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Luminosity and Tune Shift in e^+e^- Storage Rings*

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1. Introduction

Luminosity and tune shift have been the subject of numerous papers and talks since the invention of electron-positron storage rings. In this paper we derive an equation for luminosity and one for the linear tune shift based upon two simple assumptions. The first assumption is that the storage ring be designed such that the linear tune shifts in the two transverse planes, x and y, are equal; i.e., that $\Delta\nu_x = \Delta\nu_y$. The second assumption is that the maximum acceptable disruption angle, θ_D, of the colliding beams is approximately equal to the "natural" beam spread, θ_B, of the stored colliding beams at the interaction point.

We will first derive the results for round beams having transverse gaussian distribution functions and then extend the derivation to beams having elliptical cross sections. We will then compare our theoretical results with the observed results in several operating machines and with the "design" parameters of three new machines; namely KEK, BEPC, and LEP.

2. Theory

TUNE SHIFT

The linear beam-beam tune shift parameters $\Delta\nu_y$ and $\Delta\nu_x$ are defined by the equations[1]

$$\Delta\nu_y = \left(\frac{Nr_e}{2\pi\gamma}\right)\frac{\beta_y^*}{\sigma_y(\sigma_x+\sigma_y)} \tag{1}$$

and

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$$\Delta\nu_x = (\frac{Nr_e}{2\pi\gamma})\frac{\beta_x^*}{\sigma_x(\sigma_x+\sigma_y)} \tag{2}$$

N is the number of particles in each stored bunch, β_y^* and β_x^* are the lattice beta functions at the interaction point. $\sigma_y = \sqrt{\beta_y^*\epsilon_y}, \sigma_x = \sqrt{\beta_x^*\epsilon_x}$ where ϵ_y and ϵ_x are the emittances of the stored, colliding beams in the y and x planes respectively. $r_e = \frac{e^2}{mc^2}$ is the classical electron radius and $\gamma = \frac{E_0}{mc^2}$.

If the tune shifts in the two planes are equal, i.e., $\Delta\nu_x = \Delta\nu_y$ then it follows from Eq. (1) and Eq. (2) that

$$\sqrt{\frac{\epsilon_x}{\beta_x^*}} = \sqrt{\frac{\epsilon_y}{\beta_y^*}} = \theta_B \tag{3}$$

where θ_B is the angular beam spread of the stored colliding beams at the interaction point. Note that it is the same in the x and y phase planes.

DISRUPTION ANGLE

The maximum distruption angle, θ_D, resulting from two round beams, with gaussian cross sections colliding with each other has been given by Hollebeek and Minten.[2] The result is

$$\theta_D \cong \frac{Nr_e}{\gamma\sigma_R} \tag{4}$$

where for a round beam $\epsilon_x = \epsilon_y = \epsilon_R$, $\beta_x^* = \beta_y^* = \beta_R^*$ and $\sigma_R = \sqrt{\beta_R^*\epsilon_R}$. Equating Eqs. (3) and (4), we implement our second assumption that the maximum acceptable disruption angle, θ_D, of the colliding beams is approximately equal to the natural angular beam spread of the stored beams, θ_B. The result is

$$\frac{Nr_e}{\gamma\sigma_R} = K\sqrt{\frac{\epsilon_R}{\beta_R^*}}$$

or

$$\frac{Nr_e}{\gamma} = K\epsilon_R \tag{5}$$

for the round beam case. K is a factor to be determined by comparing our theoretical results with the observed luminosity in operating colliders.

Substituting Eq. (5) into Eq. (1) or (2), we find the interesting result for the linear tune shift

$$\Delta\nu_R = \frac{K}{4\pi} \tag{6}$$

This result for round beams has been mentioned by Pellegrini[3] in a 1972 review article on colliding beam accelerators.

The corresponding equation for the luminosity of two colliding round gaussian beams may now be derived using Eq. (5) and the usual definition of luminosity. The result is

$$\mathcal{L} = \frac{N^2 fb}{4\pi\sigma_R^2} = \frac{Nfb}{4\pi\beta_R^*\epsilon_R^*} \cdot \frac{K\gamma\epsilon_R^*}{r_e}$$

or

$$\mathcal{L} = \frac{(Nfb)\gamma K}{4\pi r_e \beta_R^*} \tag{7}$$

where (Nfb) $= 6{\cdot}10^{18}$ I, $\gamma = \frac{E_0}{mc^2}$ and $r_e = \frac{e^2}{mc^2}$ is the classical electron radius. N is the number of particles per bunch, f is the circulating frequency of the machine, b is the number of bunches in each beam and I is the total stored beam current in each beam. Substituting numbers into Eq. (7), we have

$$\mathcal{L} \approx 3.3 \cdot 10^{31} (\frac{IE_0K}{\beta_R^*}) \mathrm{cm}^{-2}\mathrm{sec}^{-1} \tag{8}$$

where I is in amps, E_0 in GeV and β_R^* in meters.

ELLIPTICAL BEAM CROSS SECTIONS

If the colliding beams have elliptical cross sections, then we must modify the equations to take this into account. To do this, we assume that the elliptical beams have the same cross sectional area and charge density as the round beams such that $\sigma_x\sigma_y = \sigma_R^2$, and $\sqrt{\epsilon_x\epsilon_y} = \epsilon_R$. Given this assumption, it has been determined by computer simulations[4] that the maximum disruption angle for the "equivalent" elliptical beams is essentially the same as that for the round beams. Specifically if

$$(\frac{\sigma_x}{\sigma_y}) = \frac{4}{1}, \text{then } \theta_D \approx 0.9\theta_D \ \ \text{(round)}$$

and for

$$(\frac{\sigma_x}{\sigma_y}) = \frac{6}{1}, \text{then } \theta_D \approx 0.85\theta_D \text{ round})$$

Substituting $\epsilon_R = \sqrt{\epsilon_x \epsilon_y}$ into Eq. (5), we arrive at the result

$$\frac{Nr_e}{\gamma} = K\sqrt{\epsilon_x \epsilon_y} \tag{8}$$

for colliding beams having elliptical cross sections. We are now in a position to derive expressions for the luminosity and tune shift.

For elliptical gaussian beams with sigmas of σ_x and σ_y, we define the luminosity as

$$\mathcal{L} = \frac{N^2 fb}{4\pi\sigma_x\sigma_y}$$

and substitute the results of Eq. (8), yielding

$$\mathcal{L} = \frac{(Nfb)\gamma K}{4\pi r_e} \cdot \frac{\sqrt{\epsilon_x\epsilon_y}}{\sigma_x\sigma_y} = \frac{(Nfb)\gamma K}{4\pi r_e\sqrt{\beta_x^*\beta_y^*}} \tag{9}$$

Note that Eq. (9) reduces to Eq. (7) if $\beta_x^* = \beta_y^* = \beta_R^*$ as it should.

As with Eq. (7) if we evaluate the constants and note that $(Nfb) \cong 6 \cdot 10^{18} I$, we obtain

$$\mathcal{L} = 3.3 \cdot 10^{31} \frac{IE_0K}{\sqrt{\beta_x^*\beta_y^*}} \mathrm{cm}^{-2}\mathrm{sec}^{-1} \tag{10}$$

for the predicted luminosity of electron-positron storage rings. Again I is the stored current in each beam measured in amps, E_0 is the energy in GeV, and β_x^* and β_y^* are the x and y beta functions measured in meters. K is to be determined by comparison with operating machines. The results of these comparisons are tabulated in Table 1.

Table 1
Luminosity of Storage Rings*

$$\mathcal{L} = \frac{N^2 fb}{4\pi\sigma_x\sigma_y} = \frac{(Nfb)\gamma K}{4\pi r_e\sqrt{\beta_x^*\beta_y^*}} = 3.3\cdot 10^{31}\frac{IE_oK}{\sqrt{\beta_x^*\beta_y^*}}\mathrm{cm}^{-2}\mathrm{sec}^{-1}$$

I(amps), E_0(GeV), β(meters)

Machine	I_{MAX} (mA)	E_o (GeV)	B_y^* (cm)	$\sqrt{\beta_x^*/\beta_y^*}$	$\mathcal{L}_{exp}$ ($10^{30}\mathrm{cm}^{-2}\mathrm{sec}^{-1}$)	K
VEPP-2M	20	0.51	5.8	2.6	1.2	0.53
ACO	35	0.51	400	0.5	0.1	0.34
ADONE	31	1.5	337	1.6	0.2	0.7
SPEAR	16.4	1.89	10	3.5	2.0	0.68
DCI	24	0.8	218	1.0	0.07	0.24
VEPP-4	12	4.7	12	5.0	6.0	1.93
DORIS II	45	5.0	5	3.6	30.0	0.73
CESR	18	5.28	3	6.5	15.0	0.93
PETRA	11.4	11.0	9	3.8	8.0	0.66
PEP	24.5	14.5	11	5.2	32.3	1.58
LEP (Design)	3.0	55	7	5	16	1.0
BEPC(Design)	66	2.8	10	3.6	17	1.0
KEK (Design)	6	28	10	4	14	1.0

⋆ J.T. Seeman, SLAC-PUB-3825.

BEAM-BEAM TUNE SHIFT

Substituting the results of Eqs. (8) and (3) into either of Eq. (1) or Eq. (2), we obtain the expression for the linear tune shift

$$2\pi\Delta\nu_{x,y} = \frac{K}{\sqrt{\frac{\beta_x^*}{\beta_y^*}} + \sqrt{\frac{\beta_y^*}{\beta_x^*}}} \tag{11}$$

This reduces to Eq. (6) for the tune shift in round gaussian beams if $\beta_x^* = \beta_y^*$ as is expected. Note that except for the factor K, the tune shift is just a geometric property of the aspect ratio of the elliptical cross section measured at the interaction point. All of the "interesting" physics seems to be contained in the factor K.

COMPARISON WITH EXPERIMENTS

We note from the definition of K (Eq. 5) that it is the ratio of the maximum disruption angle, θ_D, that a collider will accept to the natural angular spread, θ_B, of the stored, colliding beams at the interaction point. No attempt has been made to determine K analytically. However, by comparing the predicted luminosities and tune shifts as determined by extensive computer simulations for the recently designed machines LEP, KEK, and BEPC, we find that $K = 1$ is in good agreement with all of these "new" designs. It remains to be seen what the experimental value of K will be for these colliders!

A value of $K \cong 0.7$ predicts the observed luminosity for Adone, Spear, Doris II, and Petra. For CESR we find $K \cong 0.9$. The large values for PEP and VEPP-4 are not understood at this time. We cannot evaluate the results for DCI and ACO since they do not satisfy our first design condition that $\Delta\nu_x = \Delta\nu_y$. The low K value for VEPP-2M is also not understood at this time.

REFERENCES

1. J.T. Seeman, SLAC-PUB-3825, November 1985.
2. R. Hollebeek, Nucl. Instr. and Meth. **184**, 333-347 (1981) and A. Minten, *Disruption Angles*, Single Pass Collider Memo CN-305, June 25, 1985.
3. C. Pellegrini, Annual Review of Nuclear Science, ed. by E. Segre, Vol. 22, 1972 (Annual Reviews, Inc., Palo Alto, California) p. 5521.
4. Roger Servranckx, in a private communication.