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The Heterotic String on the Simplest Calabi-Yau Manifold and its Orbifold Limits*

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ABSTRACT

We study the heterotic string on the simplest (non-trivial) Calabi-Yau manifold, K3, and its orbifold limits. We set the background gauge connection equal to the spin connection. The massless spectrum of the field theory that is the low energy limit of the $E_8 \otimes E_8$ string is derived on K3. Some terms in the effective Lagrangian for the massless modes are shown to be determined by the topology of K3. Several orbifold limits of K3 are described as T^4/Z_l , after the singular points are removed and replaced by well behaved spaces. Massless spectra of the full string theory on these orbifolds are obtained. Orbifold and manifold results are compared. Knowing how to blow up the orbifolds determines much of the full string massless spectrum, including information about the spectra of the individual twisted sectors.

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1. INTRODUCTION

The space of vacuum solutions to the heterotic string theory is enormous and there is no known way to decide which solution the string theory will choose as its preferred vacuum. More detailed knowledge of this space, and the solutions themselves, may help in solving this dilemma.

Two classes of vacuum solutions that yield quasi-realistic effective theories are those on Calabi-Yau manifolds[1] and those on orbifolds.[2,3] These two classes are related; many of the orbifold compactifications are singular limits of Calabi-Yau compactifications. It is important to understand their relation, since much more can be said about strings on orbifolds than about strings on Calabi-Yau manifolds; perhaps some of this information can be carried over to the Calabi-Yau case. If we restrict to these two classes, there is a unique theoretical laboratory. K3 is the simplest (non-trivial) Calabi -Yau manifold. It also has several distinct orbifold limits.

In this paper we will study heterotic string propagation on K3 and its orbifold limits. We will consider the $E_8 \otimes E_8$ string and derive the massless spectra obtained by setting the background gauge connection equal to the spin connection.[1,2,3] We will consider only those orbifold limits of K3 that are obtained as the four-dimensional torus T^4 with points identified under the action of a group Z_l. We denote these orbifold limits as K3(Z_l)=T^4/Z_l .

K3 has been discussed extensively in the physics literature. A review of its properties can be found in Ref. 4 . The simplest orbifold limit of K3, K3(Z_2), has been studied.[5-7] We are aware of no discussions of the other orbifold limits K3(Z_l), $l \neq 2$, and the spectra of heterotic strings on them.

The layout of this paper is as follows. In the second chapter, we analyse the propagation of the $E_8 \otimes E_8$ string on the manifold K3. On a non-flat manifold, one can only deal with the field theory that is the low energy limit of the heterotic string. In section 2.1, the massless spectrum obtained from this theory is derived using index theory.[8] In section 2.2 these results are confirmed by writing the internal wave functions of the massless modes in terms of the harmonic forms on K3. This knowledge is also put to use by deriving terms in the six-dimensional effective lagrangian that are determined by the topology of K3.

Chapter 3 is concerned with the orbifold limits of K3. In section 3.1 the orbifolds K3(Z_l) are described. Their relation to the manifold K3 is established. Since the orbifolds K3(Z_l) are flat except at isolated singular points, the full string theory (not just the low energy limit) can be solved.[2,3] This is done in section 3.2. The results are compared with the earlier results. Finally, chapter 4 is the conclusion.

2. LOW ENERGY FIELD THEORY ON K3

The authors of Ref. 1 showed that, assuming the background torsion vanishes, requiring simple supersymmetry in four dimensions demands that the internal manifold be a Ricci-flat Calabi-Yau space, provided the background gauge connection is set equal to the spin connection. Various consequences of this scenario have been worked out.[9–11] What we do in this chapter is the analogue of the work just mentioned for the case of a four-dimensional internal manifold. The motivation for doing this is increased simplicity, and we will in fact find that much more can be said in the simpler four-dimensional case.

2.1. Massless Spectrum by Index Theory

It can be verified that the analysis of Ref. 1 may be carried directly over to the case of a four-dimensional internal space. The preservation of simple supersymmetry in six dimensions requires that the internal space be a Calabi-Yau manifold with Ricci-flat metric tensor, assuming the background torsion vanishes. Although there is an enormous number of six-dimensional Calabi-Yau manifolds, the unique (non-trivial) four-dimensional Calabi-Yau space is K3, and the only two-dimensional example is the trivial torus. Thus K3 is the unique "test bed" for study of compactification on Calabi-Yau spaces.

K3 has Euler characteristic 24, with Betti numbers $b_0 = b_4 = 1$, $b_1 = b_3 = 0$, and $b_2 = 22$. A complex Riemann tensor in four dimensions that is Ricci-flat must be either self-dual or anti-self-dual. We choose an anti-self-dual Riemann curvature, so that the signature of K3 is $\tau = b_2^+ - b_2^- = -16$, *i.e.* the $b_2 = 22$ harmonic 2-forms can be decomposed into $b_2^+ = 3$ self-dual and $b_2^- = 19$ anti-self-dual harmonic 2-forms. Since the Ricci-flat metric is Kähler, the harmonic forms can also be decomposed into harmonic forms with complex indices, counted by the Hodge numbers h^{pq} . The nonzero h^{pq} are $h^{00} = h^{22} = h^{02} = h^{20} = 1$ and $h^{11} = 20$. All the harmonic (1,1)-forms are anti-self-dual, except the self-dual Kähler form $K = \frac{i}{2} g_{j\bar{k}} dz^j \wedge dz^{\bar{k}}$. The remaining self-dual 2-forms are the (0,2)-form, given by the antisymmetric tensor ϵ_{ij}, and its complex conjugate (2,0)-form. The SU_2- invariant tensor ϵ_{ij} appears because the holonomy group of K3 is SU_2.

The analysis of Ref. 1 is done on the field theory that is the low energy limit of the heterotic string. This field theory is ten-dimensional super-Yang-Mills theory coupled to ten-dimensional supergravity.[12] The relevant supersymmetry multiplets are the ten-dimensional supergravity multiplet R(10), and super-Yang-Mills multiplet Y(10):

$$\begin{aligned} R(10) &= \{g_{MN}, \Psi_M^{(-)}, B_{MN}, \Psi^{(+)}, \varphi\} \\ Y(10) &= \{A_M, \Lambda^{(-)}\} \end{aligned} \tag{2.1}$$

R(10) contains the graviton g_{MN}, a negative chirality Majorana-Weyl gravitino $\Psi_M^{(-)}$, an antisymmetric tensor B_{MN}, a positive chirality Majorana-Weyl spinor $\Psi^{(+)}$, and the dilaton φ. Y(10) includes a vector A_M and a negative chirality Majorana-Weyl spinor $\Lambda^{(-)}$, both transforming in the adjoint representation $(248,1) \oplus (1,248)$ of the gauge group $E_8 \otimes E_8$.

The potential anomalies of this theory are cancelled because the antisymmetric tensor transforms non-trivially under the gauge and Lorentz groups.[13] Assuming the torsion H vanishes in the background implies that

$$dH = 0 = \frac{1}{30} TrF^2 - trR^2 \tag{2.2}$$

One solution of this constraint is to set the gauge connection equal to the spin connection, $A = \omega$, in the background. We will now derive the resulting massless spectrum in six dimensions obtained by doing this. This calculation has already been carried out, in Ref. 8.

Massless modes are solutions of the linearised field equations. With the vacuum of the form $M_6 \otimes K3$, M_6 being six-dimensional Minkowski space, these solutions can be written as direct products of tensors (or spinors) on M_6 with tensors (or spinors) on K3. The tensors (or spinors) on M_6 will be the wave functions of zero-mass particles in six dimensions provided the corresponding objects on K3 satisfy certain differential equations. So there is a one to one correspondence between massless six- dimensional particles and solutions of certain differential equations on the internal space, K3. The massless spectrum can be determined by counting the numbers of solutions to the internal differential equations. The internal differential equations are determined by the equations of motion of the ten-dimensional fields present in the theory.

The situation is made simpler by supersymmetry. All six-dimensional particles must be grouped into multiplets of simple supersymmetry. The possible massless multiplets are the supergravity multiplet R(6), the tensor supermultiplet T(6), the Yang-Mills supermultiplet Y(6), and the scalar supermultiplet S(6):

$$\begin{aligned} R(6) &= \{g_{\mu\nu}, \psi_\mu^{(+)}, B_{\mu\nu}^{(+)}\} \\ T(6) &= \{B_{\mu\nu}^{(-)}, \psi^{(-)}, \varphi\} \\ Y(6) &= \{A_\mu, \lambda^{(+)}\} \\ S(6) &= \{\chi^{(-)}, 4\phi\} \end{aligned} \tag{2.3}$$

R(6) contains a graviton $g_{\mu\nu}$, a Weyl gravitino $\psi_\mu^{(+)}$, and a self-dual antisymmetric tensor $B_{\mu\nu}^{(+)}$. T(6) contains an anti-self-dual antisymmetric tensor $B_{\mu\nu}^{(-)}$, a Weyl spinor $\psi^{(-)}$, and a scalar φ . Y(6) includes a vector A_μ and a gaugino $\lambda^{(+)}$; and the component fields of S(6) are a Weyl fermion $\chi^{(-)}$ and four scalars ϕ .

Since supersymmetry pairs bosons with fermions in the way we have just described, it is sufficient to count the massless fermions to deduce the full massless spectrum. To count the massless fermions, we use index theory. The index of a fermion is the difference in the numbers of normalisable negative chirality and positive chirality solutions of the equation of motion. At least this number of solutions (of the correct chirality) cannot be paired with solutions of opposite chirality, and so must be massless solutions. Assuming these are the only zero-mass solutions, we can derive the full massless spectrum.

Let us first use this method on the supergravity multiplet R(10).[14] Consider the spinor $\Psi^{(+)}(w) = \psi(x) \otimes \eta(y)$, where $\psi(x)$ is a spinor on M_6, whose coordinates we denote by x^μ, and $\eta(y)$ is a spinor on K3, whose coordinates are y^m. $\psi(x)$ will be a massless six-dimensional field provided $\eta(y)$ satisfies the zero-mass Dirac equation on K3 :

$$\gamma^m \nabla_m \, \eta(y) \;=\; \nabla\!\!\!\!/\, \eta(y) = \; 0.$$

Here ∇_m is the gravitational covariant derivative on K3. The Dirac index on K3 is[4]

$$I_{\nabla\!\!\!\!/}(K3) \;=\; n_{\frac{1}{2}}^{(-)} - n_{\frac{1}{2}}^{(+)} \;=\; \frac{1}{24(8\pi^2)} \int_{K3} tr R \wedge R \;=\; 2 \tag{2.4}$$

Each of the two unpaired negative chirality zero-modes on K3 give rise to what would have the right number of degrees of freedom to be a Majorana-Weyl positive chirality fermion in six dimensions. Majorana-Weyl fermions do not exist in six dimensions, however. So the two K3 zero-modes combine to result in one Weyl positive chirality six-dimensional fermion $\psi^{(+)}(x)$.

Similarly, the ten-dimensional gravitino $\Psi_m^{(-)}(w)$ gives rise to six-dimensional gravitinos through Dirac zero-modes on K3. This happens when the vector index M is restricted to the external space M_6 : $\Psi_\mu^{(-)}(w) = \psi_\mu^{(+)} \otimes \eta^{(-)}(y)$. The two Dirac zero-modes yield one positive chirality gravitino $\psi_\mu^{(+)}(x)$ in six dimensions.

When the gravitino index takes values on K3, $\Psi_m^{(-)}(w) \;=\; \chi(x) \otimes \psi_m(y)$, massless solutions $\psi_m(y)$ of the internal Rarita-Schwinger equation yield massless spin-$\frac{1}{2}$ fields $\psi(x)$ in six dimensions. The spin-$\frac{3}{2}$ (Rarita-Schwinger) index on K3 is

$$I_{\frac{3}{2}}(K3) \;=\; -21\, I_{\frac{1}{2}}(K3) \;=\; -42$$

But this index includes ghost contributions. There are just 40 positive chirality Rarita-Schwinger zero-modes on K3,[14] giving rise to 20 negative chirality spin-$\frac{1}{2}$ fields $\chi^{(-)}(x)$ in six dimensions.

The knowledge of these zero-modes is almost enough to give us the full massless spectrum arising from R(10). There is an ambiguity as to where to place the negative chirality fermions — in S(6) or T(6) multiplets. However, it is known[15] that there is no six-dimensional Lorentz-invariant action for a single unpaired self-dual or anti-self-dual tensor field $B^{(+)}_{\mu\nu}$ or $B^{(-)}_{\mu\nu}$. Only when they are paired into unconstrained tensors $B_{\mu\nu}$ can we write such an action. The original action for the ten-dimensional field theory is Lorentz invariant, and we have chosen a background $M_6 \otimes K3$ such that this invariance is not spontaneously broken. Therefore the single self-dual tensor $B^{(+)}_{\mu\nu}$ of R(6) must be paired with a single anti-self-dual tensor $B^{(-)}_{\mu\nu}$ of T(6). Therefore R(10) yields

$$R(10) \implies R(6) + T(6) + 20S(6). \tag{2.5}$$

We now move on to the Yang-Mills sector. The sole ten-dimensional fermion in this sector in the gaugino $\Lambda^{(-)}(w)$, transforming in the adjoint representation $(248,1) \oplus (1,248)$ of the gauge group $E_8 \otimes E_8$. With the background gauge connection set equal to the SU_2 spin connection on K3, the branching rule for the adjoint of $E_8 \otimes E_8$ into the maximal subgroup $E_7 \otimes SU_2 \otimes E_8$ is relevant:

$$(248,1) \oplus (1,248) = (133,1,1) \oplus (1,3,1) \oplus (56,2,1) \oplus (1,1,248) \qquad (2.6)$$

The SU_2 subgroup will be spontaneously broken, since the vacuum value of the gauge field is nonzero. Dirac zero-modes on K3 which are SU_2 singlets, doublets, and triplets will yield massless six-dimensional fields in the $(133,1) \oplus (1,248)$, (56,1), and (1,1) representations, respectively, of the unbroken gauge group $E_7 \otimes E_8$. The numbers of gauge nonsinglet zero-modes are given by the twisted Dirac index theorem on K3[4] :

$$I^{\mathbf{c}}_{\not{\nabla}}(K3) \;=\; n^{(-)}_{\frac{1}{2}}(\mathbf{c}) - n^{(+)}_{\frac{1}{2}}(\mathbf{c}) \;=\; \frac{1}{8\pi^2}\int\limits_{K3}\left(\frac{c}{24}trR \wedge R - tr_{\mathbf{c}}F \wedge F\right) \qquad (2.7)$$

$\mathbf{c}$ is the relevant gauge representation, and here it is a representation of SU_2. D_m is the covariant derivative on K3 including the SU_2 gauge connection. Setting the gauge connection equal to the spin connection gives $tr_{\mathbf{2}}\; F \wedge F \;=\; \frac{1}{2}tr\; R \wedge R$, since the $\mathbf{4}$ of SO_4 is two doublets of SU_2. Now $6tr_{\mathbf{c}}F \wedge F = (c-1)c(c+1)tr_{\mathbf{2}}F \wedge F$ so that using (2.4) we have

$$I^{\mathbf{c}}_{\not{\nabla}}(K3) = 2c(3 - 2c^2)$$

This result for c=1 agrees as it should with the untwisted Dirac index (2.4), and shows that we obtain positive chirality six-dimensional gauginos in the $(133,1) \oplus (1,248)$ representation of $E_7 \otimes E_8$. For **c=2**, we get 10 negative chirality fermions $\chi^{(-)}(x)$ in the (56,1) representation, and with **c=3** we obtain 45 negative chirality fermions $\chi^{(-)}$ that are gauge singlets.

In the Yang-Mills sector there is no ambiguity in placing fermions in the appropriate supermultiplets. The massless spectrum in the Yang-Mills sector is

$$\begin{aligned} Y(10) \Longrightarrow & Y(6)[(133,1) \oplus (1,248)]+ \\ & S(6)[10(56,1) \oplus 45(1,1)] \end{aligned}$$

so that the entire massless six-dimensional spectrum is

$$\begin{gathered} R(6)+T(6)+ \\ Y(6)[(133,1)\oplus(1,248)]+ \\ S(6)[10(56,1)\oplus 65(1,1)] \end{gathered} \tag{2.8}$$

Potential anomalies give us a quick check of our result. Compactification on a four-dimensional manifold will preserve the anomaly cancellation present in the ten-dimensional theory.[8] Anomalies in six dimensions are formally proportional to an 8-form. The only terms that cannot possibly be cancelled by the Green-Schwarz mechanism[13] are the "leading" gravitational and gauge anomalies, proportional to trR^4 and trF^4 , respectively.

The gauge anomaly can in our case be cancelled by the Green-Schwarz mechanism, because trF^4 is proportional to $(trF^2)^2$ for both gauge groups E_7 and E_8. The contributions of a positive chirality gravitino and a positive chirality spin-$\frac{1}{2}$ fermion to the leading gravitational anomaly are in the proportion 245:1.[16] With a single T(6) multiplet in the theory, this means that the total numbers, s and y, of S(6) and Y(6) multiplets must satisfy $s - y = 244$. This is satisfied by the spectrum of (2.8).

2.2. Zero-Modes and Harmonic Forms on K3

Many of the zero-modes we have just counted can be written explicitly in terms of objects known to exist, the harmonic forms on K3. In general, this not only confirms index theory results, but can also tell us the numbers of zero-mass positive and negative-chirality spinors separately (this happens in the six-dimensional Calabi-Yau case[1]). It gives information about bosonic zero-modes, whereas index theory can only under special circumstances, like when there is supersymmetry. Also, with the explicit forms of the zero-modes, we will be able to calculate some topological terms in the effective six-dimensional Lagrangian.[9,17]

As in the previous section, we first consider the gravitational sector, the fields in R(10). Perturbing the metric $g_{MN} \rightarrow g_{MN} + h_{MN}$ and insisting that the background remains Ricci-flat ($R_{MN} = 0$) gives the free graviton (Lichnerowicz) equation in a Ricci-flat background:

$$-\nabla^2 h_{MN} + 2R_{MNPQ}h^{PQ} = 0$$
$$\nabla^M h_{MN} = 0 \qquad g^{MN} h_{MN} = 0$$

For $h_{mn}(w) = \phi(x) \otimes h_{mn}(y)$, $\phi(x)$ is a massless scalar field if and only if h_{mn} satisfies the Lichnerowicz equation on K3. The solutions to this equation can be constructed from the 3 self-dual and 19 anti-self-dual harmonic 2-forms S_{mn} and A_{mn}. Being harmonic, A_{mn} and S_{mn} satisfy

$$\nabla^2 f_{mn} - R_{pqmn} f^{pq} = \nabla^2 f_{mn} + 2R_{mpqn} f^{pq} = 0$$

But the Riemann tensor is anti-self-dual so that the forms S_{mn} are covariantly constant. The exterior derivative d and its adjoint δ both annihilate the harmonic forms A_{mn}, so that

$$\nabla_m A_{np} + \nabla_p A_{mn} + \nabla_n A_{pm} = 0 \qquad \nabla^m A_{mn} = 0$$

Using these equations and the (anti-)self-dual property of the (A_{mn} and R_{mnpq}) S_{mn}, it can be verified that the following are $b_2^+ \times b_2^- = 57$ solutions of the Lichnerowicz equation on K3:[5]

$$h_{mn} = A_m^{\ p} S_{pn} + A_n^{\ p} S_{pm}$$

There is also the dilatational zero-mode $\delta g_{mn} = \lambda g_{mn}$, which is not traceless but still preserves the Ricci-flatness. So there are a total of 58 zero-modes of the Lichnerowicz operator on K3,[5,18] each giving rise to a massless scalar $\phi(x)$ in the external space, M_6.

In complex notation $(i, \bar{j} = 1, 2)$, the zero-modes are expressed as follows:

$$h_{i\bar{j}} = F_{i\bar{j}}$$
$$h_{ij} = (\epsilon_{ik} F_{j\bar{l}} + \epsilon_{jk} F_{i\bar{l}}) g^{\bar{l}k}$$
$$h_{\bar{i}\bar{j}} = (h_{ij})^*$$

where F_{ij} is a harmonic (1,1)-form. Note that the $h_{ij}, h_{\bar{i}\bar{j}}$ vanish if F=K, the Kähler form, so we again get 58 solutions.

With $h_{\mu\nu}(w) = h_{\mu\nu}(x) \otimes \phi(y)$, we obtain the external Lichnerowicz equation, so that $h_{\mu\nu}(x)$ represents a six-dimensional graviton, provided $\phi(y)$ is a massless scalar on K3. Of course, this is true if and only if ϕ = constant, so we get one graviton in six dimensions. In the language of forms, we say that $\phi(y)$ must be a harmonic 0-form, and since $b_0 = 1$ on K3, we get a single external graviton.

With $h_{\mu m}(w) = A_\mu(x) \otimes f_m(y), A_\mu(x)$ is massless provided $f_m(y)$ is covariantly constant on K3. Harmonic 1-forms are covariantly constant on Ricci-flat spaces. Since $b_1 = 0$ on K3, there are no massless vectors A_μ obtained in six dimensions.

The ten-dimensional gravitino satisfies the Rarita-Schwinger equation

$$\Gamma^M(\nabla_N \Psi_M - \nabla_M \Psi_N) = 0$$

With $\psi_\mu(w) = \psi_\mu(x) \otimes \Psi(y), \psi_\mu(x)$ satisfies the external Rarita-Schwinger equation provided $\Psi(y)$ satisfies the internal massless Dirac equation. From the last section we know there are 2 such solutions on K3. To construct these spinors we note the correspondence between (0,p)-forms and spinors on a complex n-dimensional manifold[1,10]: any spinor $\psi(y)$ can be written

$$\psi(y) = f^{(0)}(y)\Omega + f^{(1)}_{\bar{i}_1}(y)\Gamma^{\bar{i}_1}\Omega + ... + f^{(n)}_{\bar{i}_1\bar{i}_2...\bar{i}_n}(y)\Gamma^{\bar{i}_1\bar{i}_2...\bar{i}_n}\Omega$$

where Ω is a spinor annihilated by the internal γ-matrices Γ^i with complex indices, $\Gamma^i\Omega = 0$, and the $f^{(p)}$ are (0,p)-forms. Furthermore, if the manifold is Kähler and Ω is covariantly constant, $\psi(y)$ obeys the massless Dirac equation if and only if the $f^{(p)}$ are harmonic.

So there is a one to one correspondence between spin-$\frac{1}{2}$ zero-modes and harmonic (0,p)-forms. On K3, $h^{00} = h^{02} = 1, h^{01} = 0$, and so the two Dirac zero-modes can be written as

$$\begin{aligned} \Omega, \quad \Gamma^i\Omega = 0 \\ \omega \equiv \frac{1}{2}\epsilon_{\bar{i}\bar{j}}\Gamma^{\bar{i}\bar{j}}\Omega, \quad \Gamma^{\bar{i}}\omega = 0 \end{aligned} \tag{2.9}$$

These two zero-modes both have negative internal chirality, as required, and they give rise to one positive chirality gravitino $\psi^{(+)}_\mu(x)$ in six dimensions.

With $\Psi_m(w) = \chi(x) \otimes \psi_m(y), \chi(x)$ is a massless spinor if $\psi_m(y)$ obeys the internal Rarita-Schwinger equation. The solutions can be written using the harmonic (1,1)-forms F on K3:

$$\psi_i = F_{i\bar{j}} \Gamma^{\bar{j}} \Omega \qquad \psi_{\bar{i}} = F_{j\bar{i}} \Gamma^j \omega$$

When F=K, the Kähler form, these solutions are covariantly constant. Otherwise F is traceless (in real notation they are anti-self-dual) so that $\Gamma^i \psi_i = \Gamma^{\bar{j}} \psi_{\bar{j}} = 0$. Because the holomorphic and antiholomorphic exterior derivatives ∂ and $\bar{\partial}$, and their adjoints $\partial^\dagger$ and $\bar{\partial}^\dagger$, annihilate harmonic forms such as F,

$$\partial F = \bar{\partial} F = \partial^\dagger F = \bar{\partial}^\dagger F = 0$$

we have $\gamma^m \nabla_m \psi_i = \gamma^m \nabla_m \psi_{\bar{i}} = 0$. So there are 40 Rarita-Schwinger zero-modes as claimed above. The result is 20 negative chirality fermions $\chi^{(-)}(x)$.

The equation of motion for the antisymmetric tensor B_{MN} of R(10) is exactly the harmonic equation for 2-forms: $\nabla^2 B_{MN} - R_{MNPQ} B^{PQ} = 0$; $\nabla^P B_{PQ} = 0$. The three cases to discuss are $B_{\mu\nu}, B_{\mu n}$, and B_{mn}. For $B_{mn}(w) = \phi(x) \otimes f_{mn}(y), \phi(x)$ is massless if $f_{mn}(y)$ is harmonic. Since $b_2 = 22$ on K3, we obtain 22 scalars $\phi(x)$ in six dimensions. For $B_{\mu m}(w) = A_\mu(x) \otimes f_m(y)$ to yield a massless vector, $f_m(y)$ must again be harmonic. But $b_1 = 0$, so no massless vectors are generated from the antisymmetric tensor. With $B_{\mu\nu}(w) = B_{\mu\nu}(x) \otimes 1$, one massless tensor, or equivalently, a self-dual tensor $B^{(+)}_{\mu\nu}(x)$ and an anti-self-dual tensor $B^{(-)}_{\mu\nu}(x)$ are obtained.

The spinor $\psi^{(+)}(w) = \psi^{(-)}(x) \otimes \eta^{(-)}(y)$ yields a massless six-dimensional spinor $\psi^{(-)}(x)$ when $\eta^{(-)}(y) = \Omega + i\omega$. The remaining field in R(10), the dilaton, produces a scalar in six dimensions in the trivial way, $\varphi(w) = \varphi(x) \otimes 1$.

Gathering our results from the supergravity multiplet R(10), we have 81 scalars, 21 negative chirality spin-$\frac{1}{2}$ fermions, one gravitino, one graviton,and one antisymmetric tensor. The unique way (with six-dimensional Lorentz invariant action) to fit these zero-modes into simple supersymmetry multiplets is as in (2.5).

Moving on to the Yang-Mills sector, we will examine the fermions first. Since the gauge connection is set equal to the spin connection, there is a background SU_2 gauge field on K3. Because of (2.6), the gaugino $\Lambda^{(-)}(w)$ gives a spin-$\frac{1}{2}$ six-dimensional particle in the $E_7 \otimes E_8$ representation $(133,1) \oplus (1,248)$; (56,1); and (1,1) for every internal spinor in the $\mathbf{1}, \mathbf{2}, \mathbf{3}$ representations of SU_2 respectively, that is annihilated by the internal Dirac operator $\gamma^m D_m$. Here D_m includes the SU_2 gauge connection as well as the gravitational connection. The SU_2 singlets are Ω and ω, so that we get the positive chirality gaugino $\lambda^{(+)}(x)$ transforming in the adjoint representation $(133,1) \oplus (1,248)$ of the gauge group.

The SU_2 doublets are the most interesting. Because the gauge connection has been set equal to the spin connection, an SU_2 doublet index can be related to a complex coordinate index on K3 .[1,10,9] Consider an element V of $H^1(T)$, the first $\bar{\partial}$-cohomology group with values in the complex tangent bundle:

$$V = V_{\bar{i}}^{a} dz^{\bar{i}} \frac{\partial}{\partial z^a}$$

The a is a complex tangent index on K3. It is this index that can be identified with the SU_2 gauge doublet index, in the following sense. If V is harmonic it satisfies:

$$\begin{aligned} \bar{\partial} V &= \partial_{\bar{i}} V_{\bar{j}}^{a} dz^{\bar{i}} \wedge dz^{\bar{j}} \frac{\partial}{\partial z^a} = 0 \\ \bar{\partial}^{\dagger} V &= -2 g^{\bar{i} j} \nabla_j V_{\bar{i}}^{a} \frac{\partial}{\partial z^a} = 0 \end{aligned}$$

A spinor constructed from such an object, $\psi^a = V_{\bar{i}}^{\ a}\Gamma^{\bar{i}}\Omega$, satisfies the Dirac equation

$$(\nabla\!\!\!\!/\,\psi)^a = (\nabla_k V_{\bar{i}})^a \Gamma^k \Gamma^{\bar{i}} \Omega + (\nabla_{\bar{j}} V_{\bar{i}})^a \Gamma^{\bar{j}} \Gamma^{\bar{i}} \Omega = 0 \tag{2.10}$$

∇ is the gravitational covariant derivative, now including spin connection terms because of the tangent index a. The key is that these extra terms are exactly the required gauge connection terms for an SU_2 doublet spinor, provided the gauge connection is identical to the spin connection. Therefore, the solutions to (2.10) are the required solutions.

To count the number of solutions, we count the number of harmonic elements of $H^1(T)$. These are isomorphic to the harmonic (1,1)-forms F, in the following way:

$$F_{b\bar{j}} = V_{\bar{j}}^{\ a}\epsilon_{ab}, \quad V \in H^1(T) \tag{2.11}$$

Since $h^{11} = 20$, we obtain 10 negative chirality spinors $\chi^{(-)}(x)$ in the (56,1) representation of $E_7 \otimes E_8$, in agreement with the index theory results.

The analysis is not so easy for the triplet zero-modes. A **3** index of SU_2 is equivalent to a pair of symmetrised doublet indices. The symmetrisation makes it impossible to relate these zero-modes to forms. For the 45 $E_7 \otimes E_8$ singlets in six dimensions, we cannot construct the corresponding zero-modes on K3.

Note that the situation on K3 is nevertheless simpler than that on six-dimensional Calabi-Yau spaces. This is because on the six-dimensional spaces, index theory cannot be used to derive the massless four-dimensional spectrum. There is no general method available to count the gauge singlet zero-modes analogous to our SU_2 triplets (these are the SU_3 octets in the six-dimensional case[19]

). In the K3 case, we at least know there are 45 gauge singlet fermions in the Yang-Mills sector, by index theory.

What about the supersymmetry partners of the gauge nonsinglet fermions? These bosons come from the vector gauge boson $A_M(w)$ in Y(10). With $A_\mu(w) = A_\mu(x) \otimes 1$, we get the vector gauge boson in the $(133,1) \oplus (1,248)$ adjoint representation of the gauge group. For the gauge bosons $A_{\bar{i}}^{(56,2,1)}(w)$ in the (56,2,1) representation of the subgroup $E_7 \otimes SU_2 \otimes E_8 \subseteq E_8 \otimes E_8$, we write

$$A_{\bar{i}}^{(56,2,1)}(w) = \phi^{(56,1)}(x) \otimes V_{\bar{i}}^{a}(y)$$

where $V \in H^1(T)$. So we obtain both the 10 fermions $\chi^{(-)}(x)$ and the 40 scalars $\phi(x)$ in the (56,1) multiplet from the elements of $H^1(T)$, as it should be, by supersymmetry.

In summary, we have constructed all the zero-modes derived by index theory, except those that are SU_2 triplets. So we have partially confirmed the above results. In addition, with the explicit constructions we will be able to extract some information about the effective Lagrangian in six dimensions.

The topology of a six-dimensional Calabi-Yau space tells us much about the effective four-dimensional field theory.[9,17] The same is true for K3. If we concentrate on the Yang-Mills sector, The only non-trivial zero-modes are the SU_2 doublets given by the harmonic (1,1)-forms. Since effective Lagrangian terms are obtained by integrating over the internal manifold, we expect any such topological object to be quadratic in the fields.

A natural candidate is the intersection matrix for 2-forms on K3.[20] Its elements are given by

$$\int_{K3} f \wedge h$$

where f and h are harmonic 2-forms. The intersection matrix is topological. First of all, it depends only on the cohomology classes of f and h, not that they are harmonic. Also, one can associate with each harmonic 2-form f the two-dimensional submanifold $M(f)$ of K3 that is dual to f. The integral above is proportional to the number of points at which the submanifolds $M(f)$ and $M(h)$ intersect, weighted by their relative orientation. This number is invariant under smooth deformations of K3, *i.e.* it is topological. Because of this it doesn't depend on the detailed (unknown, in general) form of the metric on K3, and can be calculated. The result is

$$-2[e(8) \oplus e(8) \oplus \sigma^1 \oplus \sigma^1 \oplus \sigma^1]$$

where σ^1 is the usual Pauli matrix, e(8) is the Cartan matrix for the algebra of E_8, and we have divided out a factor of the K3 volume.

The intersection matrix is 22-dimensional because $b_2 = 22$. If we diagonalise each of the σ^1 matrices, then the 19 anti-self-dual 2-forms correspond to the $e(8) \oplus e(8)$ part along with the $+1$ eigenvalues of the three σ^1 matrices. The remaining (1,1)-form is the Kähler form which corresponds to one of the -1 eigenvalues of the three σ^1 matrices. The other two -1 eigenvalues come from the harmonic (0,2)- and (2,0)-forms.

We now show that the intersection matrix does determine effective field theory terms. If $V, \widetilde{V}$ are elements of $H^1(T)$, they give rise to six-dimensional negative chirality fermions $\chi, \widetilde{\chi}$, in the (56,1) representation of $E_7 \otimes E_8$ in the way we have described:

$$\Lambda^a(w) = \chi(x) \otimes V_{\bar{i}}^a(y)\Gamma^{\bar{i}}\Omega \qquad \widetilde{\Lambda}^a(w) = \widetilde{\chi}(x) \otimes \widetilde{V}_{\bar{i}}^a(y)\Gamma^{\bar{i}}\Omega$$

Substituting these expressions into the the ten-dimensional gaugino kinetic energy term gives

$$\int d^{10}w\sqrt{-g}\,\frac{1}{2}\,Tr\left(\bar{\widetilde{\Lambda}}\Gamma^M\partial_M\Lambda\right)$$
$$= N_{\widetilde{\chi}\chi}\int d^6x\sqrt{-g}\,\frac{1}{2}\,tr\left(\bar{\widetilde{\chi}}\gamma^m\partial_m\chi\right) + \ldots$$

with

$$N_{\widetilde{\chi}\chi} = \int\limits_{K3} d^4y\sqrt{g}\;\widetilde{\chi}_i^{\bar{a}}\;V_{\bar{j}}^b\;\eta_{b\bar{a}}\;g^{\bar{j}i}$$

Using the isomorphism (2.11) between elements of $H^1(T)$ and harmonic (1,1)-forms, we obtain

$$N_{\widetilde{\chi}\chi} = \frac{1}{4}\int\limits_{K3} F \wedge *\bar{\widetilde{F}}$$

$F, \widetilde{F}$ are the harmonic (1,1)-forms isomorphic to $V, \widetilde{V}$, respectively. The (1,1)-forms are real and all are anti-self-dual except the Kähler form, which is self-dual. Therefore

$$N_{\widetilde{\chi}\chi} = \mp\frac{1}{4}\int\limits_{K3} F \wedge \widetilde{F}$$

and the + sign occurs only when $\widetilde{F}$ is the Kähler form. So we see that the intersection form determines the normalisation matrix $N_{\widetilde{\chi}\chi}$ for the negative chirality spinors transforming in the (56,1) representation of the gauge group $E_7 \otimes E_8$. The result is

$$N_{\tilde{\chi}\chi} = \frac{1}{2}[e(8) \oplus e(8) \oplus \mathbf{1}_4]$$

where we have again divided out a factor of the volume of K3.

The terms related to the χ kinetic energy terms by six-dimensional supersymmetry and $E_7 \otimes E_8$ gauge invariance are also determined by the intersection matrix. Besides these terms, there are no others in the Yang-Mills sector determined solely by the topology of K3.

To close this chapter, let us mention a couple of properties of K3 that result in simplifications over the case of compactification on six-dimensional Calabi-Yau manifolds. K3 is simply connected so that no Wilson lines are possible. In the six-dimensional case, manifolds with reasonably low Euler characteristic are usually non-simply connected, and Wilson lines can be used to break the gauge group.[1,10] The non-simply connected manifolds are the spaces obtained from simply connected manifolds by identification under the action of a freely acting discrete isometry . The resulting non-simply connected manifold is always a Calabi-Yau space.[10] However, K3 is the unique 4-dimensional Calabi-Yau manifold, simply connected or otherwise. Division is not possible, therefore, and Wilson lines cannot be included.

Another simplification is that the Ricci-flat metric is a solution of the equations of motion, since K3 is a hyperkähler manifold. For the six-dimensional Calabi-Yau spaces, the four loop β-function does not vanish,[21,22] so that the metric must be modified to a non-Ricci-flat one.[23,24]

3. HETEROTIC STRINGS ON ORBIFOLD LIMITS OF K3

Some Calabi-Yau manifolds can be deformed such that all the curvature is "pinched off" into a finite number of isolated singular points. This deformation is an orbifold limit of the Calabi-Yau manifold. In general, there are many such limits for a single manifold. The subject of this chapter is a class of orbifold limits of K3.

In the limit that all the curvature is concentrated at isolated singular points, the resulting object can in some cases be identified with the orbifold constructed from a torus by modding out the action of a discrete isometry group. Then the singular points are those points fixed under the action of the isometry group. The orbifold limits K3(Z_l) can be constructed in this way: K3(Z_l)=T^4/Z_l.

In the first section of this chapter we construct these orbifolds and show how they can be "blown up" into K3. In the second section we derive the massless spectra of the $E_8 \otimes E_8$ heterotic string on these orbifolds. We then show how the string spectrum makes sense in light of the knowledge of how to blow up the orbifolds.

3.1. Orbifold Limits of K3

The simplest example of an orbifold limit of K3 is the torus T^4 divided by reflections $Q\ y^m\ Q^{-1} = -y^m$, $K3(Z_2) = T^4/Z_2$.[5] If T^4 is four-dimensional Euclidean space with the identifications $y^m \cong y^m + 1$, there are 16 points fixed under the reflection; $y^m = 0, \frac{1}{2}$, say.

To obtain K3 from K3(Z_2), surgery must be performed. Neighbourhoods containing the 16 singular points must be excised and replaced by appropriate non-compact regions containing no singular points. For a Ricci-flat K3 metric we use the Ricci-flat Eguchi-Hanson space,[25] which we denote by E(Z_2). The boundary of E(Z_2) at ∞ is S^3/Z_2, so that it fits the hole left by an excised region. However, the fit is only precise in the limit the Eguchi-Hanson region shrinks to zero size. This is the orbifold limit K3→K3(Z_2).

Using this construction, we can calculate the Euler characteristic of K3. Let $Z = (\frac{M}{G})_{F[N]}$ denote the manifold obtained by first dividing the manifold M by the discrete isometry group G of order g, then excising the fixed point set F and replacing it with appropriate non-compact well-behaved regions N. The Euler characteristic of Z is

$$\chi(Z) = \frac{1}{g}[\chi(M) - \chi(F)] + \chi(N) \tag{3.1}$$

K3 can be obtained with $M = T^4$, $F =$ 16 fixed points, and $N = 16E(Z_2)$. Since $\chi[E(Z_2)] = 2, \chi(T^4) = 0$, and $\chi = 1$ for a point, we get the correct result $\chi(K3) = 24$.

The Euler characteristic is not the only information we can deduce about the harmonic forms of K3 in the orbifold limit $K3 \to K3(Z_2)$. The signature of an Eguchi-Hanson space is $\tau = -1$, so its nonzero Betti numbers are $b_0 = b_2^- = 1$. Therefore 16 of the 19 anti-self-dual harmonic 2-forms are located in the Eguchi-Hanson regions. The remaining 3 are paired with 3 self-dual harmonic 2-forms. These 6 are not localised near the fixed points. In fact, they are the forms $dy^m \wedge dy^n$, the harmonic 2-forms of T^4, which survive because they are invariant under reflections.

The construction above can easily be generalised. Consider T^4 as the direct product of two 2-tori ($T^4 = T^2 \otimes T^2$), each being a complex plane with the identifications $z^j \cong z^j + 1 \cong z^j + e^{\frac{\pi i}{3}}$ $(j = 1,2)$. Then the discrete isometry generated by $Q\ z^1\ Q^{-1} = e^{\frac{2\pi i}{3}} z^1$ and $Q\ z^2\ Q^{-1} = e^{\frac{-2\pi i}{3}} z^2$ has 9 fixed points: $z^j = 0,\ \frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}},\ \frac{2}{\sqrt{3}} e^{\frac{\pi i}{6}};\ j = 1,2$. K3 can be obtained by replacing each of the fixed points with a non-compact Ricci-flat space whose boundary at ∞ is S^3/Z_3, and has $\tau = -2,\ \chi = 3$. Such a space exists,[26] and we denote it by $E(Z_3)$. These 9 spaces provide all the harmonic 2-forms of K3, except for 4. These 4, one anti-self-dual and the other three self-dual, survive from the torus. They are $dz^1 \wedge dz^2$, $dz^{\bar{1}} \wedge dz^{\bar{2}}$, $dz^1 \wedge dz^{\bar{1}}$, and $dz^2 \wedge dz^{\bar{2}}$; with the anti-self-dual combination being $dz^1 \wedge dz^{\bar{1}} - dz^2 \wedge dz^{\bar{2}}$.

In fact, non-compact spaces that are both Ricci-flat and asymptotically S^3/G, with G being any discrete subgroup of SU_2, are known to exist.[27] In keeping with the notation above, we denote them by $E(G)$. One can use these spaces to construct K3 from orbifolds of the general type T^4/G, in the way we have sketched.

In this paper we restrict ourselves to the groups $G = Z_l$. The Euler characteristic of $E(Z_l)$ is l, and its nonzero Betti numbers are $b_0 = 1,\ b_2 = b_2^- = l - 1$. Its only harmonic forms (besides a constant) are $l - 1$ 2-forms which are all anti-self-dual, and can also be written as (1,1)-forms.

Besides $l = 2$ and 3, $l = 4$ and 6 are possible. First we discuss the Z_4 case. For Z_4 symmetry, T^4 must be defined by the equivalences $z^j \cong z^j + 1 \cong z^j + i; j = 1,2$. The Z_4 isometry is then generated by $Q\ z^1\ Q^{-1} = iz^1$; $Q\ z^2\ q^{-1} = -iz^2$. There are four points fixed under Q:

$$(z^1, z^2) = (0,0), (0, \frac{1+i}{2}), (\frac{1+i}{2}, 0), (\frac{1+i}{2}, \frac{1+i}{2})$$

Besides these 4 points, there are also 12 points fixed under Q^2. (Note that in the cases l=2,3 it was sufficient to consider just those points fixed under Q.) These 12 points transform non-trivially under Z_4. They can be paired into 6 "doublets" whose two component points are transformed into each other by Q:

$$\begin{aligned}(z^1, z^2) &= [(0, \frac{1}{2}), (0, \frac{i}{2})] \ , [(\frac{1}{2}, 0), (\frac{i}{2}, 0)], \\ &[(\frac{1+i}{2}, \frac{1}{2}), (\frac{1+i}{2}, \frac{i}{2})] \ , [(\frac{1}{2}, \frac{1+i}{2}), (\frac{i}{2}, \frac{1+i}{2})], \\ &[(\frac{1}{2}, \frac{1}{2}), (\frac{i}{2}, \frac{i}{2})] \ , [(\frac{1}{2}, \frac{i}{2}), (\frac{i}{2}, \frac{1}{2})].\end{aligned}$$

This means that when a fundamental region of T^4 is chosen that is Z_4-invariant, it will contain 4 singular points that can be repaired using $E(Z_4)'s$, and 6 more that can be repaired with $E(Z_2)'s$. This is because the 6 points (one point of each doublet above) are fixed under Q^2, which generates a Z_2 subgroup of Z_4.

To verify that the manifold obtained is indeed K3, we calculate its Euler characteristic using (3.1). Since $F = 16$ points, and $N = 4E(Z_4) + 6E(Z_2)$, we have

$$\chi = \frac{\chi(T^4) - 16}{4} + 4\chi[E(Z_4)] + 6\chi[E(Z_2)] = 24$$

The harmonic 2-forms of K3 are either localised at the replaced fixed points, or survive from T^4 because they are Z_4-invariant. The non-localised harmonic 2-forms are 3 self-dual forms, $\epsilon, \bar{\epsilon}$, and the Kähler form K; along with the anti-self-dual form $dz^1 \wedge dz^{\bar{1}} - dz^2 \wedge dz^{\bar{2}}$. The localised 2-forms are all anti-self-dual and of type (1,1). The four $E(Z_4)$ support 3 each and each of the six $E(Z_2)'s$ support one. So we have $h^{11} = 2 + 4(3) + 6(1) = 20$, $b_2^+ = 3$, $b_2^- = 1 + 4(3) + 6(1) = 19$, as required.

Finally, for the Z_6 case we must use the torus T^4 as defined for the Z_3 case discussed above: $z^j \cong z^j + 1 \cong z^j + e^{\frac{\pi i}{3}}$. The Z_6 isometry is generated by

$Q\, z^1\, Q^{-1} \;=\; e^{\frac{\pi i}{3}} z^1$, $Q\, z^2\, Q^{-1} \;=\; e^{\frac{-\pi i}{3}} z^2$. There is just one point fixed under Q, the origin (0,0). There are 9 fixed under Q^2; besides (0,0) these can be grouped into four "doublets", whose components are transformed into each other by the action of Q:

$$
\begin{aligned}
(z^1, z^2) \;&=\; [(0, \frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}}), (0, \frac{2}{\sqrt{3}} e^{\frac{\pi i}{6}})], [(\frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}}, 0), (\frac{2}{\sqrt{3}} e^{\frac{\pi i}{6}}, 0)], \\
&[(\frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}}, \frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}}), (\frac{2}{\sqrt{3}} e^{\frac{\pi i}{6}}, \frac{2}{\sqrt{3}} e^{\frac{\pi i}{6}})], [(\frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}}, \frac{2}{\sqrt{3}} e^{\frac{\pi i}{6}}), (\frac{2}{\sqrt{3}} e^{\frac{\pi i}{6}}, \frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}})].
\end{aligned}
$$

Under the action of Q^3, there are 16 fixed points. Again excluding (0,0), they can be grouped into triplets under the action of Q:

$$
\begin{aligned}
(z^1, z^2) \;&=\; [(0, \frac{1 + e^{\frac{\pi i}{3}}}{2}), (0, \frac{1}{2}), (0, \frac{e^{\frac{\pi i}{3}}}{2})], \\
&[(\frac{1 + e^{\frac{\pi i}{3}}}{2}, 0), (\frac{1}{2}, 0), (\frac{e^{\frac{\pi i}{3}}}{2}, 0)], \\
&[(\frac{1 + e^{\frac{\pi i}{3}}}{2}, \frac{1 + e^{\frac{\pi i}{3}}}{2}), (\frac{1}{2}, \frac{1}{2}), (\frac{e^{\frac{\pi i}{3}}}{2}, \frac{e^{\frac{\pi i}{3}}}{2})], \\
&[(\frac{1 + e^{\frac{\pi i}{3}}}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{e^{\frac{\pi i}{3}}}{2}), (\frac{e^{\frac{\pi i}{3}}}{2}, \frac{1 + e^{\frac{\pi i}{3}}}{2})], \\
&[(\frac{1 + e^{\frac{\pi i}{3}}}{2}, \frac{e^{\frac{\pi i}{3}}}{2}), (\frac{1}{2}, \frac{1 + e^{\frac{\pi i}{3}}}{2}), (\frac{e^{\frac{\pi i}{3}}}{2}, \frac{1}{2})].
\end{aligned}
$$

Therefore we must use one $E(Z_6)$, four $E(Z_3)'s$, and five $E(Z_2)'s$ to blow up fixed points in a fundamental region of T^4/Z_6.

The Euler characteristic can again be seen to be 24, since 24 points are removed from T^4 and $N = E(Z_6) + 4E(Z_3) + 5E(Z_2)$ for this case (see (3.1)):

$$
\chi \;=\; \frac{0 - 24}{6} \;+\; 1(6) \;+\; 4(3) \;+\; 5(2) \;=\; 24.
$$

The harmonic 2-forms surviving from T^4 are the same that survived in the Z_3 and Z_4 cases. The remaining 18 anti-self-dual harmonic (1,1)-forms are localised in the inserted regions as follows: 5 in the single $E(Z_6)$, 2 in each of the four $E(Z_3)'s$, and 1 in each of the five $E(Z_2)'s$.

So now we have a picture of four distinct orbifold limits of K3. In each of the limits we know how to blow the orbifold up into K3. Hence we know how the harmonic forms of K3 arise in each of the orbifold limits. As shown in section 2.2, the harmonic forms are the origin of massless particles when the heterotic string is compactified on K3.

In the analysis of the low energy field theory limit of the heterotic string on K3, the detailed knowledge of the harmonic forms does not enter. For example, we do not know where any of the harmonic forms on K3 may be localised. The results were topological and therefore independent of such information. However, in the orbifold limit, one can solve the full string theory on K3.[2,3] In this case, as we will demonstrate in the next section, the detailed knowledge we have obtained is important.

3.2. THE HETEROTIC STRING ON K3 ORBIFOLDS

In this section we will derive the massless spectra obtained by compactifying the heterotic string on the orbifold limits $K3(Z_l)$ of K3. For definiteness we are considering the $E_8 \otimes E_8$ string. We will use the Green-Schwarz formulation of the superstring in the light cone gauge, and also follow Ref. 28, where the bosonic formulation of the gauge degrees of freedom was first used.

To solve string theory on an orbifold T^4/Z_l, one first solves it on T^4. Projecting out everything but the Z_l-singlet states then gives part of the theory, the states of the untwisted sector. To get the rest of the theory the procedure must be repeated for strings that are only closed after the identification under the action of the group Z_l. These open strings on T^4 obey twisted boundary conditions

$$X(\sigma+\pi,\tau) \;=\; Q^s\, X(\sigma,\tau)\, Q^{-s} \quad ; s=1,\dots l-1 \tag{3.2}$$

where here X is a generic string coordinate. The states formed using the coordinates obeying (3.2) are the states of the twisted sectors.

$X^M = X^M(\sigma,\tau)$ will denote the 8 transverse coordinates of the string. The index M splits up into 4 transverse coordinates $X^\mu(\sigma,\tau)$, describing the string in transverse six-dimensional spacetime, and 4 internal coordinates $X^m(\sigma,\tau)$. The centre of mass coordinate for the X^m takes values in the range of the y^m, the coordinates of the internal spaces, $K3(Z_l)$. We split the string coordinates X^M into complex coordinates

$$Z^i \;=\; X^{2i-1} \;+\; iX^{2i} \;; i=1,\dots 4$$

Z^3 and Z^4 are the complex coordinates associated with the transverse spacetime coordinates in six dimensions. Z^1 and Z^2 are the complex string coordinates on the orbifolds, so that their centre of mass coordinates take values in the ranges of z^1 and z^2, respectively.

In the Green-Schwarz formalism there is also the right-moving Majorana-Weyl spinor $S^{\dot a} = S^{\dot a}(\tau-\sigma)$, the spacetime superpartner of the right-moving bosonic string coordinates $X^M(\tau-\sigma)$. The index $\dot a$ is a conjugate spinor index of the rotation group SO_8 of transverse spacetime. We have also the 16 "extra" left-moving string coordinates $X^I = X^I(\tau+\sigma)$, responsible for the gauge degrees of freedom of the string. All these closed string coordinates are periodic functions of $0 \leq \sigma < \pi$. On a flat space they have the following normal mode expansions:

$$\begin{aligned} X^M(\tau-\sigma) &= \frac{1}{2}x^M + \frac{1}{2}p^M(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha_n^M}{n}\,e^{-2in(\tau-\sigma)} \\ S^{\dot{a}}(\tau-\sigma) &= \sum_n S_n^{\dot{a}}\,e^{-2in(\tau-\sigma)} \end{aligned}$$

$$\begin{aligned} X^M(\tau+\sigma) &= \frac{1}{2}x^M + \frac{1}{2}p^M(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^M}{n}\,e^{-2in(\tau+\sigma)} \\ X^I(\tau+\sigma) &= x^I + p^I(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^I}{n}\,e^{-2in(\tau+\sigma)} \end{aligned}$$

The fermionic oscillators obey

$$\{S_m^{\dot{a}}, S_n^{\dot{b}}\} = \delta_{m+n,0}\delta^{\dot{a}\dot{b}}$$

From the zero-modes $S_0^{\dot{a}}$ we can construct operators $\theta_i, \theta_{\bar{j}}$ obeying the anticommutation relations of annihilation and creation operators:

$$\{\theta_i, \theta_{\bar{j}}\} = \delta_{i\bar{j}} \qquad \{\theta_i, \theta_j\} = \{\theta_{\bar{i}}, \theta_{\bar{j}}\} = 0. \tag{3.3}$$

The indices $i, \bar{j}$ are transverse complex indices taking values from 1 to 4.

For the $E_8 \otimes E_8$ string, the centre of mass coordinates of the 16 extra string coordinates X^I lie on the self-dual lattice $\Lambda \oplus \Lambda$, where Λ is the root lattice of E_8. Vectors on the lattice Λ fall into the following two classes:

$$p_1 = (n_1, \ldots, n_8) \qquad p_2 = (n_1 + \frac{1}{2}, \ldots, n_8 + \frac{1}{2}) \tag{3.4}$$

with the restriction in both cases that the integers n_i satisfy $\sum_{i=1}^{8} n_i = 0 \bmod 2$.

In keeping with our descriptions of the orbifolds $K3(Z_l)$ in the last section, the twist Q generating Z_l is specified by

$$Q\, Z^{1,2}\, Q^{-1} \;=\; e^{\frac{\pm 2\pi i}{l}} Z^{1,2} \qquad Q\, Z^{3,4}\, Q^{-1} \;=\; Z^{3,4} \tag{3.5}$$

This twist induces a similar action on the fermionic operators θ :

$$Q\, \theta_{\bar{1},\bar{2}}\, Q^{-1} \;=\; \theta_{\bar{1},\bar{2}} \qquad Q\, \theta_{\bar{3},\bar{4}}\, Q^{-1} \;=\; e^{\frac{\pm 2\pi i}{l}} \theta_{\bar{3},\bar{4}} \tag{3.6}$$

It remains to specify the action of the twist Q on the gauge degrees of freedom. For Abelian groups like Z_l, the twist can be represented in the bosonic formulation as a shift in the extra 16 coordinates[28]:

$$Q\, X^I Q^{-1} \;=\; X^I \;+\; \pi q^I$$

Since our twist is of order l, we must have $lq \in \Lambda \oplus \Lambda$. We will write $q = (q_1, q_2)$ with $lq_1, lq_2 \in \Lambda$.

The choices of shift vectors q are limited by modular invariance,[30,31] and one can also restrict without loss of generality to those shift vectors having a length smaller than a maximum.[3] Even with these restrictions, however, many choices are still possible. Only one corresponds to setting the gauge connection equal to the spin connection. That choice is the shortest possible shift vector, with $q^2 = \frac{2}{l^2}$. For this choice, we can use $q_2 = 0$, $q_1 = \frac{1}{l}\, e_8^* \;=\; \frac{1}{l}\,(0,0,0,0,0,0,1,1)$ where e_8^* is the lattice vector on Λ that is dual to the eighth simple root e_8.[32]

Right away we can find the gauge group to which $E_8 \otimes E_8$ is broken. Since $q_2 \;=\; 0$, the second E_8 is untouched. Let us denote the E_8 root vectors by

e_i , $i = 1, \ldots 8$, and let e_0 be the negative of the highest root. For E_8, $e_0 = -e_8^*$, where the e_i^* ($i = 1, \ldots 8$) are dual to the root vectors: $e_i \cdot e_j^* = \delta_{ij}$. The 9 root vectors are depicted in the extended Dynkin diagram of E_8, drawn in Figure 1.

In the heterotic string theory, the E_8 generator corresponding to the root e, e being one of the e_i or e_0, is built from the factor : $e^{2ie\cdot X(\tau+\sigma)}$:.[33] With $q = \frac{1}{l} e_8^*$, the E_8 generator corresponding to the simple root e_8 is twisted and so will not survive the projection. The e_i, $i \neq 8$, are the simple roots of $E_7 \subset E_8$. The generator given by e_0 can be used as the simple root of SU_2, if it is a Q-singlet. But it picks up a phase $exp[\frac{2\pi i}{l}(e_8^*)^2] = exp(\frac{4\pi i}{l})$. Therefore, only for $l = 2$ does it survive the projection to a Z_l-invariant theory, and for $l = 2$ we obtain the gauge group $E_7 \otimes SU_2 \otimes E_8$. For $l \neq 2$, we have the smaller gauge group $E_7 \otimes U_1 \otimes E_8$, with U_1 charge proportional to the projection onto e_0.

We can now derive the massless spectrum. For a state $| \, \rangle_L \otimes | \, \rangle_R$ to be massless, both the left-moving state $| \, \rangle_L$ and the right-moving state $| \, \rangle_R$ must be massless. States satisfying this constraint obey

$$N_L + \sum_I \frac{(p^I)^2}{2} + C_L^{(s)} = 0 \qquad N_R + C_R^{(s)} = 0 \tag{3.7}$$

The $C_{L,R}^{(s)}$ are the normal ordering constants in the sector twisted by Q^s. If a bosonic (fermionic) string field oscillator index takes the value η ,$0 \leq \eta < 1$, then it contributes

$$(-)\,[\,-\frac{1}{24} + \frac{1}{4}\eta(1-\eta)\,] \tag{3.8}$$

to the normal ordering constant.

First consider the untwisted sector. In the right-moving sector we must have $N_R = 0$ for zero mass. The only oscillators having $N_R = 0$ are the $S_0^{\dot{a}}$, or equivalently, the $\theta_i, \theta_{\bar{j}}$. We denote by $|0\rangle_R$ the vacuum satisfying $\theta_i |0\rangle_R = 0$, $i = 1, \ldots 4$. The following are then massless right-moving states:

$$|0\rangle_R \, , \;\; \theta_{\bar{i}} |0\rangle_R \, , \;\; \theta_{\bar{i}}\theta_{\bar{j}} |0\rangle_R \, , \;\; \theta_{\bar{i}}\theta_{\bar{j}}\theta_{\bar{k}} |0\rangle_R \, , \;\; \theta_{\bar{i}}\theta_{\bar{j}}\theta_{\bar{k}}\theta_{\bar{l}} |0\rangle_R$$

The states with an even (odd) number of θ's are the physical states of the fields λ (A_M) contained in Y(10).

To perform (later) the projection to Z_l-singlet states we will need the Q-eigenvalues of the various states. We denote these eigenvalues by λ_R. Using (3.6) the $\lambda_R = 1$ states are

$$\begin{gathered} |0\rangle_R \\ \theta_{\bar{1}} |0\rangle_R \, , \;\; \theta_{\bar{2}} |0\rangle_R \\ \theta_{\bar{1}}\theta_{\bar{2}} |0\rangle_R \, , \;\; \theta_{\bar{3}}\theta_{\bar{4}} |0\rangle_R \\ \theta_{\bar{1}}\theta_{\bar{3}}\theta_{\bar{4}} |0\rangle_R \, , \;\; \theta_{\bar{2}}\theta_{\bar{3}}\theta_{\bar{4}} |0\rangle_R \\ \theta_{\bar{1}}\theta_{\bar{2}}\theta_{\bar{3}}\theta_{\bar{4}} |0\rangle_R \end{gathered}$$

while we also have states with $\lambda_R = e^{\frac{2\pi i}{l}}$:

$$\theta_{\bar{3}} |0\rangle_R \, , \quad \theta_{\bar{1}}\theta_{\bar{3}} |0\rangle_R \, , \;\; \theta_{\bar{2}}\theta_{\bar{3}} |0\rangle_R \, , \quad \theta_{\bar{1}}\theta_{\bar{2}}\theta_{\bar{3}} |0\rangle_R$$

and $\lambda_R = e^{\frac{-2\pi i}{l}}$:

$$\theta_{\bar{4}} |0\rangle_R \, , \quad \theta_{\bar{1}}\theta_{\bar{4}} |0\rangle_R \, , \;\; \theta_{\bar{2}}\theta_{\bar{4}} |0\rangle_R \, , \quad \theta_{\bar{1}}\theta_{\bar{2}}\theta_{\bar{4}} |0\rangle_R$$

The $\lambda_R = 1$ states form a six-dimensional vector supermultiplet Y(6), while each of the sets of $\lambda_R = e^{\frac{\pm 2\pi i}{l}}$ states form half of scalar supermultiplet, $\frac{1}{2}$S(6). The two halves of S(6) are related by CPT; they are antiparticles of each other. For $l = 2$, and only for $l = 2$, λ_R is the same for both halves.

It is worthwhile to express these results in the language of group theory. The little group of the ten-dimensional Lorentz group is SO_8. This group has an $SO_4 \otimes SO_4$ maximal subgroup, with one SO_4 being the little group of the six-dimensional Lorentz group, and the other corresponding to the structure group of the internal space. Both the internal SO_4 and external SO_4 are locally equivalent to $SU_2 \otimes SU_2$. Under the $(SU_2)^4$ subgroup of SO_8, the three 8-dimensional representations transform in the following way:

$$
\begin{aligned}
\mathbf{8}_c &= (1,2|1,2) \ \oplus \ (2,1|2,1) \\
\mathbf{8}_s &= (2,1|1,2) \ \oplus \ (1,2|2,1) \\
\mathbf{8}_v &= (1,1|2,2) \ \oplus \ (2,2|1,1)
\end{aligned}
\tag{3.9}
$$

Here we have separated internal representations (on the right) from the external ones by a vertical bar. The string field $S^{\dot{a}}$ transforms as $\mathbf{8}_c$, while the fields making up the supermultiplet Y(10) transform as the last two: A_M as $\mathbf{8}_v$ and Λ as $\mathbf{8}_s$. These are the states obtained as the massless right-movers in the untwisted sector.

The last SU_2 group of (3.9) tells us how the corresponding fields transform under the Z_l twist. Splitting the SU_2 representations into representations of $U_1 \subset SU_2$, we can write

$$
\begin{aligned}
\mathbf{8}_s &= (2,1|1)_{-1} \ \oplus \ (2,1|1)_1 \ \oplus (1,2|2)_0 \\
\mathbf{8}_v &= (1,1|2)_{-1} \ \oplus \ (1,1|2)_1 \ \oplus (2,2|1)_0
\end{aligned}
$$

The value of λ_R is simply $exp[\frac{2\pi i}{l}]$ raised to the power of the U_1 charge. The $\lambda_R = 1$ states are the states of the supermultiplet Y(6). Under the external SO_4 group the spinor $\lambda^{(+)}$ transforms as 2(1,2) and the vector A_μ transforms as

(2,2). $\lambda_R = exp[\frac{\pm 2\pi i}{l}]$ picks out $\frac{1}{2}$S(6). Note that this gives us a precise notion of what we mean by $\frac{1}{2}$S(6). In the physical theory, two halves of S(6) must combine to yield the full supermultiplet, which contains four scalars and a spinor $\chi^{(-)}$ transforming as 2(2,1).

The left-moving massless modes obey $N_L + \frac{p^2}{2} - 1 = 0$, since $C_L^{(0)} = -1$. With $N_L = 1,\ p = 0$, we obtain the states $\widetilde{\alpha}^I_{-1}\,|0\rangle_L$ of the Cartan subalgebra of $E_8 \otimes E_8$ with Q-eigenvalue $\lambda_L = 1$, and the 8 transverse physical states $\widetilde{\alpha}^M_{-1}\,|0\rangle_L$ of a ten-dimensional vector particle. These latter split into the transverse states $\widetilde{\alpha}^\mu_{-1}\,|0\rangle_L$ of a six-dimensional vector, having $\lambda_L = 1$, and four six-dimensional scalars with $\lambda_L \neq 1$. If $\tilde{\varsigma}^i_n,\ i = 1,2$ ($\tilde{\varsigma}^{\bar{j}}_n,\ \bar{j} = 1,2$), are the oscillators for the left-moving complex string coordinates Z^i ($Z^{\bar{j}}$), the four scalars split into two

$$\tilde{\varsigma}^1_{-1}\,|0\rangle_L\,,\ \tilde{\varsigma}^2_{-1}\,|0\rangle_L$$

with $\lambda_L = e^{\frac{2\pi i}{l}}$, and two

$$\tilde{\varsigma}^{\bar{1}}_{-1}\,|0\rangle_L\,,\ \tilde{\varsigma}^{\bar{2}}_{-1}\,|0\rangle_L$$

with $\lambda_L = e^{\frac{-2\pi i}{l}}$.

The remaining massless gauge degrees of freedom come from states with $N_L = 0$ and $\sum_I (p^I)^2 = 2$, where $p \in \Lambda \oplus \Lambda$. For $p^2 = 2,\ p$ will be nonzero in only one of the Λ, but not both. Using the notation of (3.4), the nonzero p in one of the Λ will be 112 vectors of the form $p_1 = (0,\dots 0, \pm 1, 0, \dots 0, \pm 1, 0, \dots 0)$, and 128 of the form $p_2 = (\pm\frac{1}{2}, \pm\frac{1}{2}, \dots \pm \frac{1}{2})$, with an even number of $-\frac{1}{2}$ components. With the shift vector shifting just the first E_8 root lattice Λ, all 240 states in the second Λ have $\lambda_L = 1$. For those in the first Λ, the Q-eigenvalue is $\lambda_L = exp[2\pi i p \cdot q]$.

Of the 240 states on the first Λ, 126 have $\lambda_L = 1$, while 56 have $\lambda_L = e^{\frac{2\pi i}{l}}$, 56 have $\lambda_L = e^{\frac{-2\pi i}{l}}$, one has $\lambda_L = e^{\frac{4\pi i}{l}}$, and the remaining state has $\lambda_L = e^{\frac{-4\pi i}{l}}$.

The states with $\lambda_L = 1$, along with the 16 states $\tilde{\alpha}^I_{-1} |0\rangle_L$ of the Cartan subalgebra, fill out the adjoint representation of the gauge group. We see that the gauge group is $E_7 \otimes U_1 \otimes E_8$ for $l \neq 2$, and $E_7 \otimes SU_2 \otimes E_8$ for $l = 2$, as advertised above. Along with the adjoint representation, we have also found two $\mathbf{56}'s$ of E_7, one with $\lambda_L = e^{\frac{2\pi i}{l}}$, and the other with $\lambda_L = e^{\frac{-2\pi i}{l}}$. For $l \neq 2$, we also have two singlets, with Q-eigenvalues $\lambda_L = e^{\frac{4\pi i}{l}}$ and $\lambda_L = e^{\frac{-4\pi i}{l}}$.

The states that survive to live on the orbifold are those on the torus having $\lambda_L \times \lambda_R = 1$. For $l = 2$, these are

$$\begin{aligned} &R(6) \ + \ T(6) \ + \\ &Y(6)[(133,1,1) \oplus (1,3,1) \oplus (1,1,248)] \ + \\ &S(6)[4(1,1,1) \oplus (56,2,1)] \end{aligned} \tag{3.10}$$

where representations of the gauge group $E_7 \otimes SU_2 \otimes E_8$ are given in the round brackets. For $l \neq 2$, the untwisted sector contribution to the massless spectrum is

$$\begin{aligned} &R(6) \ + \ T(6) \ + \\ &Y(6)[(133,1) \oplus (1,1) \oplus (1,248)] \ + \\ &S(6)[n(1,1) \oplus (56,1)] \end{aligned} \tag{3.11}$$

Here $n = 3$ for $l = 3$, $n = 2$ otherwise, and the numbers in the round brackets are representations of $E_7 \otimes E_8$. We will not indicate the U_1 charges.

We will now work out the massless modes in the twisted sector in the case $l = 2$. The calculations for the orbifolds $K3(Z_l)$, $l \neq 2$, proceed in a similar manner, and the results will be given later.

There is just one twisted sector for the case $l = 2$. The fields satisfy the twisted boundary conditions

$$X^m(\sigma+\pi,\tau) \;=\; Q\, X^m(\sigma,\tau)\, Q^{-1} \;=\; -X^m(\sigma,\tau). \tag{3.12}$$

with similar boundary conditions for the right-moving fermionic fields. For the right-moving modes the fermionic and bosonic contributions to the normal ordering constant cancel, so that $C_R^{(1)} = 0$. We require $N_R = 0$ again for massless states. Looking at (3.6) we see that the fermionic oscillators $\theta_{\bar{3}}$ and $\theta_{\bar{4}}$ will be twisted, with $\eta = \frac{1}{2}$. The only oscillators with $N_R = 0$ are $\theta_{\bar{1}}$ and $\theta_{\bar{2}}$. We denote the twisted sector vacuum by $|0\rangle_R^{(1)}$; it satisfies $\theta_1 |0\rangle_R^{(1)} = \theta_2 |0\rangle_R^{(1)} = 0$. The $N_R = 0$ states are then

$$|0\rangle_R^{(1)}, \quad \theta_{\bar{1}} |0\rangle_R^{(1)}, \theta_{\bar{2}} |0\rangle_R^{(1)}, \quad \theta_{\bar{1}}\theta_{\bar{2}} |0\rangle_R^{(1)} \tag{3.13}$$

These states form $\frac{1}{2}S(6)$. Under the SO_4 little group of the six-dimensional Lorentz group, the states $|0\rangle_R^{(1)}$, $\theta_{\bar{1}}\theta_{\bar{2}} |0\rangle_R^{(1)}$ transform as (2,1), and the states $\theta_{\bar{1}} |0\rangle_R^{(1)}$, $\theta_{\bar{2}} |0\rangle_R^{(1)}$ are singlets.

The physical states must combine into an integral number of scalar supermultiplets. This will happen because there are 16 points fixed under Z_2. Each of the fixed points contributes equally to the spectrum, so the result obtained is multiplied by 16.

Since by (3.12) there are 4 fields with $\eta = \frac{1}{2}$, we have $C_L^{(1)} = -\frac{3}{4}$ in the twisted sector, using (3.8). So the massless left-movers satisfy $N_L + \frac{p^2}{2} - \frac{3}{4} = 0$, *i.e.* either $N_L = 0$ and $p^2 = \frac{3}{2}$, or $N_L = \frac{1}{2}$ and $p^2 = \frac{1}{2}$. Because $X^I(\sigma+\pi,\tau) = QX^I(\sigma,\tau)Q^{-1} = X^I(\sigma,\tau) + \pi q^I$ in the twisted sector, p lies on the shifted lattice

$(\Lambda\oplus\Lambda)+q$. Using (3.4), we find two solutions to $p^2 = \frac{1}{2}$, both of the form $p = p_1+q$, with $exp[2\pi i p\cdot q] = -1$. There are four oscillators with $N_L = \frac{1}{2}$, coming from the four internal coordinates X^m. We obtain four doublets of SU_2 in the form $\widetilde{\alpha}^m_{-\frac{1}{2}} e^{2ip\cdot x}\,|0\rangle^{(1)}_L$. The Q-eigenvalue for these states is $\lambda_L = -exp[2\pi i p\cdot q] = 1$. Note that we have assumed the twisted left-moving vacuum is a Q-singlet. This need not be the case, and there are ways to find the vacuum phase.[31] The phase is 1 for the Z_2-case but in general it is not.

There remain the states with $N_L = 0$ and $p^2 = \frac{3}{2}$. There are 24 solutions to $p^2 = (p_2+q)^2 = \frac{3}{2}$ and 32 solutions to $p^2 = (p_1+q)^2 = \frac{3}{2}$. All 56 solutions have integer $p\cdot q$, so the corresponding states $e^{2ip\cdot x}\,|0\rangle^{(1)}_L$ have $\lambda_L = 1$. They form the **56** representation of the gauge group E_7.

Combining left-moving and right-moving states into physical states with $\lambda_L \times \lambda_R = 1$ yields the following contribution to the massless spectrum

$$8S(6)[4(1,2,1)\oplus(56,1,1)]$$

Note that we have multiplied by 16, the number of points on T^4 that are fixed under Q.

The entire massless spectrum is therefore

$$\begin{aligned}&R(6)\ +\ T(6)\ + \\ &Y(6)[(133,1,1)\oplus(1,3,1)\oplus(1,1,248)]\ + \\ &S(6)[4(1,1,1)\oplus(56,2,1)\oplus 32(1,2,1)\oplus 8(56,1,1)]\end{aligned} \tag{3.14}$$

It can be verified that this spectrum has no anomaly that cannot be cancelled in the Green-Schwarz[13] manner.

Our knowledge of the origin of the various harmonic forms of K3 in the orbifold limit K3(Z_2) can help us understand this spectrum. The low energy field theory on K3 produces 20S(6) in the gravitational sector. As shown in section 1.2, these massless particles arise from the 20 harmonic (1,1)-forms on K3. But there are only 4S(6) arising in the untwisted sector on K3(Z_2). This is because harmonic forms that are localised at fixed points contribute to the spectra of the twisted sectors. The only harmonic forms that are not localised at fixed points are those inherited from the torus T^4, because they are Z_2-invariant. But as noted in the previous section, 4 harmonic (1,1)-forms survive from T^4, explaining the 4S(6) in the untwisted sector.

We can see similar tendencies in the Yang-Mills sector. Note first a very important feature of the spectrum. The number of **56**'s of E_7 obtained by compactifying on K3(Z_2) is the same as that obtained from K3. This will be true of all orbifold limits K3(Z_l) of K3. The number of **56**'s is a topological number that does not change as we deform the manifold K3 into an orbifold K3(Z_l). In fact, this number is proportional to the Euler characteristic of K3, as shown in section 1.1. This will be useful later.

The 10 S(6) in the **56** representation of E_7 have the same origin on K3 as the 20 S(6) in the gravitational sector: the 20 harmonic (1,1)-forms. In the orbifold limit K3(Z_2), 16 are localised at the fixed points, or equivalently, in the Eguchi-Hanson regions that replace the singular fixed points. The remaining four do not vanish away from these regions. Our results are consistent with this: 8 of the 10 S(6) in the **56** come from the twisted sector, while 2 arise in the untwisted sector.

The spectra on K3(Z_l), $l \neq 2$, can be derived in similar fashion. We have already derived the spectra (3.11) in the untwisted sector. For the case $l = 3$, there are two twisted sectors. However, it is always true that the particles found in the sector twisted by Q^{l-s} are the antiparticles of the particles found in the sector twisted by Q^s. So for the spectrum on K3(Z_3), we need only calculate the states in the sector twisted by Q. We find $\frac{1}{2}$S(6) in the representations $(56,1) \oplus 7(1,1)$ of $E_7 \otimes E_8$ (we omit the U_1 charges) at every fixed point of Q on T^4. The sector twisted by Q^2 provides the other half of the S(6) supermultiplet, and since there are 9 points fixed under the action of Z_3 on T^4, we obtain the spectrum

$$\begin{aligned} s=0 \quad & R(6) \;+\; T(6) \;+ \\ & Y(6)[(133,1) \oplus (1,1) \oplus (1,248)] \;+ \\ & S(6)[3(1,1) \oplus (56,1)] \;+ \\ s=1,3 \quad & 9S(6)[(56,1) \oplus 7(1,1)] \end{aligned} \tag{3.15}$$

Here s labels the sector twisted by Q^s; for example $s = 0$ indicates the untwisted sector.

As in the previous section, the orbifolds K3(Z_4) and K3(Z_6) are the most interesting. The new feature is that there are points fixed under certain elements of Z_4 or Z_6 that are not fixed under others. We will work out the consequences of this on the massless spectra.

Consider first the case $l = 4$. There are 3 twisted sectors, twisted by Q^s , $s = 1,2,3$. The fields in the $s = 1$ sector are related to those in the $s = 3$ sector by CPT, while those in the $s = 2$ sector must describe particles and antiparticles both. It can easily be shown that the $s = 1$ and $s = 3$ sectors combine to yield the spectrum $S(6)[(56,1) \oplus 8(1,1)]$. How many times is this spectrum copied? In the $l = 2$ and 3 cases, the degeneracy factors in the twisted sectors were the numbers of fixed points, 16 and 9, respectively. As discussed above, for the twist group Z_4, there are 4 points on T^4 fixed under the action of Q. Thus 4 is the required degeneracy factor.

In the Q^2-twisted sector, the spectrum is a multiple of $\frac{1}{2}S(6)[(56,1)\oplus 4(1,1)]$. But here the degeneracy factor is not the number of points on T^4 fixed under Q^2. This is because only four of these sixteen fixed points are invariant under the full action of Z_4. The remaining twelve can be grouped into 6 "doublets" whose component points are transformed into each other by the action of Q, as discussed in the previous section. So to construct a Z_4-singlet spectrum from states on the torus, we must take the spectrum at each of the fixed points in each of the "doublets" and form the linear combination that has Q-eigenvalue 1. A physical state will be the sum of the states at the two fixed points. (There is also the difference, but since it is not a Q-singlet, it is projected out of the spectrum.) Therefore, these doublets make a contribution of 6 to the degeneracy factor. Adding the contributions of the 4 Q-singlet points fixed under Q^2, the total degeneracy factor is 10. We obtain, therefore, the spectrum $5S(6)[(56,1)\oplus 4(1,1)]$ in the sector twisted by Q^2.

Note that we have massless states that are actually non-local on the torus T^4. They are, however, local on the orbifold, since the two fixed points of a "doublet" are identified by modding out the isometry group.

There is a simple way to determine the degeneracy factor of each twisted sector. As noted above, the number of **56**'s is proportional to the Euler characteristic. Knowing the number of **56**'s in each sector tells us the degeneracy factor for each twisted sector. There is an orbifold formula for the Euler characteristic[2,3] which for the Z_l groups reduces to

$$\chi = \frac{1}{l} \sum_{s,t=0}^{l-1} \chi(Q^s, Q^t) \tag{3.16}$$

$\chi(Q^s, Q^t)$ is the Euler characteristic of the intersection of the points fixed under Q^s and the points fixed under Q^t. If we fix s, we obtain the contribution $\chi(s)$ to χ from the sector twisted by Q^s. For $l = 4$ we get $\chi(s = 1) = \chi(s = 3) = 4$, and $\chi(s = 2) = 10$, in agreement with the degeneracy factors used above.

In passing we should note that $\chi(s = 0)$ is always the contribution to χ from those harmonic forms of T^4 that survive because they are Z_l-invariant. Thus $\chi(s = 0) = 8$ for $l = 2$, and $\chi(s = 0) = 6$ for $l \neq 2$. These numbers do not translate so easily into the numbers of **56**'s in the untwisted sectors. This is simply because not all of the harmonic forms surviving from T^4 are (1,1)-forms, which are those needed to construct the internal wave functions of the E_7 **56** particles.

The full massless spectrum on the orbifold K3(Z_4) is

$$\begin{array}{ll} s = 0 & R(6) \ + \ T(6) \ + \\ & Y(6)[(\mathbf{133}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{248})] \ + \\ & S(6)[2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{56}, \mathbf{1})] \ + \\ s = 1,3 & 4S(6)[(\mathbf{56}, \mathbf{1}) \oplus 8(\mathbf{1}, \mathbf{1})] \ + \\ s = 2 & 5S(6)[(\mathbf{56}, \mathbf{1}) \oplus 4(\mathbf{1}, \mathbf{1})] \end{array} \tag{3.17}$$

We can understand this spectrum in terms of the cohomology of the blown-up orbifold K3(Z_4). To blow up the orbifold K3(Z_4), one must insert 4 $E(Z_4)$ regions, each having 3 harmonic (1,1)-forms, and 6 $E(Z_2)$ regions with 1 harmonic

(1,1)-form each. These 18 harmonic forms determine the internal wave functions of the 9 **56**'s of the twisted sectors. To divide these forms up into the sectors twisted by Q (and Q^{-1}) and Q^2, it is important to note that a fixed point replaced by an $E(Z_4)$ will support states that are only closed after identification by Q^2, as well as those closed only after identification by Q (and Q^{-1}). So along with the 6 harmonic (1,1)-forms localised in the $E(Z_2)$, we must include 4 harmonic (1,1)-forms localised in the 4 $E(Z_4)$. This explains the 5 **56**'s obtained as states that are closed by the identification under the action of Q^2. The 4 **56**'s of the sectors twisted by Q and Q^{-1} have the remaining harmonic (1,1)-forms localised in the $E(Z_4)$'s as their internal wave functions.

We now briefly discuss the final example, K3(Z_6). The spectrum can be worked out to be

$$\begin{array}{ll} s=0 & R(6)\ +\ T(6)\ + \\ & Y(6)[(133,1)\oplus(1,1)\oplus(1,248)]\ + \\ & S(6)[2(1,1)\oplus(56,1)]\ + \\ s=1,5 & S(6)[(56,1)\oplus 10(1,1)]\ + \\ s=2,4 & 5S(6)[(56,1)\oplus 4(1,1)]\ + \\ s=3 & 3S(6)[(56,1)\oplus 4(1,1)] \end{array} \tag{3.18}$$

The interesting numbers are the degeneracy factors in each of the twisted sectors. The degeneracy factor for the sector twisted by Q (Q^{-1}) is one, since there is only one point on T^4 fixed under Q (Q^{-1}). As discussed in the previous section, there are 9 points fixed under $Q^{\pm 2}$. Only one is a Q-singlet; the rest form 4 Q-doublets. A Z_6-invariant state is formed as the sum of a state at the fixed point that is one component of a doublet with a state at the other fixed point in the doublet. The degeneracy factor for the s=2 sector is therefore 5, as in (3.18). In the Q^3-twisted sector, the 16 fixed points break up into one singlet and 5 "triplets" under the action of Q. Q-singlet states are obtained from these triplets as the sum of states at each of the fixed points that are components of the triplets. The degeneracy factor is therefore 6, again as in (3.18).

The orbifold formula for the Euler characteristic reproduces these factors: $\chi(s=1) = \chi(s=5) = 1$, $\chi(s=2) = \chi(s=4) = 5$, and $\chi(s=3) = 6$.

Also, the cohomology of the inserted regions verifies the factors. There is one $E(Z_6)$, 4 $E(Z_3)'s$, and 5 $E(Z_2)$'s inserted. The 5 harmonic (1,1)-forms of the E(Z_6) region correspond to 1 state that is closed after identification under Q ("Q-closed"), 1 that is Q^5-closed, 1 that is Q^2-closed and another that is Q^4-closed, and finally, 1 that is Q^3-closed. The two harmonic (1,1)-forms of E(Z_3) correspond to a Q^2-closed state and a Q^4-closed state. Finally, each of the five E(Z_2)'s has 1 harmonic (1,1)-form, yielding a Q^3-closed state. So we have 1 Q(Q^5)-closed state, 5 $Q^2(Q^4)$-closed states, and 6 Q^2-closed states, reproducing the degeneracy factors of (3.18).

To close this chapter we note that it is easy to use the orbifold Euler characteristic formula (3.16) to find the numbers and types of spaces $E(Z_l)$ required in the blowing up of an orbifold K3(Z_l). This circumvents the more tedious procedure, used above, of finding the points fixed under the various elements of Z_l, and their transformation properties under the other elements. Perhaps that procedure can in this way be avoided in more general circumstances.

4. CONCLUSION

We have studied the $E_8 \otimes E_8$ heterotic string on the manifold K3 and its orbifold limits K3(Z_l)=T^4/Z_l. The topology (specifically, the cohomology) of K3 determines the six-dimensional massless spectrum obtained when K3 is the internal space, and also determines some terms in the effective six-dimensional Lagrangian. It was shown how to construct the orbifolds K3(Z_l)=T^4/Z_l. The massless spectra obtained with the K3(Z_l) as the internal spaces were calculated. We showed how to blow up the orbifolds K3(Z_l) into K3, and this knowledge made possible a fruitful comparison of the spectra on K3(Z_l) with the spectrum on K3.

This comparison shows that, in some sense, the string "knows" about how the orbifold should be blown up into a Calabi-Yau manifold. There is a correspondence between massless string states on the orbifold and harmonic forms on the manifold obtained by blowing up the orbifold. Furthermore, one can understand the distinct twisted sectors of the string massless spectrum given the number and cohomology of the spaces inserted to replace the singular points of the orbifold. Knowing how to blow up the orbifold is enough to (almost) determine the massless spectrum of string states on that orbifold, including the separate twisted sectors.

Conversely, knowledge of the string massless modes on the orbifold can yield information about the blowing-up procedure required. This has recently been demonstrated in Ref. 34, where it is noted that the existence of certain massless scalar fields (in the twisted sectors) whose potential is flat (to all orders in string perturbation theory) signals that the corresponding singularity can be blown up. Essentially the massless field can acquire a nonzero vacuum expectation value related to the nonzero size of the space inserted to repair the relevant singularity. In addition, for these blown-up Calabi-Yau manifolds, the mass spectrum, Yukawa couplings, and the other parameters of the effective Lagrangian can be calculated using the string amplitudes on the orbifold.[34]

Of great help in our analysis was the orbifold Euler characteristic formula (equation (3.16)) of References 2 and 3. This formula could tell us the numbers and types of spaces to be inserted in the repair of the orbifold singularities, at least for the class of orbifolds we considered. This is another example of string formulae on orbifolds "knowing" about the underlying Calabi-Yau manifold. It would be interesting to see if the orbifold Euler characteristic formula can be used in more general cases as a tool for finding the correct blowing up procedure.

Our study of K3 and its orbifold limits illustrates general features of Calabi-Yau manifolds and the corresponding orbifolds. There can exist many orbifold limits of a single Calabi-Yau manifold. Furthermore, there are many possible compactifications of a single such orbifold limit, resulting from different choices of shift vectors (in the Abelian case), or equivalently, different gauge group breakings. For example, for K3(Z_2) there are 2 possible gauge group breakings, and for K3(Z_3), K3(Z_4), and K3(Z_6) there are 5,10, and 30 possibilities, respectively. And we have not allowed the inclusion of Wilson lines on the non-trivial closed paths of the torus from which the orbifold is constructed.[3,35] One sees that the number of distinct orbifold compactifications for a single Calabi-Yau manifold is rather large.

Questions remain that could possibly be answered in the simple four-dimensional system we have studied. The orbifolds T^d/G, where $G \subset SU_d$, like Calabi-Yau manifolds, are guaranteed to preserve supersymmetry when used in compactifications with the background gauge connection set equal to the spin connection. Are all such orbifolds limits of Calabi-Yau manifolds? Conversely, can all orbifold limits of a Calabi-Yau manifold be written as T^d/G for some values of their moduli? We have not touched non-Abelian orbifolds; are there further complications in the blowing-up procedures required for orbifolds with G non-Abelian? The phenomena of further gauge group breaking and Wilson lines on the non-trivial closed paths of the torus could also be studied in the simple four-dimensional system.

Of course any such study, like the one presented here, provides no direct help in the most important question—how does the string select its ground state? However, it is hoped that such analyses will provide some indirect help in the solution of this difficult problem.

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FIGURE CAPTIONS

1) The extended Dynkin diagram of E_8.

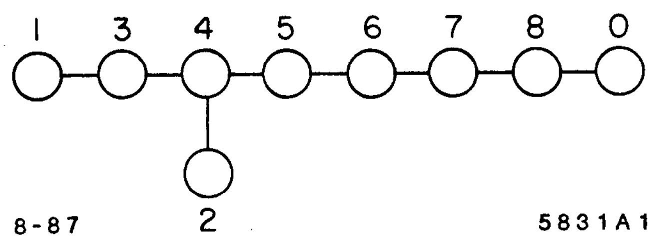

Fig. 1