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RENORMALIZATION GROUP IMPROVED YENNIE–FRAUTSCHI–SUURA THEORY FOR Z° PHYSICS*

B. F. L. WARD

Dept. of Physics and Astronomy, The University of Tennessee
Knoxville, Tennessee 37996–1200

and

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We describe a recently developed renormalization group improved version of the program of Yennie, Frautschi and Suura for the exponentiation of infrared divergences in Abelian gauge theories. Particular attention is paid to the relevance of this renormalization group improved exponentiation to Z° physics at SLC and LEP.

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I. Introduction

There is currently a substantial amount of interest in testing the $SU_{2L}\times U_1$ model to levels exceeding 1% accuracy[1] at the SLC (and LEP) on or near the Z° resonance. Such an accuracy necessarily implies that the respective $SU_{2L}\times U_1$ radiative corrections are known to $\lesssim .3\%$. Accordingly, inspired by the Mark II SLC Z° Mass and Width Physics Working Group, we have developed a methodology for achieving such an accuracy on these $SU_{2L}\times U_1$ radiative corrections.[2] This methodology is our subject in the following discussion.

More specifically, the standard $SU_{2L}\times U_1$ model, in its minimal manifestation, uses the parameters α, G_μ, M_{Z° , m_f and m_H to describe all known electroweak physical processes. Here, α is the fine structure constant of QED, G_μ is the μ decay constant, M_{Z° is the rest mass of the Z° vector boson, m_f denotes the rest mass of standard model fermion f and m_H is the rest mass of the physical Higgs boson ϕ° in this minimal manifestation of $SU_{2L}\times U_1$ (we note that, at this time, ϕ° has yet to appear explicitly in an experimental apparatus). Supersymmetry considerations,[3] for example, even in their most minimal form, would enlarge this set of parameters. It is thus a great achievement that, to date, the minimal $SU_{2L}\times U_1$ electroweak theory has encountered no obvious disagreement with observation.

Indeed, the theory has enjoyed the outstanding predictions of the $W^\pm$ and Z° bosons themselves (with masses of essentially the right value) and the attendant Z° neutral current interactions with the essentially correct magnitude and space-time structure. The stage is therefore set for precision checks of the predictions of $SU_{2L}\times U_1$ theory. Such checks are a primary aspect of the physics programs at SLC and LEP on (or near) the Z° resonance.

The type of checks envisioned are described in some detail in Refs. 1 and 4. For example, precise measurements of Γ_{Z°, M_{Z° and A_{LR} (the left-right asymmetry for $e^+e^- \to Z^\circ \to X$) can restrict the number of new light neutrinos, or give an eye toward possible new heavy particles. The type of precision required ranges from a few % down to .1% for some of the more subtle effects. As a benchmark, we may say that precision Z° physics (at SLC and LEP) requires that the respective cross sections are known to $\lesssim$ 1%, as we have noted.

One of the main contributors to the error on e^+e^- annihilation cross sections is the uncertainty associated with the respective $SU_{2L}\times U_1$ radiative corrections. Accordingly, it has been realized by many[5] that precise Z° physics will entail a substantial improvement on the methodology by which such corrections are performed in comparison with the analogous methodology used at PEP and PETRA.

One can make a straightforward assessment of the situation by considering the processes illustrated in Fig. 1 where we show only the γ exchange graphs, since they already characterize the size of the radiative corrections of interest to us here. Upon effecting the familiar cancellation of the real and virtual infrared (IR) singularities in the standard manner, we find that the size of the order α correction to the basic Born process for $e^+e^- \to Z^\circ \to X$ is expressed by

$$\frac{2\alpha}{\pi}\left(\ln\frac{s}{m_e^2}-1\right)\ln\frac{\sqrt{s}}{2\bar{k}_0}$$

where $\sqrt{s}$ = total c.m. energy and $\bar{k}_0$ is a typical energy resolution type (detector) parameter. One notices that the pure large ultraviolet (UV) corrections are characterized by $t \equiv (2\alpha/\pi)(\ln(s/m_e^2)-1) \cong .108$ for $\sqrt{s} = M_{Z^\circ}$ and, hence, that the infrared effects are possibly $\sim$ 100% corrections in each order of α since generally $\sqrt{s}/2 \gg \bar{k}_0$, for example. It follows that .3% $SU_{2L}\times U_1$ radiative corrections in $e^+e^- \to X$ near the Z° resonance involves summing all large IR effects and summing $\gtrsim$ 3 loops of the large UV effects.

Accordingly, we have used the method of Yennie, Frautschi and Suura[6] (Y–F–S) to sum the respective large IR effects and the renormalization group method of Weinberg and 't Hooft[7] to sum the respective large UV effects. In this way, we have arrived at the consideration of renormalization group improved Y–F–S theory. (The entire development has been motivated by precision Z° physics at SLC and LEP.)

Thus, in the next section we shall present a brief review of the elements of the Y–F–S program. Section III then presents a description of the basic renormalization group improved Y–F–S theory. Section IV illustrates the type of application we have in mind for our theory—the exponentiation of Monte Carlo electroweak event generators. Section V contains some concluding remarks.

II. Yennie–Frautschi–Suura Theory—A Short Review

With the ultimate purpose of achieving high-precision radiative corrections at SLC and LEP energies, we shall review the relevant elements of the Yennie–Frautschi–Suura (Y–F–S) theory as it relates to $e^+e^- \to Z^\circ \to X$ near the Z° resonance itself. We have, then, the full $SU_{2L}\times U_1$ theory in mind.

More precisely, consider the situations illustrated in Figs. 2 and 3. In Fig. 2, we show a typical contribution to the expansion of the full connected amplitude $\mathcal{M}$ for $e^+e^- \to X$ at $\sqrt{s} = M_{Z^\circ}$ in terms of the number of virtual photon loops. In Fig. 3, we show a typical contribution to $\mathcal{M}$ which involves $X = n(\gamma) + X'$, an n-real photon final state. The key results of Y–F–S theory are that

$$\mathcal{M}\left(P_e, P_{\bar{e}}\right) = \exp\{\alpha B\} \sum_{n=0}^{\infty} m_n \tag{1}$$

where m_n are free of virtual infrared divergences and, if $m_j^{(n)}$ is the n-real photon case of m_j,

$$\begin{aligned}\left|\sum_{n'=0}^{\infty} m_{n'}^{(n)}\right|^2 &= \widetilde{S}\left(\vec{k}_1\right)\ldots\widetilde{S}\left(\vec{k}_n\right)\overline{\beta}_0 \\ &+ \sum_{i=1}^{n}\widetilde{S}\left(\vec{k}_1\right)\ldots\widetilde{S}\left(\vec{k}_{i-1}\right)\widetilde{S}\left(\vec{k}_{i+1}\right)\ldots\widetilde{S}\left(\vec{k}_n\right)\overline{\beta}_1\left(\vec{k}_i\right) \\ &+ \ldots + \sum_{i=1}^{n}\widetilde{S}\left(\vec{k}_i\right)\overline{\beta}_{n-1}\left(\vec{k}_1,\ldots,\vec{k}_{i-1},\vec{k}_{i+1},\ldots,\vec{k}_n\right) \\ &+ \overline{\beta}_n\left(\vec{k}_1,\ldots,\vec{k}_n\right)\end{aligned} \tag{2}$$

where $\overline{\beta}_j$ have no real or virtual infrared divergences so that, for $X' = f\bar{f}$ and $e_f e_R =$ electric charge of f,

$$d\sigma = \exp\left\{2\alpha\left(\mathrm{ReB} + \widetilde{B}\right)\right\} \frac{1}{(2\pi)^4} \int d^4y \exp\left\{iy \cdot (P_e + P_{\bar{e}} - P_{X'}) + D\right\}$$
$$\times \left\{\overline{\beta}_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{j=1}^{n} \frac{d^3k_j}{k_j} \exp\left\{-iy \cdot k_j\right\} \overline{\beta}_n\right\} dE_{X'}\, d^3P_{X'} \tag{3}$$

where the functions B and $\widetilde{B}$ are given by (here, m_γ is our photon mass infrared cutoff)

$$2\alpha\widetilde{B} = \frac{\alpha}{4\pi^2} \frac{\int^{k \le K_{\max}} d^3k}{(k^2 + m_\gamma^2)^{1/2}} \left\{ -\left(\frac{P_{\bar{e}\mu}}{P_{\bar{e}} \cdot k} - \frac{P_{e\mu}}{P_e \cdot k}\right)^2 + e_f\left(\frac{P_{f\mu}}{P_f \cdot k} - \frac{P_{e\mu}}{P_e \cdot k}\right)^2 \right.$$
$$- e_f\left(\frac{P_{\bar{f}\mu}}{P_{\bar{f}} \cdot k} - \frac{P_{e\mu}}{P_e \cdot k}\right)^2 - e_f\left(\frac{P_{f\mu}}{P_f \cdot k} - \frac{P_{\bar{e}\mu}}{P_{\bar{e}} \cdot k}\right)^2 \tag{4}$$
$$\left. + e_f\left(\frac{P_{\bar{f}\mu}}{P_{\bar{f}} \cdot k} - \frac{P_{\bar{e}\mu}}{P_{\bar{e}} \cdot k}\right)^2 - e_f^2\left(\frac{P_{\bar{f}\mu}}{P_{\bar{f}} \cdot k} - \frac{P_{f\mu}}{P_f \cdot k}\right)^2 \right\} ,$$

$$B = \frac{-i}{8\pi^3} \frac{\int d^4k}{k^2 - m_\gamma^2 + i\epsilon} \left\{ - \left(\frac{-2P_{e\mu} - k_\mu}{k^2 + 2k\cdot P_e + i\epsilon} + \frac{-2P_{\bar{e}\mu} + k_\mu}{k^2 - 2k\cdot P_{\bar{e}} + i\epsilon} \right)^2 \right.$$

$$+ e_f \left(\frac{-2P_{e\mu} - k_\mu}{k^2 + 2k\cdot P_e + i\epsilon} + \frac{2P_{f\mu} + k_\mu}{k^2 + 2k\cdot P_f + i\epsilon} \right)^2$$

$$- e_f \left(\frac{-2P_{e\mu} - k_\mu}{k^2 + 2k\cdot P_e + i\epsilon} + \frac{2P_{\bar{f}\mu} + k_\mu}{k^2 + 2k\cdot P_{\bar{f}} + i\epsilon} \right)^2$$

$$- e_f \left(\frac{-2P_{\bar{e}\mu} - k_\mu}{k^2 + 2k\cdot P_{\bar{e}} + i\epsilon} + \frac{2P_{f\mu} + k_\mu}{k^2 + 2k\cdot P_f + i\epsilon} \right)^2 \tag{5}$$

$$+ e_f \left(\frac{-2P_{\bar{e}\mu} - k_\mu}{k^2 + 2k\cdot P_{\bar{e}} + i\epsilon} + \frac{2P_{\bar{f}\mu} + k_\mu}{k^2 + 2k\cdot P_{\bar{f}} + i\epsilon} \right)^2$$

$$\left. - e_f^2 \left(\frac{2P_{f\mu} - k_\mu}{k^2 - 2k\cdot P_f + i\epsilon} + \frac{2P_{\bar{f}\mu} + k_\mu}{k^2 + 2k\cdot P_{\bar{f}} + i\epsilon} \right)^2 \right\}$$

and

$$D = \int \frac{d^3k}{k} \, \widetilde{S} \times \left(e^{-iy\cdot k} - \theta\left(K_{\max} - k\right) \right) \tag{6}$$

with

$$2\alpha\widetilde{B} \equiv \int^{k\leq K_{\max}} \frac{d^3k}{\left(k^2 + m_\gamma^2\right)^{1/2}} \, \widetilde{S} \quad . \tag{7}$$

Here, $K_{\max}$ may depend on the direction of $\overrightarrow{k}$. It is result (3) which we will use in our study of $e^+e^- \to Z^\circ \to X$. In (3), all infrared divergences are cancelled in the sum $\mathrm{ReB}+\tilde{\mathrm{B}}$ to all orders in α.

As we (and others) have discussed elsewhere,[2,5] there remain large ultraviolet effects in $\overline{\beta}_n$ in (3). These may be analyzed using the methods of Weinberg and 't Hooft[7] as we illustrate in the next section.

III. Renormalization Group Improved Yennie–Frautschi–Suura Theory

In this section we shall illustrate how one uses the partial differential equation of Weinberg and 't Hooft to sum the large ultraviolet effects in $\overline{\beta}_n$ in (3). We begin by recapitulating this equation.

Specifically, Weinberg and 't Hooft have shown that the multiplicatively renormalized Green's functions $\{\Gamma\}$ of a theory may be subtracted with the massless limits of the subtraction constants for the theory at a Euclidean scale μ. The fact that the unrenormalized theory is independent of μ then implies the equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\left(g_R\right)\ \frac{\partial}{\partial g_R} - \gamma_\Theta(g_R)\ m_R\ \frac{\partial}{\partial m_R} - \gamma_\Gamma\left(g_R\right)\right)\Gamma = 0 \qquad (8)$$

where for simplicity we imagine we have one renormalized coupling g_R and one renormalized mass m_R. In the $SU_{2L}\times\ U_1$ theory, we would have two couplings, e_R and g_{WR}, where g_{WR} is the SU_{2L} coupling and e_R is the electric charge of the positron, we would have renormalized mass parameters for the fermions in the presumed three families of quarks and leptons, we would have the mass parameter of the $W^\pm$ and Z° bosons, and the mass parameter of the physical Higgs particle (or the quartic coupling of the physical Higgs particle), as a minimal set of masses and couplings. The physics beyond the standard model would enlarge this set. The coefficient functions β, γ_Θ and γ_Γ are computable to in renormalized perturbation theory. The detailed application of (8) to (3) for the full $SU_{2L}\times\ U_1$ theory has been illustrated in Ref. 2 and will be taken-up in more detail elsewhere.

More precisely, we have shown in Ref. 2 that (8) yields the following form of (3), where, here, we focus on the QED aspect of the $SU_{2L}\times\ U_1$ theory for purposes of illustration,

$$d\sigma = \exp\left\{2\alpha(1)\left[\mathrm{ReB}\left(P_i(1), m_{iR}(\lambda)\right) + \widetilde{B}\left(P_i(1), m_{iR}(\lambda), \frac{K_{\max}}{\lambda}\right)\right]\right\}$$

$$\frac{1}{(2\pi)^4}\int d^4y\ \exp\left\{iy\cdot(P_e + P_{\bar{e}} - P_{X'}) + D\right\} \tag{9}$$

$$\left\{\widetilde{\overline{\beta}}_0(q_0) + \sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{\ell'=1}^{n}\frac{d^3k_{\ell'}}{k_{\ell'}}\exp\left\{-iy\cdot k_{\ell'}\right\}\widetilde{\overline{\beta}}_n(q_n)\right\} dE_{X'}d^3P_{X'}$$

where

$$\widetilde{\overline{\beta}}_n(q_n) \equiv \lambda^{2\bar{D}_{\mathcal{M}^{(n)}}}\overline{\beta}_n\left(P_i(1), k_{o_j}, m_{iR}(\lambda), \alpha(\lambda), \mu\right)\left(\frac{e_R(\lambda)}{e_R(1)}\right)^{-2n} \tag{10}$$

and

$$D \equiv D\left(P_i(1), m_{iR}(\lambda), \alpha(\lambda), \frac{K_{\max}}{\lambda}\right) \quad . \tag{11}$$

The running charge $e_R(\lambda)$ and the running masses $m_{iR}(\lambda)$ have their familiar[7] definitions; these definitions are reviewed in Ref. 2. The normalization of (9) is such that $\overline{D}_{\mathcal{M}^{(n)}}$ is the engineering dimension of the amputated amplitude $\mathcal{M}^{(n)}$ which describes the connected contribution to $e^+e^- \to n(\gamma) + X'$ (see Ref. 2). Our scale parameter λ is such that

$$\begin{aligned} P_e &\equiv \left(\lambda\sqrt{s_0}/2,\ \sqrt{\lambda^2 s_0/4 - m_e^2}\ \hat{z}\right) \\ P_{\bar{e}} &\equiv \left(\lambda\sqrt{s_0}/2,\ -\sqrt{\lambda^2 s_0/4 - m_e^2}\ \hat{z}\right) \end{aligned} \tag{12}$$

and, in $\mathcal{M}^{(n)}$,

$$\begin{aligned} P_f^0 + P_{\bar{f}}^0 &= \lambda\left(\sqrt{s}_0 - \sum_{i=1}^{n} k_{oi}^0\right) \quad , \quad k_i \equiv \lambda k_{oi} \quad , \\ \vec{P}_f + \vec{P}_{\bar{f}} &= -\lambda\sum_{i=1}^{n}\vec{k}_{oi} \quad . \end{aligned} \tag{13}$$

We can always presume this in the physical region provided that $\lambda\sqrt{s}_0 > 2m_f$ and $\lambda\sqrt{s}_0 > 2m_e$. We will always imagine that $\sqrt{s}_0 > 2m_e$ and that $\lambda > 1$.

The result (9) is central to our approach to high-precision radiative corrections in $e^+e^- \to Z^\circ \to X$ near the Z° resonance. It is a rigorous consequence of the renormalization group equation. The detailed application of (9) to $e^+e^- \to Z^\circ \to X$ will be taken-up elsewhere by the author and S. Jadach[8] and the Mark II SLC Z° Mass and Width Physics Working Group.[9] In the next section we wish to illustrate, in a pertinent way, the type of application we have in mind for (9).

IV. Exponentiation of Monte Carlo Event Generators

A primary use of a formula like (9) would be in exponentiating large infrared (IR) effects and summing large ultraviolet (UV) effects in a way which allows an event generator, such as MMG1 in Ref. 10, to reflect the respective net effects in $e^+e^- \to X$ near the Z° resonance, for example. Accordingly, in this section we wish to show how (9) would be applied to the results in Ref. 11 for $e^+e^- \to \mu^+\mu^-(\gamma)$, which are the basis of the event generator MMG1. (The application of (9) to the general one-loop calculation of $e^+e^- \to X$ from the standpoint of event generators is one of the details which will be taken-up elsewhere by Jadach and the author[8] and by the Mark II SLC Z° Mass and Width Physics Working Group.[9]) In this way, we hope to clarify the relationship between (9) and the results in Ref. 5, for example, and to illustrate the type of applications we have in mind for (9).

More precisely, in specializing (9) to the results in Ref. 11, we may identify $\overline{\beta}_0(q_0)$ as

$$\overline{\beta}_0\,(q_0) = \frac{d\sigma}{d\Omega_\mu}(1\text{–loop}) - 2\,\mathrm{Re}\,(\alpha(1)B)\,\frac{d\sigma_0}{d\Omega_\mu} \tag{14}$$

where $d\sigma(\text{1-loop})/d\Omega_\mu$ is the one–loop cross section in Eq. (2.27) of Ref. 11 and $d\sigma_0/d\Omega_\mu$ is the lowest order cross section in Eq. (2.2) in Ref. 11. $\alpha(1)$ is the fine structure constant at $\sqrt{s} = 2m_{\mu,\mathrm{phys}}$, for example.

Similarly, the cross section $\overline{\beta}_1$ is identified as $(k = k_1)$

$$\overline{\beta}_1 = \frac{d\sigma^{B1}}{d\Omega_\mu\, d\Omega_\gamma\, kdk} - \widetilde{S}(k)\frac{d\sigma_0}{d\Omega_\mu} \tag{15}$$

where $d\sigma^{B1}$ is given by Eq. (3.13) of Ref. 11 and $\widetilde{S}(k)$ is given by (7).

Clearly, the virtual infrared function B should be computed in a complete way in order to make (14) as precise as it is desired. We find

$$\begin{aligned} B = & B_1(P_e, P_{\bar{e}}) + B_2(P_e, P_f) - B_2(P_f \to P_{\bar{f}}) \\ & - B_2(P_e \to P_{\bar{e}}) + B_2(P_e \to P_{\bar{e}}, P_f \to P_{\bar{f}}) \\ & + e_f^2 B_1(P_e \to P_f, P_{\bar{e}} \to P_{\bar{f}}, m_e \to m_f) \end{aligned} \tag{16}$$

where[2]

$$\begin{aligned} B_1 = & -\frac{1}{2\pi} + \frac{1}{4\pi}\left(1 - 4m_e^2/s\right)^{1/2} \ln\left(\frac{1+\beta_e}{1-\beta_e}\right) \\ & - \frac{1}{4} i\beta_e \theta(s - 4m_e^2) - \frac{1}{2\pi} \ln\frac{m_\gamma^2}{m_e^2} \\ & + \frac{s - 2m_e^2}{\pi s \beta_e}\left[\frac{1}{2}\ln\left(\frac{\beta_e^2}{(m_\gamma^2/s)}\right)\ln\left(\frac{1+\beta_e}{1-\beta_e}\right)\right. \\ & + \frac{1}{2} Li_2\left(-\frac{1-\beta_e}{1+\beta_e}\right) \\ & + \frac{1}{4}\left\{\ln^2\left(\frac{1-\beta_e}{1+\beta_e}\right) - \ln^2\left(\frac{(m_\gamma^2/s)(1-\beta_e)}{\beta_e^2(1+\beta_e)}\right)\right\} \\ & + \frac{1}{2} Li_2\left(-\frac{1-\beta_e}{1+\beta_e}\right) + \frac{1}{4}\ln^2\left(\frac{(1+\beta_e)(m_\gamma^2/s)}{(1-\beta_e)\beta_e^2}\right) \\ & \left. + \frac{1}{2}\ln^2(1+\beta_e) + \frac{\pi^2}{4} + Li_2\left(\frac{1}{1+\beta_e}\right) + Li_2(1-\beta_e)\right] \\ & + i\frac{(s - 2m_e^2)}{s\beta_e}\theta(s - 4m_e^2)\left\{\frac{1}{2}\ln\beta_e^2 - \frac{1}{2}\ln\frac{m_\gamma^2}{s}\right\} \end{aligned} \tag{17}$$

and

$$B_2 = \frac{3e_f}{4\pi} + \frac{e_f}{4\pi}\left[\ln\frac{m_\gamma^2}{m_e^2} + \ln\frac{m_\gamma^2}{m_f^2}\right] + \frac{e_f}{8\pi}\left[\ln\frac{s}{m_e^2} + \ln\frac{s}{m_f^2}\right]$$

$$+ \frac{e_f\left(s_{ef} - m_e^2 - m_f^2\right)}{2\pi\sqrt{-t_{ef}\, s_{ef} + \left(m_e^2 - m_f^2\right)^2}} \left\{-\frac{1}{2}\ln\left(\frac{(1+\Sigma_+)(1+\Sigma_-)}{(1-\Sigma_-)(1-\Sigma_+)}\right)\right.$$

$$\ln\left(\frac{m_\gamma^2\, t_{ef}}{s_{ef}\, t_{ef} - \left(m_e^2 - m_f^2\right)^2}\right) - \frac{1}{2}Li_2\left(-\frac{2\Sigma_-}{1-\Sigma_-}\right) \tag{18}$$

$$+ \frac{1}{2}Li_2\left(\frac{2\Sigma_+}{1+\Sigma_+}\right) + Li_2\left(-\frac{\Sigma_-}{1-\Sigma_-}\right)$$

$$+ Li_2\left(-\frac{\Sigma_+}{1-\Sigma_+}\right) - Li_2\left(\frac{\Sigma_-}{1+\Sigma_-}\right)$$

$$- Li_2\left(\frac{\Sigma_+}{1+\Sigma_+}\right) + \ln 2\ln\left(\frac{(1-\Sigma_-)(1-\Sigma_+)}{(1+\Sigma_-)(1+\Sigma_+)}\right)$$

$$\left.- \frac{1}{2}Li_2\left(\frac{-2\Sigma_+}{1-\Sigma_+}\right) + \frac{1}{2}Li_2\left(\frac{2\Sigma_-}{1+\Sigma_-}\right)\right\}$$

$$- \frac{e_f}{4\pi}\left\{\ln\left(\frac{-t_{ef}}{m_f^2}\right) + \ln\left(\frac{-t_{ef}}{m_e^2}\right) + b_{21}\left(P_e, P_f, m_e, m_f\right)\right.$$

$$\left. + b_{21}\left(P_f, P_e, m_f, m_e\right)\right\} - \frac{e_f}{2\pi}b_{22}\left(P_e, P_f, m_e, m_f\right) \quad ,$$

where

$$\sum\nolimits_{\mp} = \sqrt{-t_{ef}}\ \frac{1 \mp \left(m_e^2 - m_f^2\right)\Big/\left(-t_{ef}\right)}{\left(s_{ef} - \left(m_e^2 - m_f^2\right)^2 / t_{ef}\right)^{1/2}} \quad , \tag{19}$$

$$
\begin{aligned}
&b_{21}\left(P_e, P_f, m_e, m_f\right) = \\
&\left[1-\left(m_e^2-m_f^2-t_{ef}\right)\Big/\left(-2t_{ef}\right)-\left(\left(m_e^2-m_f^2-t_{ef}\right)^2\Big/4t_{ef}^2-m_f^2\Big/t_{ef}\right)^{1/2}\right] \\
&\ln\left[\left(m_e^2-m_f^2-t_{ef}\right)\Big/\left(-2t_{ef}\right)+\left(\left(m_e^2-m_f^2-t_{ef}\right)^2\Big/4t_{ef}^2-m_f^2\Big/t_{ef}\right)^{1/2}-1\right] \\
&+\left[\left(m_e^2-m_f^2-t_{ef}\right)\Big/\left(-2t_{ef}\right)+\left(\left(m_e^2-m_f^2-t_{ef}\right)^2\Big/4t_{ef}^2-m_f^2\Big/t_{ef}\right)^{1/2}\right] \\
&\ln\left[\left(m_e^2-m_f^2-t_{ef}\right)\Big/\left(-2t_{ef}\right)+\left(\left(m_e^2-m_f^2-t_{ef}\right)^2\Big/4t_{ef}^2-m_f^2\Big/t_{ef}\right)^{1/2}\right] \\
&+\left[1-\left(m_e^2-m_f^2-t_{ef}\right)\Big/\left(-2t_{ef}\right)+\left(\left(m_e^2-m_f^2-t_{ef}\right)^2\Big/4t_{ef}^2-m_f^2\Big/t_{ef}\right)^{1/2}\right] \\
&\ln\left[1-\left(m_e^2-m_f^2-t_{ef}\right)\Big/\left(-2t_{ef}\right)+\left(\left(m_e^2-m_f^2-t_{ef}\right)^2\Big/4t_{ef}^2-m_f^2\Big/t_{ef}\right)^{1/2}\right] \\
&+\left[\left(m_e^2-m_f^2-t_{ef}\right)\Big/\left(-2t_{ef}\right)-\left(\left(m_e^2-m_f^2-t_{ef}\right)^2\Big/4t_{ef}^2-m_f^2\Big/t_{ef}\right)^{1/2}\right] \\
&\ln\left[-\left(m_e^2-m_f^2-t_{ef}\right)\Big/\left(-2t_{ef}\right)+\left(\left(m_e^2-m_f^2-t_{ef}\right)^2\Big/4t_{ef}^2-m_f^2\Big/t_{ef}\right)^{1/2}\right]
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
&b_{22}\left(P_e, P_f, m_e, m_f\right) = \\
&1+\frac{1}{2}\ln 2+\frac{\sqrt{s}}{4\sqrt{-t_{ef}}}\ \Bigg\{\frac{1}{\sqrt{s}}\Bigg(\left(s_{ef}-\left(m_e^2-m_f^2\right)^2/t_{ef}\right)^{1/2} \\
&-\sqrt{-t_{ef}}\left(1+\left(m_e^2-m_f^2\right)/t_{ef}\right)\Bigg)\ \left[-\frac{1}{2}(1+\ln 2)\right. \\
&\left.+\ln\left(\frac{1}{\sqrt{s}}\left(s_{ef}-\left(m_e^2-m_f^2\right)^2/t_{ef}\right)^{1/2}-\frac{\sqrt{-t_{ef}}}{\sqrt{s}}\left(1+\left(m_e^2-m_f^2\right)/t_{ef}\right)\right)\right] \\
&-\frac{1}{\sqrt{s}}\left(\left(s_{ef}-\left(m_e^2-m_f^2\right)^2/t_{ef}\right)^{1/2}+\sqrt{-t_{ef}}\left(1-\left(m_e^2-m_f^2\right)/t_{ef}\right)\right) \\
&\left[-\frac{1}{2}(1+\ln 2)+\ln\left(\frac{1}{\sqrt{s}}\left(s_{ef}-\left(m_e^2-m_f^2\right)^2/t_{ef}\right)^{1/2}\right.\right. \\
&\left.\left.+\frac{\sqrt{-t_{ef}}}{\sqrt{s}}\left(1-\left(m_e^2-m_f^2\right)/t_{ef}\right)\right)\right]-\frac{1}{\sqrt{s}}\left[\left(s_{ef}-\left(m_e^2-m_f^2\right)^2/t_{ef}\right)^{1/2}\right. \\
&\left.+\sqrt{-t_{ef}}\left(1+\left(m_e^2-m_f^2\right)/t_{ef}\right)\right]\ \left[-\frac{1}{2}(1+\ln 2)\right. \\
&\left.+\ln\left(\frac{1}{\sqrt{s}}\left(s_{ef}-\left(m_e^2-m_f^2\right)^2/t_{ef}\right)^{1/2}+\frac{\sqrt{-t_{ef}}}{\sqrt{s}}\left(1+\left(m_e^2-m_f^2\right)/t_{ef}\right)\right)\right] \\
&+\frac{1}{\sqrt{s}}\left[\left(s_{ef}-\left(m_e^2-m_f^2\right)^2/t_{ef}\right)^{1/2}-\sqrt{-t_{ef}}\left(1-\left(m_e^2-m_f^2\right)/t_{ef}\right)\right] \\
&\left[-\frac{1}{2}(1+\ln 2)+\ln\left(\frac{1}{\sqrt{s}}\left(s_{ef}-\left(m_e^2-m_f^2\right)^2/t_{ef}\right)^{1/2}\right.\right. \\
&\left.\left.-\frac{\sqrt{-t_{ef}}}{\sqrt{s}}\left(1-\left(m_e^2-m_f^2\right)/t_{ef}\right)\right)\right]\Bigg\}
\end{aligned}
\tag{21}
$$

with

$$s = (P_e + P_{\bar{e}})^2 \quad , \quad s_{ef} \equiv \left(P_e + P_f\right)^2 \quad ,$$
$$t_{ef} \equiv \left(P_e - P_f\right)^2 \quad , \quad \beta_e = \left(1 - 4m_e^2/s\right)^{1/2} \quad . \tag{22}$$

e_f is the electric charge of fermion f.

Similarly, we note that the real infrared function $\widetilde{B}$ which cancels the infrared singularities in ReB may be represented as

$$\begin{aligned}\widetilde{B}\left(P_e, P_{\bar{e}}, P_f, P_{\bar{f}}\right) = {} & \widetilde{B}_1\left(P_e, P_{\bar{e}}, m_e\right) + \widetilde{B}_2\left(P_e, P_f, m_e, m_f\right) - \widetilde{B}_2\left(P_f \to P_{\bar{f}}\right) \\ & - \widetilde{B}_2\left(P_e \to P_{\bar{e}}\right) + \widetilde{B}_2\left(P_e \to P_{\bar{e}}, P_f \to P_{\bar{f}}\right) \\ & + e_f^2 \widetilde{B}_1\left(P_e \to P_f, P_{\bar{e}} \to P_{\bar{f}}, m_e \to m_f\right) \quad ,\end{aligned} \tag{23}$$

where, for a spherical cutoff $K_{\max}$ for the photon momentum magnitude,

$$\begin{aligned}\widetilde{B}_1\left(P_e, P_{\bar{e}}, m_e\right) = {} & \frac{-2m_e^2 \ln\left(2K_{\max}/m_\gamma\right)}{\pi s \beta_e}\left(\frac{1}{1-\beta_e} - \frac{1}{1+\beta_e}\right) \\ & - \frac{\left(s - 2m_e^2\right)}{\pi s \beta_e} \ln\left(2K_{\max}/m_\gamma\right) \ln\left(\frac{1-\beta_e}{1+\beta_e}\right)\end{aligned} \tag{24}$$

$$\widetilde{B}_2\left(P_e,P_f,m_e,m_f\right)=\frac{e_f m_f^2}{\pi s\beta_f}\ln\left(2K_{\max}/m_\gamma\right)\left(\frac{1}{1-\beta_f}-\frac{1}{1+\beta_f}\right)$$

$$+\frac{e_f m_e^2}{\pi s\beta_e}\ln\left(2K_{\max}/m_\gamma\right)\left(\frac{1}{1-\beta_e}-\frac{1}{1+\beta_e}\right) \tag{25}$$

$$+\frac{\left(s_{ef}-m_e^2-m_f^2\right)e_f\ln\left(2K_{\max}/m_\gamma\right)}{4\pi\left(\left(m_f^2-m_e^2-t_{ef}\right)^2/4+m_e^2\left(s_{ef}-2\left(m_e^2+m_f^2\right)\right)\right)^{1/2}}$$

$$\cdot\left\{\ln\left|\frac{s_{ef}-2\left(m_e^2+m_f^2\right)-\left(m_f^2-m_e^2-t_{ef}\right)/2-\left(\left(m_f^2-m_e^2-t_{ef}\right)^2/4+m_e^2\left(s_{ef}-2\left(m_e^2+m_f^2\right)\right)\right)^{1/2}}{\left(m_f^2-m_e^2-t_{ef}\right)/2+\left(\left(m_f^2-m_e^2-t_{ef}\right)^2/4+m_e^2\left(s_{ef}-2\left(m_e^2+m_f^2\right)\right)\right)^{1/2}}\right|\right.$$

$$\left.-\ln\left|\frac{s_{ef}-2\left(m_e^2+m_f^2\right)-\left(m_f^2-m_e^2-t_{ef}\right)/2+\left(\left(m_f^2-m_e^2-t_{ef}\right)^2/4+m_e^2\left(s_{ef}-2\left(m_e^2+m_f^2\right)\right)\right)^{1/2}}{\left(m_f^2-m_e^2-t_{ef}\right)/2-\left(\left(m_f^2-m_e^2-t_{ef}\right)^2/4+m_e^2\left(s_{ef}-2\left(m_e^2+m_f^2\right)\right)\right)^{1/2}}\right|\right\}$$

where (note that $(1/(1-\beta_f)-1/(1+\beta_f))/s\beta_f \equiv 1/2m_f^2$)

$$\beta_f \equiv \left(1-4m_f^2/s\right)^{1/2} \quad . \tag{26}$$

Hence, we have completely specified $\overline{\beta}_0$ and $\overline{\beta}_1$; we now turn to $\tilde{\overline{\beta}}_0$ and $\tilde{\overline{\beta}}_1$.

Considering first $\tilde{\overline{\beta}}_0$, we have (the $\overline{\beta}_i$ in (14) and (15) contain a standard phase space factor relative to those in (9))

$$\tilde{\overline{\beta}}_0=\lambda^{-2}\,\overline{\beta}_0\left[P_e(1),P_{\bar{e}}(1),P_\mu(1),P_{\bar{\mu}}(1),m_{e,\mathrm{phys}}/\lambda,m_{\mu,\mathrm{phys}}/\lambda,\alpha(\lambda)\right] \tag{27}$$

where $\sqrt{s_0}\equiv 2m_{\mu,\mathrm{phys}}$ and

$$P_f(1)\equiv\left(\sqrt{s_0}/2 \quad , \quad \hat{z}_f\sqrt{s_0/4-m_{f,\mathrm{phys}}^2/\lambda^2}\right) \quad , \quad f=e,\bar{e},\mu,\bar{\mu} \tag{28}$$

with $\hat{z}_e=-\hat{z}_{\bar{e}}\equiv\hat{z}$ and $\hat{z}_\mu=-\hat{z}_{\bar{\mu}}$. Here, $\lambda=M_{Z^\circ}/2m_{\mu,\mathrm{phys}}$.

Similarly, for $\tilde{\bar{\beta}}_1$, we have

$$\tilde{\bar{\beta}}_1 = \lambda^{-4}\,\bar{\beta}_1\left[P_i(1),\ m_{i,\mathrm{phys}}/\lambda,\ k_1/\lambda,\ \alpha^3 \to \alpha(1)\alpha^2(\lambda)\right] \quad . \tag{29}$$

This, then, completely specifies $\tilde{\bar{\beta}}_0$ and $\tilde{\bar{\beta}}_1$.

Thus, in our example $E_{X'} = P_f^\circ + P_{\bar{f}}^\circ$, $\vec{P}_{X'} = \vec{P}_f + \vec{P}_{\bar{f}}$ and we have

$$d\sigma = \exp\left\{2\alpha(1)(\mathrm{ReB} + \tilde{B})\right\} \frac{1}{(2\pi)^4} \int d^4y \exp\left\{iy\cdot(P_e + P_{\bar{e}} - P_{X'}) + D\right\}$$
$$\left\{\tilde{\bar{\beta}}_0\,(q_0) + \int \frac{d^3k_1}{k_1}\, e^{-iy\cdot k_1}\,\tilde{\bar{\beta}}_1\,(q_1)\right\} dE_{X'}\, d^3P_{X'} \tag{30}$$

where

$$D = \int \frac{d^3k}{k}\left(e^{-iy\cdot k} - \theta(K_{\max} - k)\right)\tilde{S} \quad . \tag{31}$$

We note that, as one may check from (16)–(26), ReB+$\tilde{\mathrm{B}}$ does not contain infrared singularities.

The effect of e^D in (3) has been discussed in detail by Jadach in Ref. 12. The basic result is that, for Monte Carlo simulation, one should write (30) as

$$
\begin{aligned}
d\sigma \doteq & \exp\left\{2\alpha(1)\left(\mathrm{ReB}\ +\widetilde{B}\left(P_i(1),m_{iR}(\lambda),E_{\gamma,\max}/\lambda\right)\right)\right\} \\
& \left[\delta\left(\sqrt{s}-E_{X'}\right)\widetilde{\widetilde{\beta}}_0\left(\sqrt{s}\right)\int_0^{K_{\max}}\rho(\epsilon')d\epsilon'\right. \\
& \left.+\theta\left(\epsilon-K_{\max}\right)\widetilde{\widetilde{\beta}}_0\left(\sqrt{s}\right)\frac{(\alpha(1)A)}{\epsilon}\left(\epsilon/E_{\gamma,\max}\right)^{\alpha(1)A}\right]dE_{X'} \\
& +\exp\left\{2\alpha(1)\left(\mathrm{ReB}\ +\widetilde{B}(P_i(1),m_{iR}(\lambda),E'_{\gamma,\max}/\lambda)\right)\right\} \\
& \widetilde{\widetilde{\beta}}_1(k')\frac{d^3k'}{k'}\left\{\delta(\epsilon-k')\int_0^{K_{\max}}\rho'(\epsilon'-k')d(\epsilon'-k')\right. \\
& \left.+\theta\left(\epsilon-k'-K_{\max}\right)\frac{\alpha(1)A}{\epsilon-k'}\left(\frac{\epsilon-k'}{E'_{\gamma,\max}}\right)^{\alpha(1)A}\right\}dE_{X'} \quad ,
\end{aligned} \tag{32}
$$

where we have introduced

$$
\alpha(1)A \equiv 2\alpha(1)\widetilde{B}\left(P_i(1),m_{iR}(\lambda),K_{\max}/\lambda\right)\Big/\ln\left(2K_{\max}\Big/\lambda m_\gamma\right) \quad , \tag{33}
$$

$$
\epsilon = \sqrt{s}-E_{X'} = \sqrt{s}-E_f-E_{\bar{f}} \tag{34}
$$

and

$$
\rho(\epsilon) = \frac{\alpha(1)A}{\epsilon}\left(\frac{\epsilon}{E_{\gamma,\max}}\right)^{\alpha(1)A} \quad , \quad \rho'(\epsilon) = \frac{\alpha(1)A}{\epsilon}\left(\frac{\epsilon}{E'_{\gamma,\max}}\right)^{\alpha(1)A} \quad , \tag{35}
$$

with

$$
E_{\gamma,\max} = \sqrt{s}/2 - 2m_f^2/\sqrt{s} \quad , \quad E'_{\gamma,\max} = \frac{\left(s-2k'\sqrt{s}-4m_f^2\right)}{2\left(\sqrt{s}-2k'\right)} \quad . \tag{36}
$$

[Note that $f=\mu$ in (30).]

Hence, here, $K_{\max}$ is the maximum energy of a photon which cannot be detected by the respective detector. In order to implement (32), one proceeds as follows. One uses $\rho(\epsilon)$ $(\rho'(\epsilon - k'))$ to choose a value for $\epsilon(\epsilon - k')$ by standard Monte Carlo methods. One sets the number n of Yennie–Frautschi–Suura *soft* photons equal to 0 if $\epsilon \leq K_{\max}(\epsilon - k' \leq K_{max})$. For $\epsilon > K_{\max}(\epsilon - k' > K_{\max})$ one picks n according to the Poisson distribution

$$P_{n-1} = \frac{e^{-\bar{n}}(\bar{n})^{n-1}}{(n-1)!} \quad , \quad \bar{n} = \alpha(1)A\ln(\epsilon/K_{\max}) \left(\bar{n} = \alpha(1)A\ln\left(\frac{\epsilon - k'}{K_{\max}}\right)\right) \tag{37}$$

where the $n-1$ variables that generate P_{n-1} in Ref. 12 may be used to choose the photon energies $k_1, \ldots, k_n$ such that $\sum_i k_i = \epsilon(\sum_i k_i = \epsilon - k')$. The angular distribution of the n photons is then chosen, by standard Monte Carlo methods, according to $\widetilde{S}(k)d^3k/k$. In this way, a one-loop event generator based on results like those in Ref. 11 may be rigorously exponentiated.

We have used Ref. 11 as a pedagogical example. The method illustrated by (14)–(37) applies to any electroweak Monte Carlo event generator.

Currently, there is an effort by Berends' group[13] to create an order α^4 event generator for e^+e^- annihilation into $\mu^+\mu^-(\gamma, \gamma\gamma)$. Thus, it is of some interest to record the analoga of (14) and (15) at order α^4. In (14) we would use

$$\overline{\beta}_0\,(q_0) = \frac{d\sigma(\alpha^4)}{d\Omega_\mu} - 2Re(\alpha(1)B)\frac{d\sigma(\alpha^3)}{d\Omega_\mu} + \frac{(2Re(\alpha(1)B))^2}{2}\frac{d\sigma_0}{d\Omega_\mu} \tag{38}$$

and in (15), we would use

$$\overline{\beta}_1(k_1) = \frac{d\sigma^{B1}(\alpha^4)}{d\Omega_\mu\, d\Omega_\gamma\, k_1\, dk_1} - 2Re(\alpha(1)B)\frac{d\sigma^{B1}(\alpha^3)}{d\Omega_\mu\, d\Omega_\gamma\, k_1\, dk_1} - \widetilde{S}(k_1)\overline{\beta}_0 \tag{39}$$

where $d\sigma^{B1}(\alpha^4)$ is the cross section for single bremsstrahlung through order α^4; the analogous definition holds for $d\sigma(\alpha^n)$. In addition, to order α^4, the cross section $\overline{\beta}_2(k_1, k_2)$ may be identified as

$$\overline{\beta}_2 = \frac{d\sigma^{B2}}{d\Omega_\mu (d^3k_1/k_1)(d^3k_2/k_2)} - \widetilde{S}(k_1)\widetilde{S}(k_2)\frac{d\sigma_0}{d\Omega_\mu} \\ - \widetilde{S}(k_1)\overline{\beta}_1(k_2) - \widetilde{S}(k_2)\overline{\beta}_1(k_1) \tag{40}$$

where $d\sigma^{B2}$ is the respective order α^4 double bremsstrahlung cross section. Formulas analogous to (27)–(29) may then be used to obtain $\tilde{\overline{\beta}}_0$, $\tilde{\overline{\beta}}_1$ and $\tilde{\overline{\beta}}_2$. The steps leading from (30) to (32) may then be repeated. The net result is to add to (32) the term

$$\exp\left\{2\alpha(1)\Big(\mathrm{ReB} + \widetilde{B}\left(P_i(1), m_{iR}(\lambda), E''_{\gamma,\max}/\lambda\right)\Big)\right\} \frac{\tilde{\overline{\beta}}_2(k', k'')}{2} \frac{d^3k'}{k'} \frac{d^3k''}{k''} \\ \left\{\delta\left(\epsilon - k' - k''\right) \int_0^{K_{\max}} \rho''\left(\epsilon' - k' - k''\right) \, d\left(\epsilon' - k' - k''\right) \right. \\ \left. + \theta\left(\epsilon - k' - k'' - K_{\max}\right) \frac{\alpha(1)A}{\epsilon - k' - k''} \left(\frac{\epsilon - k' - k''}{E''_{\gamma,\max}}\right)^{\alpha(1)A}\right\} dE_{X'} \tag{41}$$

where (here $k' \cdot k'' = k'k'' - \vec{k'} \cdot \vec{k''}$ so that $k^\circ = |\vec{k}| \equiv k$ for all k)

$$E''_{\gamma,\max} = \\ \frac{\left(\sqrt{s} - k' - k'' + \left|\vec{k'} + \vec{k''}\right|\right)\left(s - 2\sqrt{s}(k' + k'') + 2k' \cdot k'' - 4m_f^2\right)}{2\left(s - 2\sqrt{s}(k' + k'') + 2k' \cdot k''\right)} ,$$

$$\rho''(\epsilon) = \rho(\epsilon)|_{E_{\gamma,\max} = E''_{\gamma,\max}} \tag{42}$$

Thus, it is clear how to extend (32) to order α^4 input. (We ignore, here, the processes $e^+e^- \to \mu^+\mu^- + f\bar{f}$, $f = e, \mu$, for pedagogical reasons; they pose no particular problem, but are expected to be insignificant at the level of accuracy of interest to us here.)

Several comments are in order. First, the use of $E'_{\gamma,\max}$ and $E''_{\gamma,\max}$ for the respective upper limits of the radiated photon energy is a refinement; these two can both be replaced by their maximum value, which is just $E_{\gamma,\max}$. Secondly, we have not allowed $K_{\max}$ to depend on the spherical angles (θ, ϕ) of the respective photons. This we have done for simplicity. The expression (32) is flexible enough to allow one to include a possible angular dependence of $K_{\max}$. Indeed, let $\overline{K}_{\max}$ = minimum value of $K_{\max}(\theta, \phi)$ for the respective detector. Then, if we set $K_{\max} = \overline{K}_{\max}$ in (32), we have a correct formula. We can then include the effect of $K_{\max}(\theta, \phi)$ by amending our prescription for choosing $\epsilon(\epsilon - k')$ and n: if $\epsilon > \overline{K}_{\max}(\epsilon - k' > \overline{K}_{\max})$ and $n > 0$, use the bremsstrahlung distribution $\widetilde{S}(k)d^3k/k$ to pick the respective angles (θ_i, ϕ_i) of the n photons with energies $\{k_i\}$ as determined by the procedure in Ref. 12. Due to the angular dependence of $K_{\max}(\theta, \phi)$, some subset of the n photons with energies $\{k_{i_1}, \ldots, k_{i_j}\}$ may not be detected. Let the energies of the detected photons be $\{k_{i_{j+1}}, \ldots, k_{i_n}\}$. Then, treat the event as an event with n–j detected photons with $\sqrt{s} - E_{X'} = \epsilon$ where only $\sum_{l=j+1}^{n} k_{i_l}$ of $\epsilon(\epsilon - k')$ is detected. In this way, we maintain a realistic description of the cross section in (32).

Finally, in the interest of completeness, we would like to describe the procedure[12] which one uses to choose the photon energies associated with (37). Specifically, these energies are generated as

$$k_i = \epsilon e^{z_i} \Big/ \left(\sum_{j=1}^{n} e^{z_j}\right) \qquad \left(k_i = (\epsilon - k')e^{z_i} \Big/ \left(\sum_{j=1}^{n} e^{z_j}\right)\right) \tag{43}$$

where the z_i are such that $z_i = \ln k_i + Y$. Here, for $i = 2, \ldots, n$, we take

$$\begin{gathered} z_i = \ln \epsilon + (\ln K_{\max} - \ln \epsilon)\,(R_i/\bar{n}) \\ \left(z_i = \ln\left(\epsilon - k'\right) + \left(\ln K_{\max} - \ln\left(\epsilon - k'\right)\right)(R_i/\bar{n})\right) \end{gathered} \tag{44}$$

where we recall that $\bar{n}$ is defined in (37) and we note that the R_{i+1} are generated from a series of uniformly distributed random numbers $r_i \in (0,1)$ with $R_{N+1} =$

$-\sum_{i=1}^{N} \ln r_i, 1 \leq N \leq n-1$, where $(n-1)$ is the value of N for which R_{N+2} first exceeds $\bar{n}$. Then, Y is fixed so that $k_1 = \epsilon - \sum_{i=1}^{n-1} k_{i+1}$ $(k_1 = \epsilon - k' - \sum_{i=1}^{n-1} k_{i+1})$ and

$$z_1 = \ln \epsilon \, (z_1 = \ln(\epsilon - k')) \quad , \tag{45}$$

which means that, at the end of the process, we must reject the entire event if $z_n - Y \leq \ln K_{\max}$, i.e., if $k_n \leq K_{\max}$. The prescription represented by (14)–(45) is now a practical way to implement (9).

What we see is that (14)–(45) afford one a method for summing the large IR and UV effects in $e^+e^- \to \mu\bar{\mu}(\gamma)$ without encountering mass singularity problems and without presuming the parton model, at the level of a realistic Monte Carlo event generator. To repeat, the general application of such "exponentiated" event generators will be taken-up elsewhere.[8,9]

V. Conclusion

We have derived a rigorous renormalization group improved version of the Yennie–Frautschi–Suura program using the renormalization group equation of Weinberg and 't Hooft. The detailed application of our formalism to the $SU_{2L} \times U_1$ theory for the processes $e^+e^- \to Z^\circ \to X$ will be discussed elsewhere.[8,9] We have, however, illustrated how one would use our formalism by giving an explicit recipe for the renormalization group improved exponentiation of the popular Monte Carlo event generator MMG1 in Ref. 10.

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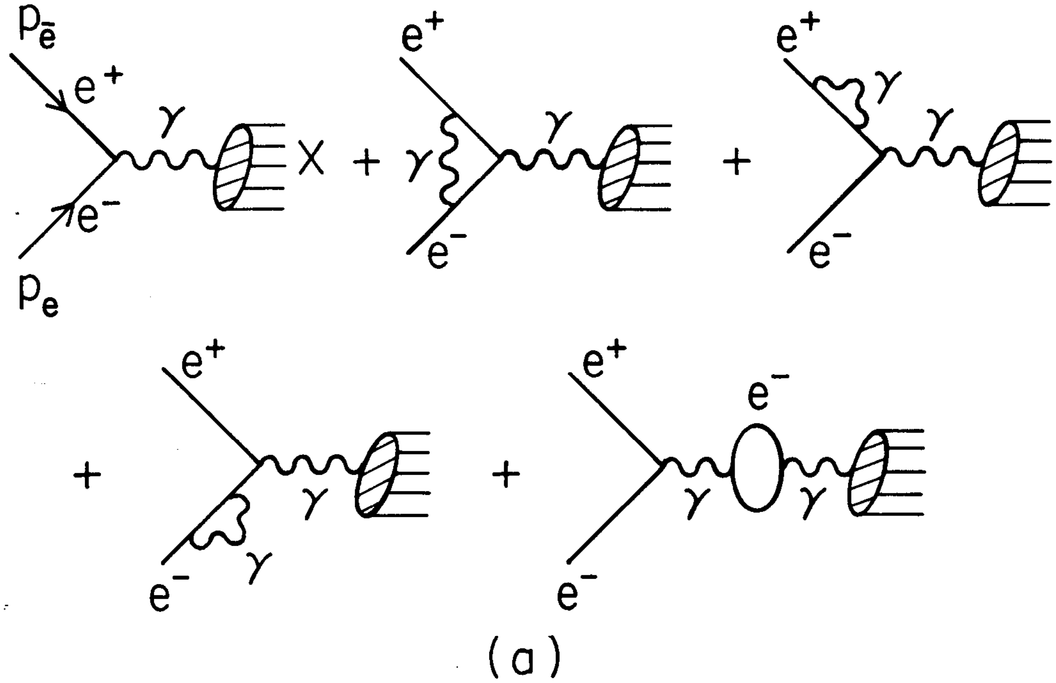
Figure Captions

Fig. 1. Order α radiative corrections to the initial state in $e^+e^- \to X$ in QED: a) virtual effects; b) bremsstrahlung.

Fig. 2. Virtual photon correction to $e^+e^- \to X$. This is a typical graph.

Fig. 3. Real photon emission in $e^+e^- \to n\,(\gamma) + X'$. This is a typical graph.



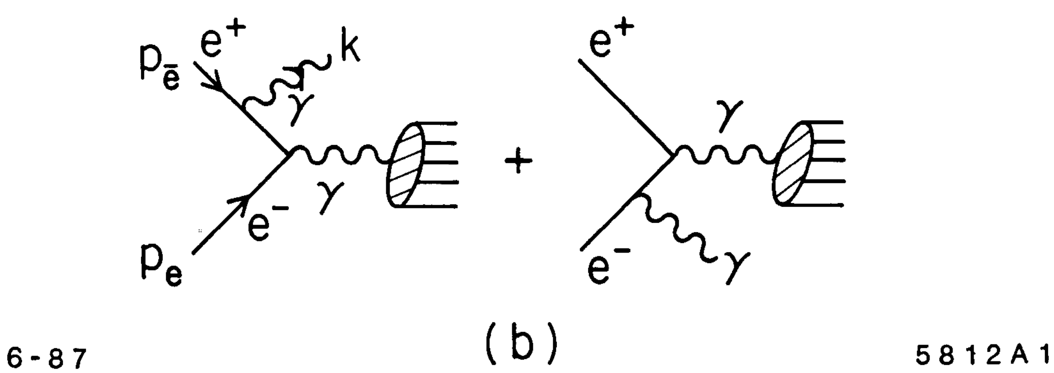


Fig. 1

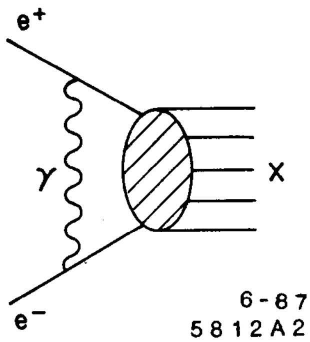


Fig. 2

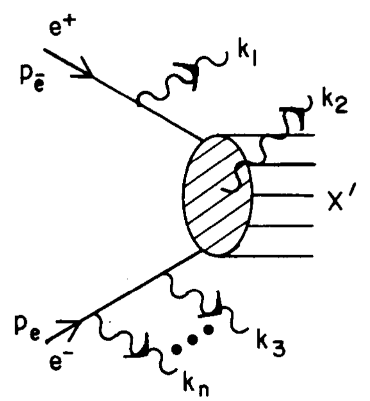


Fig. 3