results.

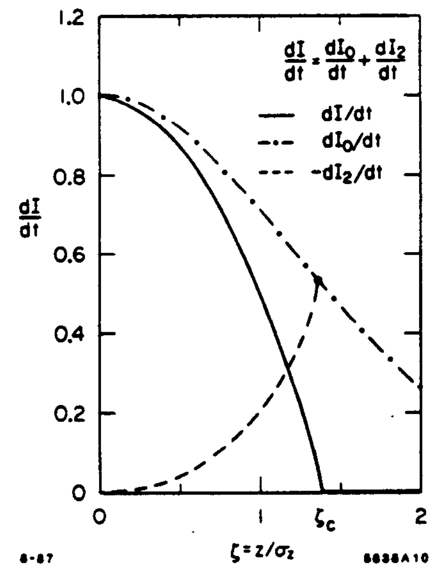

Fig. 2. Radiation intensities as function of longitudinal target bunch coordinate ς. The dash-dot curve is the Sokolov–Ternov radiation. The dash curve is the negative of our gaussian slope correction. The net intensity is represented by the solid curve. Beyond the point ς_c where dI_0/dt and - dI_2/dt meet, we assume a total suppression.

5. DISCUSSION

Due to the constraints of the perturbative approach to the gaussian slope correction, we can only estimate the upper bound of the reduction. The pathology lies in that while the B–field strength decreases exponentially to the head and the tail in the oncoming target bunch, $\dot{B}\tau$ increases more than exponentially. These facts force $\dot{B}\tau/B$ ceases to be much less than one at some point. The symptom is actually rather generic in beamstrahlung. For example, consider replacing the gaussian distribution by a parabolic one. Though we may have the advantage of terminating the field at some finite distance, but the fact that B–field vanishes forces upon us that somewhere before the bunch ends, $\dot{B}\tau/B \ll 1$ must be violated.

As long as the cut-off ς_c is several standard deviations from the bunch center, however, the correction that we calculated would be valid since the field beyond ς_c would contribute very little to the radiation in the first place. For the case of Himel–Siegrest parameters the situation is a little awkward. As shown in the previous section, the average energy loss within ς_c accounts for only $\sim$ 78% of the total loss based on Sokolov–Ternov formula. To improve the calculation, methods other than perturbation should be pursuit.

Aside from this technical difficulty, the physics involved is rather clear: In addition to the beamstrahlung parameter Υ_0, one more parameter, the radiation formation length parameter F_0, is essential in determining beamstrahlung properties. When F_0 approaches unity, the reduction of the average energy loss in the quantum regime becomes nonnegligible. Actually, our claim is that not only $\langle\epsilon\rangle$ is reduced, but also the energy resolution is improved because there would be less hard photons radiated as can be seen from Fig. 1. These features suggest that the situation is in favor of ultrashort bunches in future linear colliders [18] if this is technically attainable.

We should point out that for the purpose of estimating the energy loss we calculated the second order correction in $\dot{B}\tau/B$ for an ideal symmetric bunch. In reality, the bunches would be continuously deformed during collision, and the head-tail symmetry is destroyed. In that case there will even be first order correction in $\dot{B}\tau/B$. Recently there has been interest to consider colliding beams at an angle in future linear colliders. In this scenario the head-tail symmetry is naturally broken, and the first order correction would appear inherently.

ACKNOWLEDGEMENTS

The author has benefitted from various helpful discussions with J. S. Bell and A. Hoffmann of CERN, T. T. Wu of Harvard, and R. Blankenbecler, S. D. Drell, T. Himel, R. D., Ruth and P. B. Wilson of SLAC. A special acknowledgement goes to K. Yokoya of KEK, who took the pain of checking some of the calculations involved in this paper.

REFERENCES

[1] See, for example, B. Richter, IEEE Trans. Nucl. Sci. **32**, 3828 (1985).

[2] T. Himel and J. Siegrest in *Laser Acceleration of Particles,* editors C. Joshi and T. Katsouleas, AIP Conf. Proceedings No. 130 (1985).

[3] A. A. Sokolov and I. M. Ternov, *Synchrotron Radiation*, Pergamon Press (1967).

[4] See Ref. 2. A trivial mistake of a factor of 2 was made by the authors.

[5] R. Blankenbecler and S. D. Drell, Phys. Rev. D **36**, 277 (1987).

[6] M. Bell and J. S. Bell, CERN Report (1987) [unpublished].

[7] R. J. Noble, SLAC–PUB–3871 (1986). To appear in Nucl. Inst. Meth.

[8] K. Yokoya, Nucl. Inst. Meth., **A251**, 1 (1986) and K. Yokoya, KEK Report 85–9 (1985).

[9] V. N. Baier and V. M. Katkov, Soviet Phys. JETP, **26**, 854 (1968).

[10] V. N. Baier and V. M. Katkov, Soviet Phys. JETP, **28**, 807 (1969).

[11] J. Schwinger, Proc. National Academy of Sci. **40**, 132 (1954).

[12] A. A. Sokolov, N. P. Klepikov and I. M. Ternov, Soviet Phys. Doklady, **89**, 665 (1953).

[13] I. S. Gradshteyn and I. M. Ryzlik, *Tables of Integrals, Series and Products,* Academic Press (1980).

[14] R. Coïsson, Optics Comm. **22**, 135 (1977) and Phys. Rev. A **20**, 524 (1979).

[15] V. G. Bagrov, N. I. Fedosov and I. M. Ternov, Phys. Rev. D **28**, 2464 (1983).

[16] R. Bossart, J. Bosser, L. Burnod, R. Coïsson, E. D'Amico, A. Hoffmann and J. Mann, Nucl. Inst. Meth. **164**, 374 (1979).

[17] The validity of this assumption in the quantum limit has been examined by P. Chen and R. J. Noble, SLAC–PUB–4050 (Aug. 1986).

[18] P. Chen, in *Physics at Future Accelerators,* ed. J. Mulvey, CERN 87–07, Vol. I (1987).