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Fun and Frustration with Hydrogen in 1 + 1 Dimension*

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ABSTRACT

The Coulomb potential is derived in "one space - one time" dimension, and introduced into Dirac and Klein–Gordon equations. The equations are solved, and somewhat surprising result – nonexistence of bound state solutions in the lower dimension – discussed and identified as another fine example of the "Klein paradox".

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1. Introduction

The success and quick acceptance of the Quantum Mechanics (QM) in both nonrelativistic and relativistic form, and later of Quantum Field Theory (QFTh), is to a good degree related to their successfulness in the analyses of hydrogen. It is mainly through the repetition of hydrogen related calculations from our textbooks that we gain a confidence in the sequence

$$\text{Nonrelativistic QM} \rightarrow \text{Relativistic QM} \rightarrow \text{Quantum Field Theory} \ . \qquad (1.1)$$

Indeed, the Schrödinger (nonrelativistic) equation correctly predicts $m\alpha^2$ terms in hydrogen spectrum, the Dirac (relativistic) equation gives successfully the fine structure (order $m\alpha^4$), while QFTh can be used to calculate energy terms to even higher orders [1,2]. (Here, m denotes the mass of electron, and α is the fine structure constant).

However, had we lived in an environment in which e.g., *positronium* and not hydrogen was readily available, the history of physics would most likely have been very different. The positronium is a bound state of an electron and a positron [3-5], with essentially nonrelativistic spectrum. One can again try to use the Dirac equation as a middle step in an analysis, hoping that this will reveal fine corrections to energy levels. If the mass of the electron in the Dirac equation for hydrogen is simply replaced by the reduced mass $m/2$, and the proton mass changed to m, the expression for the energy becomes

$$E^S_{n,L,J} = 2mc^2 - \frac{\alpha^2 mc^2}{4n^2} - \frac{\alpha^4 mc^2}{2n^3(2J+1)} + \frac{3\alpha^4 mc^2}{16n^4} + O(\alpha^6) \ . \qquad (1.2)$$

n is the principal quantum number. In hydrogen J, L, S indicated angular momentum, orbital angular momentum and spin of the *electron* , respectively. In positronium one might try an interpretation in which the above quantum numbers denote the total angular momentum, relative orbital angular momentum,

and the total spin of *two constituents* . Eq. (1.2) then can be compared to the experimental results. However, the observed spectrum of parapositronium (a state of positronium with total spin 0, and $L = J$) is to order α^4 given by [5]

$$E^0_{n,J,J} = 2mc^2 - \frac{\alpha^2 mc^2}{4n^2} - \frac{\alpha^4 mc^2}{2n^3(2J+1)} + \frac{11\alpha^4 mc^2}{64n^4} + \ldots \quad . \qquad (1.3)$$

We see that Eq. (1.2) does not reproduce all α^4 terms correctly! The disagreement between the predictions of the Dirac "reduced mass" equation and the observed spectrum is even bigger and more dramatic for orthopositronium (a state with $S = 1$): not only that levels are displaced relative to parapositronium, but an additional splitting of magnitude α^4 and depending on projections S_z of the total spin appears. Eq. (1.2) clearly can not account for such a situation.

It does not take long to understand why the sequence (1.1) is valid for hydrogen but fails for positronium. In hydrogen, radiative corrections, interactions of magnetic moments and relativistic corrections due to the recoil, are all of the order $(m/M)\alpha^4 mc^2$ or higher and thus not important if only terms up to order α^4 are considered. (M stands for mass of proton). On the contrary, no small mass ratio appears in positronium ($m/M \to 1$), and the relativistic recoil and electromagnetic interactions other than the Coulomb attraction induce terms of magnitude comparable to Dirac's $\alpha^4 mc^2$ terms. Consequently, fine and hyperfine structures are of the same order, and mostly due to the spin–orbit and spin–spin interactions the spectrum of positronium is considerably more elaborate than the one given by Eq. (1.2). We can conclude that the Dirac approach, though perfect (to the order α^4) for hydrogen, is not particularly useful in analyses of bound states with constituents of nearly equal masses.

In this paper we shall examine a situation in which even the hydrogen presents a problem for the Dirac equation. This happens if the dimension of space becomes lower than 3. The source of trouble this time is unrelated to obstacles met in the analysis of positronium. The problem can most clearly be seen in a study of a world with one space and one time degree of freedom. There is no spin

or orbital angular momenta, and no magnetic field in $1+1$ dimension. This eliminates a possibility of spin–related splitting. Furthermore, the Schrödinger equation with the Coulomb potential, which is attractive but not proportional to $1/x$ in $1+1$ dimension, can be easily solved. The Dirac equation is simpler than in $3+1$ dimensions, and QFTh (known as Schwinger massive model) is relatively well understood[6] at least in the "weak coupling" and "strong coupling" limits. Nothing so far signals a failure of the sequence (1.1). Still, some surprises are waiting for us.

We shall begin with a brief review of electrodynamics and Dirac's theory in $1+1$ dimension. In Section 3 the Dirac (and Klein–Gordon) equation with Coulomb potential will be solved. Somewhat unexpected result of Section 3 will be analysed in Section 4. Two appendices are added: in Appendix A the nonrelativistic limit is derived, and Appendix B gives some basic information on the parabolic cylinder equation. The present lack of experimental data from $1+1$ dimensional universe is regrettable, but – fortunately – will not be a serious drawback in what follows.

2. Life in one dimension

In a not too likely case of a sudden collapse of two space degrees of freedom, the electrodynamics in the resulting one space and one time $(1+1)$ dimension would be described by only two Maxwell equations, compactly written as

$$\partial_\alpha F^{\alpha\beta} = \frac{1}{\epsilon_0 c^2}\, j^\beta \qquad (2.1)$$

The field tensor is defined by $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, the vector field A^α has two components, $A^\alpha = (\frac{1}{c}U, A)$, and the conserved current is $j^\alpha = (c\rho, j)$, $\partial_\alpha j^\alpha = 0$. Symbol ∂_α stands for the two–vector $(\partial/\partial\, ct\,,\ \partial/\partial x)$. The metric tensor is $g_{00} = -g_{11} = 1$, and summation over repeated Lorentz indices is understood: $ab = a^\alpha b^\beta g_{\alpha\beta} = a^0 b^0 - a^1 b^1$. The constant ϵ_0 appears in the MKS system of units and denotes the permittivity of a free space.

The gauge transformation $A^\alpha \rightarrow A^\alpha + \partial^\alpha \Lambda$ does not affect equations of motion (2.1). A gauge invariant function

$$E(x,t) = -cF^{01} = -\frac{\partial U}{\partial x} - \frac{\partial A}{\partial t} \quad , \tag{2.2}$$

can be called "electric field" of the theory. Note that there is no room for another gauge invariant function, and "magnetic field" is therefore absent in $1+1$ dimension. In terms of E, Maxwell equations (2.1) read

$$\frac{\partial E}{\partial x} = \frac{1}{\epsilon_0}\rho(x,t) \quad , \qquad \frac{\partial E}{\partial t} = -\frac{1}{\epsilon_0} j(x,t) \quad . \tag{2.3}$$

These equations determine the electric field in presence of external charges and currents. The solution of system (2.3) is

$$E(x,t) = \frac{1}{2\epsilon_0} \int_{-\infty}^{+\infty} dx'\, \rho(x',t)\, [\, \Theta(x-x') - \Theta(-x+x')\,] \, + \, G \quad . \tag{2.4}$$

This is a gauge independent expression in which Θ is the Heavyside function, and G a constant (in space) background electric field[7]. G disappears in the three–dimensional world, and comes out not to be too important in the study of single–particle relativistic equations in one–dimensional space. However, it plays a significant role in $1+1$ dimensional field theory[6].

For our later study of hydrogen–like systems we have to determine the static potential between two fixed charges. (By imitating the successful procedure from the analysis of three–dimensional hydrogen, we neglect all retardation effects and velocity dependent contributions to the potential). It will be sufficient to find a vector potential A^α of a fixed point charge q at the origin. The correct interaction term in relativistic equations is then obtained by using the "minimal coupling"

(see later). With $\rho(x',t) = q\,\delta(x')$, Eq. (2.4) gives the electric field

$$E(x,t) = \frac{q}{2\epsilon_0}\,[\,\Theta(x) - \Theta(-x)\,] + G \qquad . \tag{2.5}$$

While the electric field is unique, it can be generated by various potentials A^α. Eq. (2.2) shows that a possible choice is

$$U(x,t) = -\frac{q}{2\epsilon_0}\,|x| - Gx \qquad , \qquad A(x,t) = 0 \qquad . \tag{2.6}$$

As long as $|G| < |q|/2\epsilon_0$, the potential (2.6) induces an attractive force between particles of opposite charges. As a further simplification, we shall assume that the above inequality is satisfied, and neglect the background field thereafter. (This will in no way affect the conclusions of the paper). The choice (2.6) with $G = 0$ will be referred to as the Coulomb potential. So, the Coulomb potential in $1+1$ dimension changes linearly with the distance, giving rise to a constant, infinite–range force[8]. Using terminology from three–dimensional electrodynamics, we can say that (2.6) corresponds to both axial ($A = 0$) and Coulomb ($\partial A/\partial x = 0$) gauge. The gauge transformation may generate A^α for some other gauge.

Having found the Coulomb potential, we can proceed by constructing relativistic single–particle equations of motion. The Dirac equation for a free fermion is – like in $3+1$ dimension – based on a linearization of the energy relation $E = \sqrt{c^2p^2 + m^2c^4}$. This requires an introduction of traceless, Hermitian, 2×2 matrices α and β. The Dirac equation for a particle with mass m becomes

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(\frac{\hbar c}{i}\alpha\,\frac{\partial}{\partial x} + mc^2\beta\right)\Psi(x,t) \qquad , \tag{2.7}$$

where $\Psi(x,t)$ is a two–component spinor, and Dirac's matrices satisfy $\alpha^2 = \beta^2 = 1$, $\alpha\beta + \beta\alpha = 0$. The "standard choice" (β is diagonal) in $1+1$ dimension is

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad , \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad . \tag{2.8}$$

If a fermion has a charge q', and is placed in an external field A^α, the minimal

substitution ($\partial^\alpha \to \partial^\alpha + \frac{i}{\hbar} q' A^\alpha$) should be introduced into the equation of motion. Thus we have to replace space and time derivatives in Eq. (2.7) by

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{iq'}{\hbar} U \quad , \qquad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} - \frac{iq'}{\hbar} A \quad . \tag{2.9}$$

The resulting equation for the Coulomb potential (2.6) will be considered in the next section.

Another relativistic equation can be constructed following the Klein–Gordon procedure. We quantize the theory starting with $E^2 = c^2 p^2 + m^2 c^4$ relation. The wave equation becomes

$$-\hbar^2 \frac{\partial^2 \Phi}{\partial t^2} = (-\hbar^2 c^2 \frac{\partial^2}{\partial x^2} + m^2 c^4) \Phi(x,t) \quad . \tag{2.10}$$

To obtain the equation of motion for a charged particle (charge q') in an external field, the minimal substitution (2.9) again must be used.

Only a brief sketch of main equations relevant for $1+1$ dimensional world is given above. The reader might wish to study these equations more thoroughly. Standard textbooks on quantum mechanics often contain detailed studies of $3+1$ dimensional Maxwell, Dirac and Klein–Gordon equations (see e.g., Chapters XX, Part 2 and XXI, Part 3 in Ref. 1). A simple repetition of the steps from three–dimensional space reveals many interesting aspects of the compactified theory, and it is funny and instructive to convert as much of the material as possible.

3. Dirac and Klein–Gordon equations with Coulomb potential

From now on we shall neglect all but instantaneous Coulomb interaction between two fermions of opposite charges, assume that one of the particles is much heavier ("proton"), and then calculate the spectrum of the lighter particle ("electron") with relativistic equations. It should also be noted that most of the analysis could be repeated for two particles of nearly equal masses, by using the relative coordinates and the reduced mass system. We could hope that the procedure would fill a gap between a simple nonrelativistic approach and a more complete (and more complicated) QFTh solution.

When the Coulomb potential $U = -q|x|/2\epsilon_0\,,\ A = 0$ is introduced into the Dirac equation for a particle of mass m and charge q', and a separation of space and time coordinates performed by a substitution

$$\Psi(x,t) = e^{-iEt/\hbar}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} \qquad , \tag{3.1}$$

a system of two coupled equations results. With the standard choice (2.8) for Dirac's matrices, and $q' = -q$, one obtains

$$\begin{aligned} \psi_1' + i\,[\,\frac{q^2}{2\epsilon_0\hbar c}|x| - \frac{E+mc^2}{\hbar c}\,]\,\psi_2 = 0 \qquad , \\ \psi_2' + i\,[\,\frac{q^2}{2\epsilon_0\hbar c}|x| - \frac{E-mc^2}{\hbar c}\,]\,\psi_1 = 0 \qquad . \end{aligned} \tag{3.2}$$

Eq. (3.2) allows an easy treatment of the nonrelativistic limit (see Appendix A). However, for a complete solution it is more convenient to introduce

$$\varphi_1 = (\psi_1 + \psi_2)/\sqrt{2} \qquad , \qquad \varphi_2 = (\psi_1 - \psi_2)/\sqrt{2} \qquad . \tag{3.3}$$

Technically, this corresponds to another choice of Dirac's matrices, the one with

a diagonal α and a real β. Eq. (3.2) then becomes

$$\begin{aligned} \varphi_1' + i\,[\,\frac{q^2}{2\epsilon_0\hbar c}|x| - \frac{E}{\hbar c}\,]\,\varphi_1 &= -\,i\frac{mc}{\hbar}\varphi_2 \quad , \\ \varphi_2' - i\,[\,\frac{q^2}{2\epsilon_0\hbar c}|x| - \frac{E}{\hbar c}\,]\,\varphi_2 &= +\,i\frac{mc}{\hbar}\varphi_1 \quad . \end{aligned} \tag{3.4}$$

To solve this system, one must find general solutions in $x \leq 0$ and $x \geq 0$ regions, and match them at the boundary so that the wave function and its derivative is continuous. We shall look closely at $x \geq 0$ region. (A similar analysis can be performed in the other region).

The notation is simplified by introduction of a dimensionless coupling constant Δ, and a new dimensionless variable y,

$$\Delta = \frac{q^2}{m^2}\frac{\hbar}{\epsilon_0\, c^3} \quad , \qquad y = x\sqrt{\frac{q^2}{\epsilon_0\hbar c}} - 2E\sqrt{\frac{\epsilon_0}{q^2\hbar c}} \quad . \tag{3.5}$$

The system (3.4) now becomes (note that $-\frac{2E}{mc^2}\sqrt{\frac{1}{\Delta}} \leq y < \infty$),

$$\begin{aligned} \frac{d\varphi_1}{dy} + i\,\frac{y}{2}\,\varphi_1 &= -\,\frac{i}{\sqrt{\Delta}}\varphi_2 \quad , \\ \frac{d\varphi_2}{dy} - i\,\frac{y}{2}\,\varphi_2 &= +\,\frac{i}{\sqrt{\Delta}}\varphi_1 \quad . \end{aligned} \tag{3.6}$$

The elimination of e.g., φ_2 from (3.6) finally leads to

$$\frac{d^2\varphi_1}{dy^2} + [\,\frac{y^2}{4} - (\frac{1}{\Delta} - \frac{i}{2})\,]\,\varphi_1 = 0 \quad . \tag{3.7}$$

The good news is that (3.7) is one of the known second order differential equations, called "parabolic cylinder equation". The general solution can be

expressed (see Appendix B) in terms of parabolic cylinder functions $D_\nu(z)$:

$$\varphi_1(y) = \gamma\, D_{i/\Delta}(ye^{i\pi/4}) - \delta\sqrt{\frac{-i}{\Delta}}\, D_{-1-i/\Delta}(ye^{-i\pi/4}) \quad . \tag{3.8}$$

The first equation in (3.6) then gives

$$\varphi_2(y) = -\gamma\sqrt{\frac{i}{\Delta}}\, D_{-1+i/\Delta}(ye^{i\pi/4}) + \delta\, D_{-i/\Delta}(ye^{-i\pi/4}) \quad . \tag{3.9}$$

γ and δ above are two arbitrary constants. The bad news is that solutions (3.8) and (3.9) can never represent a bound state: although the potential is attractive, the Dirac equation allows no hydrogen (and similarly – no positronium) in $1+1$ dimension! This claim will be substantiated below.

Bound states are characterized by normalizable wave functions. Therefore, only if a certain choice of γ , δ , and E produces a finite value of the integral

$$\int^{y} dy'(\,|\varphi_1|^2 + |\varphi_2|^2\,) \tag{3.10}$$

when $y \to \infty$, we have a bound state. However, the asymptotic behavior of $D_\nu(z)$ is known (see Appendix B), and one easily finds

$$\begin{aligned}
\varphi_1(y \to +\infty) &= \gamma\, e^{-\pi/4\Delta}\, \exp\,(-\frac{iy^2}{4} + \frac{i}{\Delta}\ln y)\;\; [\,1 + \mathcal{O}(y^{-1})\,] \;\;, \\
\varphi_2(y \to +\infty) &= \delta\, e^{-\pi/4\Delta}\, \exp\,(+\frac{iy^2}{4} - \frac{i}{\Delta}\ln y)\;\; [\,1 + \mathcal{O}(y^{-1})\,] \;\;.
\end{aligned} \tag{3.11}$$

Thus regardless of values of constants, the integral (3.10) reduces to

$$e^{-\pi/2\Delta}\;(\,|\gamma|^2 + |\delta|^2\,)\int^{y} dy' \quad , \tag{3.12}$$

and diverges when $y \to \infty$. Consequently, the system (3.6) has a solution for an arbitrary energy, but does not provide genuine bound states. This answer could have been guessed already from the form of equation (3.7): positive y^2 dominates, and prevents any exponentially decaying amplitude at infinity.

A different insight comes from the analysis of the Klein–Gordon equation with the Coulomb potential. After the factorization $\Phi(x,t) = \exp(-iEt/\hbar)\,\varphi$, the Eq. (2.10) with the minimal substitution (2.9) and $q' = -q$ becomes

$$\varphi'' + [\,(\frac{E}{\hbar c} - \frac{q^2}{2\epsilon_0 \hbar c}|x|\,)^2 - (\frac{mc}{\hbar})^2\,]\varphi = 0 \qquad . \tag{3.13}$$

The equation has a familiar nonrelativistic form. Indeed, with

$$\mathcal{E} = \frac{(E - mc^2)(E + mc^2)}{2mc^2} \quad , \quad \mathcal{V}_{eff}(x) = \frac{q^2 E|x|}{2mc^2\epsilon_0} - \frac{q^4 x^2}{8mc^2{\epsilon_0}^2} \quad , \tag{3.14}$$

Eq. (3.13) can be written as

$$\frac{d^2\varphi}{dx^2} + \frac{2m}{\hbar^2}(\mathcal{E} - \mathcal{V}_{eff}\,)\varphi = 0 \qquad . \tag{3.15}$$

The effective potential is sketched in Fig. 1. When $c = \infty$ (nonrelativistic limit), $\mathcal{V}_{eff}$ is truly a confining potential ($\mathcal{V}_{eff} \to q^2|x|/2\epsilon_0$), but for any finite value of c, would–be bound state solutions leak through the potential barrier to continuum. It comes as no surprise that bound states are absent in such a potential. The formal proof easily follows after an analysis of Eq. (3.13) in the $x \geq 0$ region. The replacement (3.5) transforms the equation to

$$\frac{d^2\varphi}{dy^2} + (\,\frac{y^2}{4} - \frac{1}{\Delta}\,)\varphi = 0 \qquad , \tag{3.16}$$

and the general solution is

$$\varphi(y) = \sigma\, D_{i/\Delta - 1/2}(ye^{i\pi/4}) + \tau\, D_{-i/\Delta - 1/2}(ye^{-i\pi/4}) \qquad . \tag{3.17}$$

With the known asymptotic forms of parabolic cylinder functions, one obtains

$$\varphi(y \to \infty) = \frac{e^{-\pi/4\Delta}}{\sqrt{y}}\,[\sigma \exp(-\frac{iy^2}{4} + \frac{i}{\Delta}\ln y - \frac{i\pi}{8}) + \tau \exp(\frac{iy^2}{4} - \frac{i}{\Delta}\ln y + \frac{i\pi}{8})\,]. \tag{3.18}$$

Thus regardless of the choice for constants σ, τ, the normalization integral di-

verges for large y :

$$\int^{y} dy' |\varphi(y')|^2 \sim \int^{y} dy'/y' \to \infty . \tag{3.19}$$

To summarize, we have run into an apparent paradox. The Coulomb potential, which is confining for nonrelativistic particles in $1+1$ dimension, provides no bound states when applied to Dirac or Klein–Gordon equation. The equations have solutions, but these are not discrete. That happens not only for hydrogen atoms with an infinitely heavy source of potential, but also for positronium–like systems. However, the last step in sequence (1.1), Schwinger model, again *does* produce[6] bound states! What could have caused the failure of the middle step? Could it be that neglected retardation effects and some two–body relativistic corrections played such a crucial role in $1+1$ dimension, though they were unimportant in $3+1$ dimensions? We shall study these questions more thoroughly in the next section.

4. Paradox resolved; Conclusion

The object of interest in the previous section were two particles of opposite charges in an attractive potential. In hydrogen one of these two particles is much heavier than the other, but this comes out not to be a crucial detail. When we assumed an unlimited velocity for propagation of signals (i.e., in the nonrelativistic limit – see Appendix A), equations of motion revealed an infinite set of bound states. The particles became permanently confined in a hydrogen–like system characterized by discrete energy levels, and exponentially decaying wave functions. However, when we tried to use theories that accommodated *finite* speed of light, we found the following: equations of motion had solutions for *continuous* set of energies, and the probability of finding the particles did not fall off with their mutual distance quickly enough to provide bound states. In other words, despite the attractive force, the probability of finding these two particles infinitely separated, remained finite at all times.

One can try to blame the retardation for this strange result. Maybe, by correctly including the time of propagation of the signal from one particle to the other, we could reintroduce bound states into the relativistic picture. However, the explanation lies somewhere else. After all, the retardation effects do not play a crucial role in analyses of three–dimensional objects, so why would they be so important in $1+1$ dimension?

The real source of troubles is a paradox named after Oscar Klein and formulated as early as 1929, immediately after the publication of the Dirac theory. Since the paradox has a form slightly different from the one usually described in textbooks, we shall examine it more closely. An electron in the central potential will be considered, but the argument is very similar for two particles analysed in the center of mass system. Imagine first a free Dirac's particle. The equation (2.7) has solutions for all energies for which $|E| \geq mc^2$, and there is a "forbidden" band in between. From mathematical point of view, the negative energy solutions, $E \leq -mc^2$, are therefore perfectly acceptable. It is only through the "hole theory" reinterpretation[1,2] of the Dirac equation that states with such energies are eliminated: in order to stabilize the vacuum, we say that negative continuum is filled up with particles forming the so-called "sea".

The situation changes completely when charges (no matter how small !) are "switched on". Regardless of a sign of its energy, the electron becomes attracted to the center of the potential, and allowed regions now get separated by a V–shaped forbidden region (Fig. 2). Let the energy of the electron be $\bar{E} = mc^2 + \varepsilon$. For a free particle, ε corresponds to the kinetic energy. In the attractive potential (Fig. 2), ε represents a sum of kinetic ε_K, and potential ε_P energies. If the kinetic part is positive, the potential energy has an upper bound and classically, the electron is confined to the inner area, $|x| \leq 2\epsilon_0 \hbar c(\bar{E} - mc^2)/q^2$. Another possibility is to write $\bar{E} = -mc^2 + \varepsilon'$, where ε' is again a sum of two pieces, $\varepsilon'_K + \varepsilon'_P$. We see that if the potential energy satisfies $\varepsilon'_P \geq \varepsilon'$, an electron can have $\varepsilon'_K < 0$ (whatever this means), and still keep the positive total energy. In other words, a particle previously in the negative continuum (characterized

by negative kinetic energies ε'_K), can remain in the sea and have $\bar{E} > mc^2$, provided it is far enough from the center, $|x| \geq 2\epsilon_0\hbar c(\bar{E} + mc^2)/q^2$. Had it not be for tunneling, electrons with positive kinetic energies would have always remained isolated from the continuum. Dirac equation would have two kinds of clearly separated solutions. One with discrete spectrum and localized within the inner allowed region, and the other describing the excited particles from sea, with all possible energies and restricted to the outer region.

However, due to the quantum tunneling, there is always a possibility of communication between the two allowed regions. Tunneling occurs for an arbitrary energy $\bar{E} > mc^2$, and one can never be sure where exactly the electron is localized. Therefore, when we solve the Dirac equation for energies where tunneling occurs, we *do not* obtain two separated solutions. On the contrary, a single, continuous wave function emerges, containing information on probability of finding the particle in all three regions: inner, "forbidden", and outer. By no means can we distinguish the would–be bound states. Solutions will exist for all energies, they will not be normalizable, and will have a similar asymptotic behavior imposed by characteristics of the outer region. And this is just what we discovered in Section 3. The Klein paradox, which is always related to mixing of states with positive and negative kinetic energies in a strong field, is reflected in Section 3 as an absence of bound state solutions where naively we would expect them [9].

In the hole theory the consequence of the above mentioned tunneling is a creation of particle–antiparticle pairs. A particle from sea, with an energy $\bar{E}$ (Fig. 2), can e.g., penetrate through the forbidden zone into the central region, leaving a hole in the sea. The hole is pushed away from the center of potential, and behaves in every respect as an antiparticle. It is this creation of pairs what discredits Dirac and Klein–Gordon equations as tools in a description of one–dimensional hydrogen and similar bound states. Namely, relativistic equations of motion are single–particle equations, and they are meant to cover only the situations in which a pulling of particle–antiparticle pairs out of vacuum is suppressed. In three–dimensional hydrogen for example, a creation of virtual pairs

is a higher order process, a small quantum fluctuation that can be treated by perturbation theory, and therefore the Dirac equation works so good to the order α^4. On the contrary, in our one–dimensional analysis the creation of pairs was not a small quantum fluctuation but an ever–present process of nonperturbative nature: tunneling occurs for any nonzero value of the coupling constant Δ, defined in (3.5). Single–particle equations cannot cope with situations in which particle creation is a prominent process. Thus, when Dirac equation says: there are no hydrogen atoms in one–dimensional electrodynamics, it does not mean that such relativistic bound states are really absent. It only means that this equation is not applicable to the problem.

Finally, the described situation is not a speciality of $1+1$ dimension. It has been known for a while[10] that the Klein paradox also in $3+1$ dimensions prevents the use of Dirac equation with a linear potential. Note however that such a linear potential (which is popular in analyses of quarkonia states) is put into equations by hand, and unlike in $1+1$ dimension is not arising from a gauge theory. But even the three–dimensional Coulomb potential, which *is* derived from a field theory, suffers in certain situations of the Klein paradox! Imagine an atom with the atomic number Z. When the nucleus is pointlike, the highest possible Z for which the Dirac equation can be solved is known to be 137. However, with an extended nucleus one finds discrete levels even beyond that limit. It can be shown that for $Z > 172$ an electron acquires the ionization energy larger than $2mc^2$. Its ground orbit therefore immerges into the negative energy sea, and a variety of new effects might be expected. (For an extensive review see Ref. 11)

In summary, an unsuccessful attempt of demonstration of validity of the sequence (1.1) for the one–dimensional hydrogen was presented. While the transition to the lower dimensional space eliminated problems related to couplings of spin, and made the equations easily solvable, the price payed was much higher. By tunneling through the classically forbidden region, electrons were able to accumulate enough energy to create pairs of new particles, thus totally eliminating a possibility of using the middle step in the (1.1) sequence. When we need more

precise results than the Schrödinger approach can give, the only alternative remains to step into the QFTh. In a way, physicists in the late twenties and the thirties were quite lucky. Not only that they lived in a world with plenty of hydrogen (and not e.g., positronium), but they also lived in three–dimensional space which is about the only one in which Dirac's theory when combined with electrodynamics successfully describes hydrogen–like systems.

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APPENDIX A – Nonrelativistic Limit

In this Appendix a formal expansion in $1/c$ of the Dirac and Klein–Gordon wave functions will be performed, and the resulting Schrödinger equation for the lowest order components will be analysed.

Consider the Dirac equation (3.2). The form of the equation allows a solution for which ψ_1 is even, and ψ_2 odd upon the exchange $c \to -c$: $\psi_1(c) = \psi_1(-c)$ and $\psi_2(c) = -\psi_2(-c)$. This suggests a substitution

$$\psi_1 = f_0 + \frac{1}{c^2} f_1 + \frac{1}{c^4} f_2 + \dots \; ; \qquad \psi_2 = \frac{1}{c} g_0 + \frac{1}{c^3} g_1 + \frac{1}{c^5} g_2 + \dots \quad . \qquad (A.1)$$

With $E = mc^2 + E_{NR} + E_1/c^2 + E_2/c^4 + \dots$, one finds a set of equations corresponding to various powers of c. Particularly, by comparing terms of orders c^0, and c^{-1}, one obtains

$$f_0{}' - \frac{2im}{\hbar} g_0 = 0 \qquad , \qquad g_0{}' - \frac{i}{\hbar} (E_{NR} - V_0) \, f_0 = 0 \qquad , \qquad (A.2)$$

and by combining these two equations,

$$f_0{}'' + \frac{2m}{\hbar} (E_{NR} - V_0(x) \;) f_0 = 0 \qquad . \qquad (A.3)$$

Therefore, f_0 (which is the leading term in the "big" component ψ_1) is determined by the Schrödinger nonrelativistic equation (A.3), with the potential $V_0(x) = q^2|x|/2\epsilon_0$.

In a similar way, a substitution $\varphi = \bar{f}_0 + \bar{f}_1/c^2 + \bar{f}_2/c^4 + \dots$, into the Klein–Gordon equation (3.13) leads to the Schrödinger equation for $\bar{f}_0$.

Equation (A.3) has eigensolutions that could be expressed in terms of Airy functions. One finds that solutions are characterized by a discrete index $n \geq 1$,

and *normalizable*[12]:

$$f_0^{(n)}(x \geq 0) = \mathrm{Ai}[\, x\,(\frac{mq^2}{\epsilon_0\hbar^2})^{1/3} - c_n\,] \quad , \quad E_{NR}^{(n)} = c_n\,(\frac{\hbar^2 q^4}{8m\epsilon_0})^{1/3} \ . \qquad (A.4)$$

Here,

$$c_n = \begin{cases} \bar{a}_{(n+1)/2} & \text{for } n = 1,3,5,\ldots \\ a_{n/2} & \text{for } n = 2,4,6,\ldots \end{cases} \quad , \qquad (A.5)$$

and $-\bar{a}$, $-a$ are zeros (on the negative t axis) of $\mathrm{Ai}'(t)$ and $\mathrm{Ai}(t)$ respectively. It is the property of the Airy functions that $0 < \bar{a}_1 < a_1 < \bar{a}_2 < a_2 \ldots$. When n is odd, the wave function $f_0^{(n)}$ is even, and vice versa. When n is much larger than one, c_n reduces to

$$c_{n>>1} \approx (\frac{3\pi n}{4})^{2/3} \quad . \qquad (A.6)$$

With (A.6) one easily rediscovers the results of (i) WKB approximation and (ii) old Bohr–Sommerfeld quantization procedure for the potential $q^2|x|/2\epsilon_0$, namely $E^{(n)} \sim (n^2q^4/m)^{1/3}$.

The study of f_1, g_1, and higher order terms in (A.1) however shows that these components are not normalizable and thus (except for $c \to \infty$) the whole expansion in $1/c$ is ill–defined, which only confirms results from Section 3.

APPENDIX B – Parabolic Cylinder Functions

This Appendix describes some of the properties of the parabolic cylinder functions. Only the formulas relevant to the paper are presented. More details can be found e.g., in Ref. 13.

Parabolic cylinder equation

Consider the differential equation

$$\frac{d^2w}{dz^2} + (\nu + \frac{1}{2} - \frac{1}{4}z^2)\, w = 0 \qquad . \qquad (B.1)$$

The solutions of (B.1) , denoted by $D_\nu(z)$ and characterized by some fixed values at $z = 0$,

$$\begin{aligned} D_\nu(0) &= 2^{\nu/2}\, \Gamma(1/2)/\, \Gamma(1/2 - \nu/2) \qquad , \\ D_\nu{}'(0) &= 2^{(\nu-1)/2}\, \Gamma(-1/2)/\, \Gamma(-\nu/2) \qquad , \end{aligned} \qquad (B.2)$$

are called the parabolic cylinder functions. If $D_\nu(z)$ is a solution of (B.1) , so are $D_\nu(-z)$, $D_{-\nu-1}(iz)$ and $D_{-\nu-1}(-iz)$, and two linearly independent solutions may always be chosen among these four functions.

Closely related to (B.1) is an equation

$$\frac{d^2u}{dz^2} + (-\lambda + \frac{1}{4}z^2)u = 0 \qquad . \qquad (B.3)$$

Two solutions, $D_{i\lambda-1/2}(ze^{i\pi/4})$ and $D_{-i\lambda-1/2}(ze^{-i\pi/4})$, which are linearly independent for any finite value of λ, were used in this paper.

Useful recurrence relations

$$\frac{dD_\nu(z)}{dz} + \frac{z}{2} D_\nu(z) = \nu D_{\nu-1}(z) \quad ,$$
$$\frac{dD_\nu(z)}{dz} - \frac{z}{2} D_\nu(z) = - D_{\nu+1}(z) \quad . \tag{B.4}$$

Asymptotic values of $D_\nu(z)$

For large and positive y , and when $y >> |\nu|$, it is possible to show that

$$D_\nu(ye^{\pm i\pi/4}) \approx y^\nu \exp(\mp iy^2/4 \pm i\pi\nu/4) \{1 + O(1/y^2)\} \quad . \tag{B.5}$$

Special values

$$D_0(z) = e^{-z^2/4} \quad ; \quad D_{-1/2}(z) = \sqrt{\frac{1}{2\pi}}\, z^{1/2} K_{1/4}(z^2/4) \quad . \tag{B.6}$$

Therefore,

$$D_0(ye^{\pm i\pi/4}) = e^{\mp iy^2/4} \quad , \tag{B.7}$$

and

$$D_{-1/2}(ye^{+i\pi/4}) = -i\sqrt{\frac{y\pi}{8}}\, H^{(2)}_{1/4}(y^2/4) \quad ,$$
$$D_{-1/2}(ye^{-i\pi/4}) = +i\sqrt{\frac{y\pi}{8}}\, H^{(1)}_{1/4}(y^2/4) \quad . \tag{B.8}$$

K, $H^{(1)}$, and $H^{(2)}$ are a modified Bessel function, and two Hankel functions respectively.

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FIGURE CAPTIONS

1) Effective potential in the Klein–Gordon equation. Nonrelativistic limit ($c = \infty$) is denoted by dashed lines.

2) Klein paradox. Forbidden (shaded) and allowed regions, when the linear potential is turned on.

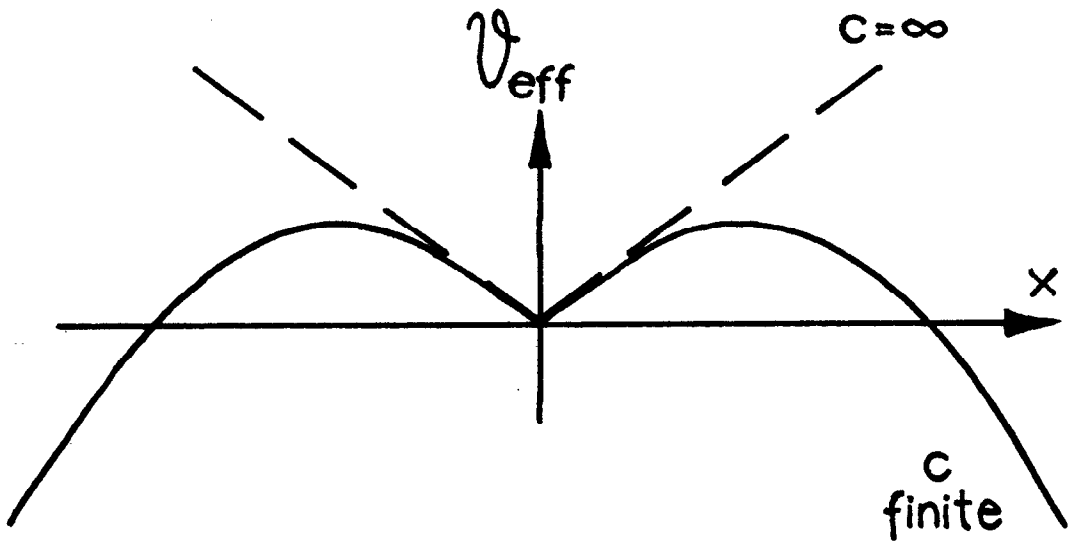

Fig. 1

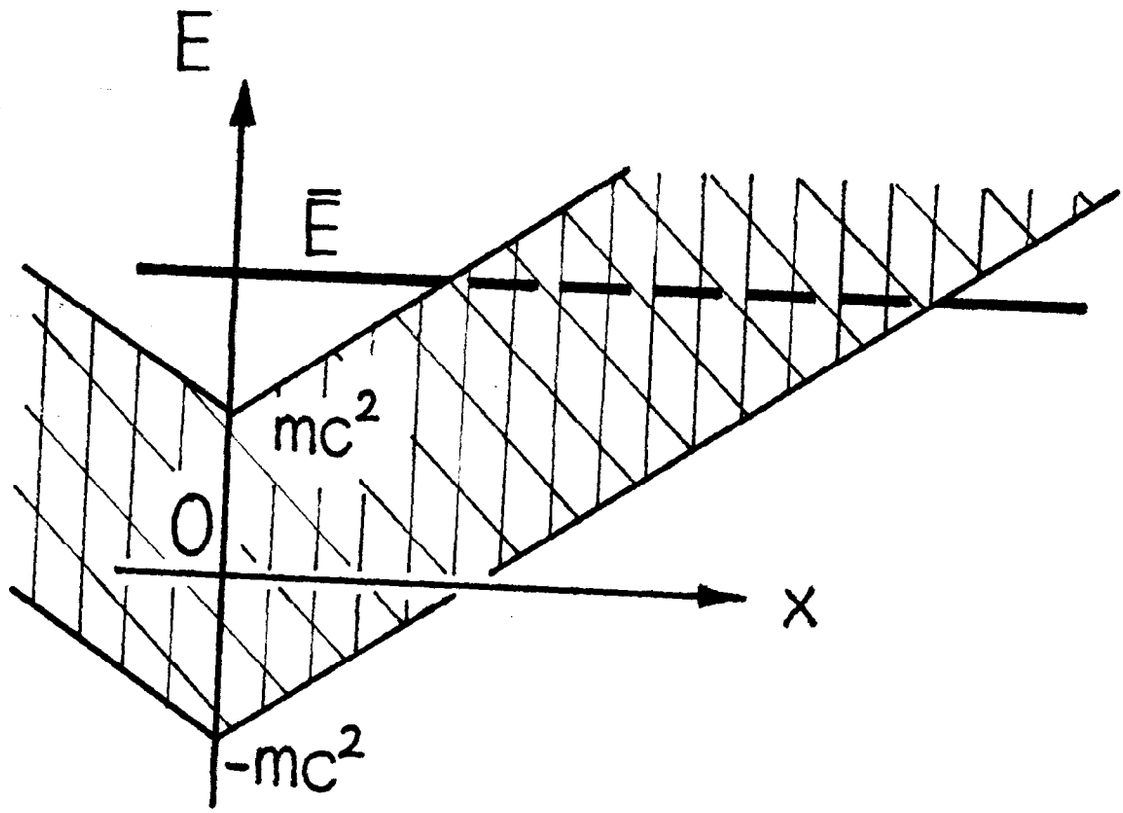

Fig. 2