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Superstrings on Hyperelliptic Surfaces and the Two Loop Vanishing of the Cosmological constant*

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ABSTRACT

We calculate the supersymmetric type II string partition function in a flat background for the situation where the world-sheet is hyperelliptic. We show explicitly that the contribution to the cosmological constant from hyperelliptic world-surfaces vanishes for genus$\leq$ 20. This implies, in particular, that the cosmological constant vanishes completely to two loops. Due to the decoupling of the holomorphic and antiholomorphic sectors, this conclusion holds equally well for the heterotic string.

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1. Introduction

Recently, there has been much interest in calculating multiloop superstring amplitudes. This has been strongly motivated by the effort to explicitly prove the finiteness of superstring theory and explain the vanishing of the cosmological constant. Hyperelliptic surfaces are a subclass of compact Riemann surfaces defined as two-sheeted coverings of the sphere. The bases of holomorphic differentials on hyperelliptic surfaces can be easily parametrized, and thus these surfaces provide a simple setting for explicit string calculations. Such calculations would be complete up to genus two, since all such surfaces are hyperelliptic. Bershadsky and Radul took advantage of this observation to calculate the two loop bosonic-string amplitude.[1] We have extended their methods so as to be able to calculate fermionic correlations using the techniques of conformal field theory.[2][3]

In this paper we have calculated the superstring (type II) vacuum amplitude on the hyperelliptic surfaces and have found that the amplitude displays cancellations in a way which generalizes the one loop result. The vacuum amplitude cancels completely for genus $g \leq 20$. We find this same cancellation in all higher orders for the subclass of spin structures with no Dirac zero modes. E. Verlinde and H. Verlinde have recently shown that the integrand of the string partition function is generally nonzero, but a total derivative on moduli space. We expect (though we have not shown) that our nonvanishing results at high genus have this interpretation .[4]

Since the partition function of the type II superstring decouples into holomorphic and antiholomorphic sectors, our results on the vanishing of the cosmological constant extend to the case of the heterotic string. Our conclusions are dependent on having an uncompactified, flat background metric. It is likely that this analysis can be generalized to any orbifold background; these calculations are in progress. Recently, J. Atick and A. Sen have calculated the two loop dilaton tadpole for string theories compactified on arbitrary supersymmetry preserving backgrounds.[6]

In the course of our analysis, we will note that hyperelliptic surfaces provide a wonderfully simple setting for studying harmonic spinors and spin structures. For an arbitrary Riemann surface very little is known about the dimensionality of the space of harmonic spinors, since an index theorem does not exist for the Dirac operator.[7][8] On hyperelliptic surfaces it is not difficult to find the number of Dirac zero modes (harmonic spinors) in each spin structure. Before proceeding into the heart of the paper we will give a short overview of the theory of hyperelliptic surfaces.

When this work was completed, we received a preprint by D. Lebedev and A. Morozov in which similar results were obtained.[9]

2. Hyperelliptic Surfaces

In this section we will summarize some of the relevant mathematics of hyperelliptic surfaces. A more detailed discussion can be found in standard texts.[10][11] A hyperelliptic surface of genus g is a two sheeted covering of the sphere with $2g+2$ branch points, or more concretely it is the Riemann surface of the algebraic curve

$$w^2 = \prod_{n=1}^{2g+2} (z - z(P_n))$$

where $P_k \neq P_j$ for $k \neq j$. These branch points are referred to in the literature as the Weierstrass points and carry information about the possible meromorphic functions on a Riemann surface.[10] They can be parametrized by coordinates in the plane $z(P_n) = a_n$, where z is the meromorphic function mapping points on the Riemann surface to the complex plane. Note that $z^{-1}(P)$ is a double valued mapping, while $w(P)$ is defined to be single-valued. Thus, under a sheet interchange $z \to z$, $w \to -w$. This mapping allows us to do our calculation on the complex plane, z, with a flat metric, except for a finite number of singularities at the Weierstrass points.

Hyperelliptic surfaces are the simplest Riemann surfaces and have many attractive properties. For example, modular transformations can be generated by permutations of the branch points. This is significant because it simplifies the construction of modular invariant quantities (e.g. the superstring partition function). The dimensionality of this set of surfaces, as a subspace of moduli space, can be found as follows: Analytic mappings of the plane into itself (elements of SL(2,C)) can be used to fix 3 of the Weierstrass points. This leaves $(2g-1)$ points, giving $(2g-1)$ complex parameters.

In this paper, we will be mainly concerned with the holomorphic differentials. The quickest way to determine them is by using the divisors of the algebraic functions defining the surface. The divisor of a function (which I will denote by: (f)) is defined by $(f) = \prod_i P_i^{\alpha(P_i)}$ where the P_i denote the zeros and poles of $f(P)$ and the $\alpha(P_i)$ denote their order. Holomorphic functions must have all of the $\alpha(P_i) \geq 0$ (i.e., they have *no* poles). Such a divisor is called integral. The algebraic functions defining the hyperelliptic surface have the following zeros and poles on the Riemann surface:[10]

$$If\ w = \sqrt{\Pi_{j=1}^{2g+2}(z - z(P_j))}\ ,then$$

$$(w) = \frac{P_1 \ldots P_{2g+2}}{Q_1^{g+1} Q_2^{g+1}}$$

and

$$(z) = \frac{Q_3 Q_4}{Q_1 Q_2},\ (dz) = \frac{P_1 \ldots P_{2g+2}}{Q_1^2 Q_2^2} \tag{2.1}$$

where Q_1, Q_2 are the poles of z (that is, the points of the surface mapped to $z = \infty$) , and Q_3, Q_4 are its zeros. The abelian, quadratic (Beltrami), and (-1)-differentials are necessary for constructing the bosonic string amplitudes (as done by Bershadsky and Radul[1]). The quadratic differentials describe deformations of the metric. It is easy to see that, for genus g, they are given by :

$$\frac{a_0 + \ldots + a_{2g-2}z^{2g-2}}{w^2}(dz)^2 \tag{2.2}$$

$$\frac{b_0 + \ldots + b_{g-3}z^{g-3}}{w}(dz)^2 \tag{2.3}$$

The power in the numerator is limited so that their divisor will be integral. It is interesting to observe that the $2g-1$ differentials given in (2.2) represent deformations within the set of hyperelliptic surfaces, since they are symmetric under a sheet interchange, while the $g-2$ differentials in (2.3) , being antisymmetric under a sheet interchange, are deformations off the subspace of hyperelliptic surfaces. This makes manifest that all surfaces of genus≤ 2 are hyperelliptic. More information about integer differentials on Z_N symmetric Riemann surfaces (i.e. N-sheeted coverings of the sphere which are invariant under a permutation of the sheets; these are the natural generalizations of the hyperelliptic surfaces) is given in the work of Bershadsky and Radul.[1]

3. The Fermionic String

3.1 The Partition Function

We will now evaluate the fermionic (i.e. supersymmetric type II) string partition function . It has been shown that the formula for the partition function, Z, reduces to:[12] [13] [14]

$$\begin{aligned} Z = \int_{Moduli} d\mu_{wp} \sum_{spin\ str.} \left(det'\nabla_{1/2}\right)^5 \left(\frac{2\pi det'\Delta}{\int d^2\xi\sqrt{g}}\right)^{-5} \\ \times \left(det'\nabla^{+}_{3/2}\nabla_{3/2}\right)^{-1/2} \left(det'\nabla^{+}_{1}\nabla_1\right)^{1/2} \langle K\rangle \end{aligned} \tag{3.1}$$

The ingredients in this equation are the following: $d\mu_{wp}$ is the Weil-Peterson measure on moduli space. $\nabla_{1/2}$ is the Dirac operator; ∇_1 and $\nabla_{3/2}$ are the covariant derivatives on the graviton (spin 2 field) and gravitino (spin 3/2) ghosts, respectively.

$$\langle K\rangle = \int d\psi_0 \ \epsilon_{l_1\cdots l_N\bar{l}_1\cdots\bar{l}_N}\langle\prod_1^N d^2z_n\chi^{(l_n)}(z_n)T_F(z_n) \times\prod_1^N d^2w_n\overline{\chi}^{(\bar{l}_n)}(w_n)\overline{T}_F(w_n)\rangle \tag{3.2}$$

and where ψ_0 are the Dirac zero modes and $\chi^{(l)}$ run over a basis dual to the holomorphic 3/2-differentials (i.e. the gravitino zero modes) where $l = 1,\dots,N = 2g-2$ enumerates the chosen basis. T_F is the fermionic stress energy tensor given by:[2]

$$T_F(z) = -\frac{1}{2}\psi^\mu\partial_z X^\mu + \frac{1}{2}b_{zz}\gamma - c^z\partial_z\beta_z - \frac{3}{2}(\partial_z c^z)\beta_z$$

with b_{zz}, c^z being the dimension 2, -1 ghost fields, and β_z, γ being the dimension $\frac{3}{2}$, $-\frac{1}{2}$ superconformal ghosts. From eqs. (3.1) and (3.2) , we can see without much difficulty that the holomorphic and antiholomorphic sectors decouple, so that we may consider each sector independently.

One would naively expect that the integrand of Z should vanish for spin structures containing Dirac zero modes. Indeed, if we have more Dirac zero modes than can be absorbed by the gravitino zero modes, then the the expression (3.2) trivially vanishes. If there are more gravitino than Dirac zero modes, however, it is possible that the Dirac zero modes may be absorbed by the T_F and $\overline{T}_F$ insertions to give a nonzero result for $\langle K\rangle$. It is not difficult to check that $\langle K\rangle$ can be nonzero only if each uncompactified dimension $\mu = 1,\cdots,10$ has an even number of zero modes. It will turn out that the first case where this remark applies appears when $g > 20$, as will be shown in section (3.4). It is straight forward to calculate $\langle K\rangle$ in these cases , and the expressions do not vanish. We expect that the integrand of Z should be a total divergence on moduli space[4] , but we have not been able to demonstrate this explicitly.

In the remainder of the paper, we will concentrate our attention on spin structures where there are no Dirac zero modes . In the following sections we will show how to identify the subset of the even spin structures on hyperelliptic surfaces which have no Dirac zero modes. We will argue that, within this class, $\langle K \rangle$ is independent of the particular spin structure. This can be understood explicitly from the description of the spin structure of differentials on hyperelliptic surfaces which will be elaborated upon in the following sections. This will allow us to combine the integrands for these spin structures and show that their sum is zero.

3.2 HALF-INTEGER DIFFERENTIALS AND SPIN STRUCTURE

Before studying superstrings on hyperelliptic surfaces, we must determine the holomorphic half-integer differentials on such surfaces. These differentials are necessary ingredients in calculating the fermionic determinants. A holomorphic differential must have an integral divisor (as defined above (2.1)). Therefore the possible half-integer differentials of weight n+1/2 (where n is an integer) are given by :

$$z^j \frac{\prod_{i=1}^{2g+2} [z - z(P_i)]^{\frac{\alpha(P_i)}{2}} (dz)^{n+1/2}}{[w]^{n+1/2}} \tag{3.3}$$

where $\alpha(P_i) = 0$ or 1 $(i = 1, \cdots, 2g+2)$ which define a spin structure. Holomorphicity requires that we have only integer or half- integer powers of z. The only restriction on the $\alpha(P_i)$ is that for even(odd) genus, $\sum_i \alpha(P_i)$ must be odd(even), so that the differential will not have an additional square root singularity at infinity. The condition that the divisor is integral is a restriction on the value of j:

$$2j + \sum_i \alpha(P_i) \leq (2n+1)(g-1) \tag{3.4}$$

which follows from the general formula for the divisor of (3.3):

$$[Q_1Q_2]^{(2n+1)(g-1)/2-j-\frac{1}{2}\sum_i \alpha(P_i)}[Q_3Q_4]^{j+\frac{1}{2}\sum_i \alpha(P_i)}$$

where Q_1Q_2 and Q_3Q_4 were defined earlier as the poles and zeros of z (see eq. (2.1)). We know that for (n+1/2)-differentials to have a finite norm they must asymptotically vanish as $z^{-(2n+1)}$; this statement is ,of course, equivalent to (3.4). We will verify in the next section that have not left out any half-integer differentials by observing that the dimension of the above basis corresponds to that given by the Riemann-Roch theorem.

A nonzero $\alpha(P_i)$ gives a nontrivial boundary condition around P_i, such that the differential changes by (-1) when going around any any loop about P_i. Thus , each set of $\alpha(P_i)$ define a spin structure. Note, though, that the set of exponents $\{(1-\alpha(P_i)\}$ give the same phases about all closed loops on the Riemann surface as $\{\alpha(P_i)\}$ and thus defines the same spin structure. (The two sectors are related in the same way as the sectors of integer differentials even and odd under sheet interchange.) With these observations, it is now simple to determine the number of allowed spin structures.The pairing of sectors allows us to limit ourselves to $N = \sum_i \alpha(P_i) \leq (g+1)$. For a given $N < (g+1)$ we have $\binom{2g+2}{N}$ different spin structures. When $N = (g+1)$, the sectors pair with one another, and we have $\frac{1}{2}\binom{2g+2}{g+1}$ spin structures. Thus, we have all together for odd g:

$$\frac{1}{2}\sum_{k=0}^{k=g+1}\binom{2g+2}{2k} = 2^{2g}$$

spin structures, and for even g we similarly have

$$\frac{1}{2}\sum_{k=0}^{k=g+1}\binom{2g+2}{2k-1} = 2^{2g}$$

spin structures. This is ,of course, as expected.[15] Alternating possible values of $\sum_i \alpha(P_i)$ correspond to the odd and even spin structures . These are the spin structures where the number of holomorphic Dirac zero modes is odd or even, respectively. It can be shown that: $\sum_i \alpha(P_i) = g+1 \ (mod\ 4)$ for the even spin structures, and $\sum_i \alpha(P_i) = g-1 \ (mod\ 4)$ for the odd spin structures. This will be proven in the following section where we explicitly count Dirac zero modes. Now note the following identities:[16]

$$\frac{1}{2}\sum_n \binom{2g+2}{g+1-4n} = 2^{g-1}(2^g+1)$$

$$\frac{1}{2}\sum_n \binom{2g+2}{g-1-4n} = 2^{g-1}(2^g-1)$$

We have thus verified that there are $2^{g-1}\,(2^g \pm 1)$ even(odd) spin structures. It should also be noted that spin structures defined by the same value of $\sum_i \alpha(P_i)$ (or its complement $2g+2-\sum_i \alpha(P_i)$) form a restricted modular orbit, in the sense that they transform into each other under permutations of the branch points and have the same number of Dirac zero modes (as will be shown). We then have $\left[\frac{g+3}{2}\right]$ such orbits for each genus g, where $[x]$ is the integer part of x.

3.3 Counting Fermion Zero Modes

We will now count the dimension of the given basis of (n+1/2)-differentials and verify that it agrees with the Riemann-Roch theorem. Recall from (3.4)that

$$j \le \frac{1}{2}\left((2n+1)(g-1) - \sum_i \alpha(P_i)\right) \tag{3.5}$$

Since $2g+2-\sum_i \alpha(P_i)$ corresponds to the same spin structure, we can also permit in the paired sector powers of z with the exponent satisfying:

$$j' \le \frac{1}{2}\left[(2n+1)(g-1) - (2g+2-\sum_i \alpha(P_i))\right] \tag{3.6}$$

if $j' \ge 0$. Note that $j' \ge 0$ implies $\sum_i \alpha(P_i) \ge (1-2n)g + 2n + 3$. For $n \ge 1, g \ge 2$ the inequality (3.6) cannot be satisfied for only two cases: $n=1, g=2$ and

$\sum_i \alpha(P_i) = 1$, which has 2 zero modes in the original sector , and also $n = 1$, $g = 3$ and $\sum_i \alpha(P_i) = 0$, which has 4 zero modes. Otherwise, we get that for $n \geq 1, g \geq 2$ the dimension is the maximum of $j + j' + 2 = 2n(g-1)$. Thus, we find that for all cases $n \geq 1, g \geq 2$, we always have $2n(g-1)$ zero modes. For $n < 0$ we have zero modes only for $g = 0, 1$. For $g = 1$ we always have only one zero mode corresponding to $\sum_i \alpha(P_i) = 0$. For $g = 0$, $n < 0$ we clearly get $2|n|$ zero modes. This is all as expected from the Riemann-Roch theorem. Still, the really interesting case is $n = 0$ where the Riemann-Roch theorem does not apply. This corresponds to the Dirac zero modes (or harmonic spinors) on hyperelliptic surfaces.

3.4 Dirac Zero Modes on Hyperelliptic Surfaces

For $n = 0$ we find from eq. (3.3) that $j \leq \frac{1}{2}(g - 1 - \sum_i \alpha(P_i))$ where $j \geq 0$. (We neglect j' since it is always negative for $\sum_i \alpha(P_i) \leq g+1$.) Hence, the number of Dirac zero modes is the maximum of $(j+1)$:

$$\frac{1}{2}\left(g + 1 - \sum_i \alpha(P_i)\right) \tag{3.7}$$

This equation verifies our previous claim that for even(odd) spin structures we have $\sum_i \alpha(P_i) = g + 1 (mod\ 4)$ $(\sum_i \alpha(P_i) = g - 1 (mod\ 4))$. It is important to observe that we will *always* have Dirac zero modes *except* in the modular orbit defined by $\sum_i \alpha(P_i) = g + 1$. We then see that some even spin structures have Dirac zero modes for all values of the hyperelliptic parameters.

The first set of even spin structures where we have Dirac zero modes corresponds to having $\sum_i \alpha(P_i) = g - 3$. For this value of $\sum_i \alpha(P_i)$ we can see from eq. (3.5) that we have $g + 1$ holomorphic 3/2-differentials. Then, in 10 dimensions we will need $g > 20$ in order to have enough gravitino zero modes to absorb the even Dirac zero modes in eq (3.2). This explains our comment in section (3.1).

Now we are prepared to see that $\langle K \rangle$ as defined by (3.2) is independent of the individual spin structures in a modular orbit . This easily follows because the spin structure dependence of $\langle K \rangle$ arises in terms of the form $\chi(z)\psi(z)$. In a modular orbit the spin structures are given by placing branch cuts at $\sum_i \alpha(P_i)$ points. Since $\chi(z)$ is dual to the holomorphic 3/2-differentials, the branch cuts describing its spin structure will occur in the denominator and will thus cancel with the numerator spin structure branch cuts in $\psi(z)$. Thus, $\langle K \rangle$ will only depend on the *number* of branch cuts describing the spin structure. Since the number of branch cuts is all that defines a restricted modular orbit , $\langle K \rangle$ will be invariant within an orbit.

4. Calculating the Fermion Determinants

Since the partition function vanishes when Dirac zero modes are present (for $g \leq 20$), we are only interested in calculating the fermionic determinants in the spin structures defined by $\sum_i \alpha(P_i) = g+1$. We will now calculate the relevant determinants on genus 1 and then in arbitrary genus, using the methods of references [1], [17], [18].

4.1 GENUS 1

The even spin structures (where $\sum_i \alpha(P_i) = g+1 = 2$) correspond to placing square-root branch cuts at a pair of Weierstrass points. Note that placing branch cuts at the complementary pair, which corresponds to a sheet interchange, gives the same spin structure, and this must be taken into account. This is only of relevance for the ghost determinant, since the complementary pair gives the same correlation function for spinors. Let us first choose the pair $(a_2, a_3)(a_1, a_4)$. Let $O_{14}(a_i)$ be the primary conformal operator which defines the proper monodromy behavior of fermion fields $\psi(z)$ transported around a_i and the spin structure defined by (a_1, a_4). We will then be interested in calculating the vacuum expectation value:

$$\langle O_{14}(a_i)\rangle \sim \left[det\nabla_{1/2}\right]^{1/2}_{14} \tag{4.1}$$

where $\nabla_{1/2}$ is the Dirac operator acting on spin-1/2 fields.

To evaluate (4.1) , the authors of references [17] , [18] suggest starting from the Green function defined by:

$$G_{14}(z,w) = \frac{\langle\overline{\psi}(z)\psi(w)O_{14}(a_i)\rangle}{\langle O_{14}(a_i)\rangle} \tag{4.2}$$

where $\psi(z)$ are the 1/2-differentials. This Green function is uniquely determined by its poles and the chosen spin structure. The only pole occurs as $z \to w$:

$$\lim_{z\to w} G(z,w) = \frac{1}{z-w} + \dots$$

The rest of the analytic structure of $G_{14}(z,w)$'s is given by the 1/2- differential nature of $\psi(z)$and the spin structure. We then clearly get:

$$G_{14}(z,w) = \frac{1}{z-w}\left[\frac{(z-a_2)(z-a_3)}{(z-a_1)(z-a_4)}\right]^{1/4}\left[\frac{(w-a_1)(w-a_4)}{(w-a_2)(w-a_3)}\right]^{1/4} \tag{4.3}$$

The Green functions G_{12} and G_{13} may be obtained from (4.3) by permutatiing the branch points a_i. The expectation value of the stress energy tensor:

$$\langle T(z)\rangle_{14} = \frac{\langle T(z)O_{14}(a_i)\rangle}{\langle O_{14}(a_i)\rangle}$$

may be computed from (4.2) and the definition:

$$T(z) = \lim_{w\to z}\left\{\frac{1}{2}\left[\partial_z\langle\overline{\psi}(z)\psi(w)\rangle - \partial_w\langle\overline{\psi}(z)\psi(w)\rangle\right] + \frac{1}{(z-w)^2}\right\} \tag{4.4}$$

A simple calculation gives:

$$\langle T(z)\rangle_{14} = \frac{1}{32}\left(\sum_{j=1}^{4}\frac{1}{z-a_i}\right)^2 = \frac{1}{32}\sum_i\frac{1}{(z-a_i)^2} - \frac{1}{16}\sum_{j>i}\frac{(-1)^{\delta^{ij}_{14}+\delta^{ij}_{23}}}{(z-a_i)(z-a_j)} \tag{4.5}$$

where[2]

$$\delta^{ij}_{kl} = \begin{Bmatrix} 1 \; if \; (ij) = (kl) \\ 0 \; otherwise \end{Bmatrix}$$

Recalling the form of the operator product expansion of T(z) with any primary conformal field[2] [3] ,

$$T(z)O(w) \sim \frac{h}{(z-w)^2}O(w) + \frac{1}{z-w}\partial_w O(w)$$

where h is the conformal dimension of $O(w)$. We can interpret the residue of a single pole in (4.5) as the logarithmic derivative of (4.1) :

$$\partial_{a_i} log\langle 0_{14}\rangle = -\frac{1}{16}\sum_{j>i}\frac{(-1)^{\delta^{ij}_{14}+\delta^{ij}_{23}}}{(z-a_j)}$$

Integrating this equation, we find:

$$\begin{aligned}\left(det_{14}\nabla_{1/2}\right)^{1/2} &\sim \left[\prod_{i>j}(a_i-a_j)^{-\delta^{ij}_{14}-\delta^{ij}_{23}}\right]^{-1/16} \\ &= \left[\frac{(a_4-a_1)(a_3-a_2)}{(a_2-a_1)(a_3-a_1)}\right]^{1/16}\end{aligned} \tag{4.6}$$

For any other even spin structure, we will find:

$$\left(det_{kl}\nabla_{1/2}\right)^{1/2} \sim \left[\prod_{i>j}(a_i-a_j)^{-\delta^{ij}_{kl}}\right]^{-1/16} \tag{4.7}$$

where (kl) is any pair defining the same spin structure. We can now use SL(2,C) to fix three of the branch points . Let $a_1 = 0$, $a_2 = x$, $a_3 = 1$, $a_4 \to \infty$,then

$$\begin{aligned}\left(det_{14}\nabla_{1/2}\right)^{1/2} &\sim \left(\frac{1-x}{x}\right)^{1/16}, \\ \left(det_{13}\nabla_{1/2}\right)^{1/2} &\sim (x(1-x))^{1/16}, \\ \left(det_{12}\nabla_{1/2}\right)^{1/2} &\sim \left(\frac{x}{1-x}\right)^{1/16},\end{aligned} \tag{4.8}$$

We now proceed to calculate the superconformal ghost contribution which is given by $\left(det'_\nu \nabla_{3/2}\right)^{-1}$ where $\nabla_{3/2}$ is the differential acting on spin-3/2 fields and ν represents the spin structure. This is the vacuum expectation value of the superconformal ghost action.[2] Let $O_\nu^{ghost}(a_i)$ be the primary field which defines the vacuum of the ghost action with spin structure ν. We observe that we have no spin-3/2 zero modes in the even spin structures (at genus 1). The relevant Green function for the (14) spin structure is:

$$\begin{aligned} G_{14}(z,w) &= \frac{\langle \gamma(z)\beta(w)O_{14}^{ghost}(a_i)\rangle}{\langle O_{14}^{ghost}(a_i)\rangle} \\ &= \frac{1}{z-w}\left[\frac{(z-a_1)(z-a_4)}{(w-a_1)(w-a_4)}\right]^{3/4}\left[\frac{(z-a_2)(z-a_3)}{(w-a_2)(w-a_3)}\right]^{1/4} \end{aligned} \tag{4.9}$$

where $\gamma(z)$ and $\beta(w)$ are the -1/2 and 3/2 spin ghost fields, respectively. It should be noted that we take γ and β to be anticommuting fields. This is just the simplest way to evaluate the necessary ghost determinants using this stress energy method. The stress tensor is given by:[2]

$$\langle T(z)\rangle_{14} = \lim_{w\to z}\{-3/2\partial_z G_{14}(z,w) - 1/2\partial_w G_{14}(z,w) - \frac{1}{(z-w)^2}\} \tag{4.10}$$

where $G_{14}(z,w)$ is given by eq. (4.9). We then obtain

$$\begin{aligned} \langle T(z)\rangle_{14} =& \frac{1}{4}\left[\frac{1}{(z-a_2)^2}+\frac{1}{(z-a_3)^2}+\frac{3}{(z-a_1)^2}+\frac{3}{(z-a_4)^2}\right] \\ &-\frac{1}{32}\left(\frac{1}{(z-a_2)}+\frac{1}{(z-a_3)}+\frac{3}{(z-a_1)}+\frac{3}{(z-a_4)}\right)^2 \end{aligned} \tag{4.11}$$

Converting this equation to a differential equation for $\langle O_{14}^{ghost}\rangle$, integrating, and fixing three a_i, we find: $\langle O_{14}^{ghost}(a_i)\rangle \sim \left[x^3(1-x)\right]^{-1/16}$. Interchanging (14) and (23) gives the second sector of this spin structure; in this sector $G(z,w)$ is changed by interchanging the exponents $\frac{3}{4}$ and $\frac{1}{4}$. We then obtain a different value for $\langle T(z)\rangle$, and ,finally, $\langle O_{23}^{ghost}(a_i)\rangle \sim \left[x^3(1-x)^9\right]^{-1/16}$. The total ghost contribution is then:

$$\begin{aligned}\left(det_{(14)(23)}\nabla_{3/2}\right)^{-1} &\sim \langle O^{ghost}_{14}(a_i)\rangle\langle O^{ghost}_{23}(a_i)\rangle,\\ &\sim \left[x^6(1-x)^{10}\right]^{-1/16}.\end{aligned} \tag{4.12}$$

In the same way, one can verify that:

$$\begin{aligned}\left(det_{(13)(24)}\nabla_{3/2}\right)^{-1} &\sim \left[x^6(1-x)^{6}\right]^{-1/16},\\ \left(det_{(12)(34)}\nabla_{3/2}\right)^{-1} &\sim \left[x^{10}(1-x)^{6}\right]^{-1/16}\end{aligned} \tag{4.13}$$

The cut sphere can be mapped explicitly onto the torus by:[17]

$$z(t) = \frac{\wp(t)-e_1}{e_2-e_3}, \quad e_1+e_2+e_3=0$$

where $\wp(t)$ is the classical Weierstrass function. We may identify

$$x \equiv \frac{e_3-e_1}{e_2-e_1}.$$

It is now straight forward to express x in terms of the Jacobi theta constants, $\vartheta_i(\tau)$:

$$x = \left(\frac{\vartheta_4(\tau)}{\vartheta_3(\tau)}\right)^4 \tag{4.14}$$

Now using eq. (4.14) and combining eqs. (4.13) , (4.8):

$$\begin{aligned}\left(det_{12}\nabla_{1/2}\right)^5\left(det_{12}\nabla_{3/2}\right)^{-1} &\sim \left[\frac{x}{1-x}\right]^{1/2}[x(1-x)]^{-1/2}\\ &= [x(1-x)]^{-2/3}\,\frac{\vartheta_4(\tau)^4}{\vartheta_1'(\tau)^{4/3}}\\ \left(det_{13}\nabla_{1/2}\right)^5\left(det_{13}\nabla_{3/2}\right)^{-1} &\sim [x(1-x)]^{-1/2}\,[x(1-x)]^{-1/2}\\ &= [x(1-x)]^{-2/3}\,\frac{\vartheta_3(\tau)^4}{\vartheta_1'(\tau)^{4/3}}\\ \left(det_{14}\nabla_{1/2}\right)^5\left(det_{14}\nabla_{3/2}\right)^{-1} &\sim \left[\frac{1-x}{x}\right]^{1/2}[x(1-x)]^{-1/2}\\ &= [x(1-x)]^{-2/3}\,\frac{\vartheta_2(\tau)^4}{\vartheta_1'(\tau)^{4/3}}\end{aligned} \tag{4.15}$$

The relative coefficients of the terms in eq. (4.15) are determined by requiring that they transform into each other under modular transformations. The modular transformations of the theta functions are well-known and we find the established result that the integrand of the one loop partition function is proportional to

$$\sum_{\nu} \left(det_{\nu}\nabla_{1/2}\right)^5 \left(det_{\nu}\nabla_{3/2}\right)^{-1} \sim \vartheta_2^4 + \vartheta_4^4 - \vartheta_3^4$$

which vanishes by the well-known Jacobi theta identity. It is interesting to ask how this cacellation would have appeared if we had not converted to the language of theta functions. In our original parameterization, modular transformations correspond to permutations of the Weierstrass points. Requiring that the different spin structures transform into each other under permutations we get simply:

$$\sum_{\nu} \left(det_{\nu}\nabla_{1/2}\right)^5 \left(det_{\nu}\nabla_{3/2}\right)^{-1} \sim Pfaff\left[A_{ij}\right] \tag{4.16}$$

where $A_{ij} = a_i - a_j$, $i,j = 1,\cdots,4$, and Pf denotes the Pfaffian. $Pf(A_{ij})$ vanishes trivially since the rows or columns of A_{ij} are not all linearly independent (e.g. $A_{ij} = A_{ik} + A_{kj}$). We can also see this explicitly as follows:

$$Pf(A_{ij}) = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23};$$

and using $A_{13} = A_{12} + A_{23}$ we obtain

$$Pf(A_{ij}) = A_{12}A_{32} + A_{23}A_{12} = 0$$

This concludes the discusion of genus 1.

4.2 GENUS> 1

We will now calculate the corresponding quantities on higher genus hyperelliptic surfaces. Since all genus two surfaces are hyperelliptic, the following results will give the complete contribution to (3.1) for genus two. We remember that because of the presence of harmonic spinors the partition function vanishes for all spin structures except for the modular orbit defined by $\sum_i \alpha(P_i) = g+1$. We place branch cuts on $g+1$ points (i.e., half of the Weierstrass points) in order to describe the relevant spin structures. The Green function is then given by:

$$
\begin{aligned}
G_{\{l_i\}}(z,w) &= \frac{\langle \overline{\psi}(z)\psi(w)O_{\{l_i\}}(a_i)\rangle}{\langle O_{\{l_i\}}(a_i)\rangle} \\
&= \frac{1}{z-w}\left[\frac{\prod_{a_i\notin\{l_i\}}(z-a_i)}{\prod_{a_i\in\{l_i\}}(z-a_i)}\right]^{1/4}\left[\frac{\prod_{a_i\in\{l_i\}}(w-a_i)}{\prod_{a_i\notin\{l_i\}}(w-a_i)}\right]^{1/4}
\end{aligned} \tag{4.17}
$$

where the set $\{l_i\}$ contains the $g+1$ Weierstrass points where we have chosen to place branch cuts to describe our spin structure. The number of inequivalent sets of $\{l_i\}$ is $\frac{1}{2}\binom{2g+2}{g+1}$, which is just the number of spin structures. The stress energy tensor is then given by from eq. (4.4):

$$
\langle T(z)\rangle_{\{l_i\}} = \frac{-1}{32}\left[\sum_{j=1}^{2g+2}\frac{(-1)^{\delta_{j\{l_i\}}}}{z-a_j}\right]^2 \tag{4.18}
$$

where

$$
\delta_{j\{l_i\}} = \left\{\begin{matrix} 1\ if\ a_j\in\{l_i\} \\ 0\ otherwise \end{matrix}\right\}
$$

and, consequently, following the same procedure as in eq.(4.7) :

$$
\left(det_{\{l_i\}}\nabla_{1/2}\right)^{1/2} \sim \left[\prod_{i>j}(a_i-a_j)^{-\delta_{ij}^{\{l_i\}}}\right]^{-1/16} \tag{4.19}
$$

where

$$\delta_{ij}^{\{l_i\}} = \begin{Bmatrix} 1 \; if \; a_i \; and \; a_j \in \{l_i\} \; or \; a_i \; and \; a_j \notin \{l_i\} \\ 0 \; otherwise \end{Bmatrix}$$

We will now calculate the superconformal ghost contribution. Again, we work with anticommuting β and γ fields, applied to produce the required determinants. We have $2(g-1)$ β zero modes which must be taken into account. In our spin structures we divide the Weierstrass points into two complementary sets of $g+1$ points: the $a_i \in \{l_i\}$ and the $a_i \notin \{l_i\}$. $g-1$ of the β zero modes correspond to placing branch cuts at $a_i \in \{l_i\}$ while the other $g-1$ correspond to putting branch cuts at the complementary set of points. Since we calculate the Green function for a particular set $\{l_i\}$ and then multiply by the complement, we only have $g-1$ β zero modes in each calculation. The relevant Green function is then:

$$\begin{aligned} G_{\{l_i\}}(z,w) =& \frac{\langle \gamma(z)\beta(w) \prod_{i=1}^{g-1} \beta(y_i) O^{ghost}_{\{l_i\}}(a_i)\rangle}{\langle \prod_{i=1}^{g-1} \beta(y_i) O^{ghost}_{\{l_i\}}(a_i)\rangle} \\ =& \frac{1}{z-w} \prod_{i=1}^{g-1} \left(\frac{w-y_i}{z-y_i}\right) \prod_{i=1}^{2g+2} \frac{(z-a_i)^{1/4}}{(w-a_i)^{3/4}} \\ & \times \prod_{a_i \in \{l_i\}} (z-a_i)^{1/2} \prod_{a_i \notin \{l_i\}} (w-a_i)^{1/2} \end{aligned} \tag{4.20}$$

We can now use the stress energy tensor as before to obtain:

$$\begin{aligned} &\langle \prod_{i=1}^{g-1} \beta(y_i) O^{ghost}_{\{l_i\}}(a_i)\rangle \langle \prod_{g}^{2g-2} \beta(y_i) O^{ghost}_{\{l'_i\}}(a_i)\rangle \\ &\sim \prod_{i>j} (a_{l_i} - a_{l_j})^{-5/8} \prod_{i>j} (a_{l'_i} - a_{l'_j})^{-5/8} \prod_{i,j} (a_{l'_i} - a_{l_j})^{-3/8} \left[det \beta_n(y_m)\right]^{-1} \end{aligned} \tag{4.21}$$

where

$$[det\beta_n(y_m)]^{-1} = \prod_{k=g}^{2g-2}\prod_{l=1}^{g-1}\prod_{a_i\in\{l_i\}}(a_i-y_k)^{1/4}\prod_{a_i\notin\{l_i\}}(a_i-y_l)^{1/4}$$
$$\times\prod_{a_i\in\{l_i\}}(a_i-y_l)^{3/4}\prod_{a_i\notin\{l_i\}}(a_i-y_k)^{3/4}\prod_{m\neq n}(y_m-y_n)^{-1}$$

is the determinant of the $2g-2$ supermoduli,$\{l_i\}$ and $\{l'_i\}$ are complementary partitions, and $a_{l_i}\in\{l_i\}$, $a_{l'_i}\in\{l'_i\}$. This is related to the determinant of $\nabla_{3/2}$ with the zero modes, $\beta(y_i)$, absorbed:

$$\left(det'_{\{l_i\}}\nabla_{3/2}\right)^{-1} = \frac{\langle\prod_{1}^{g-1}\beta(y_i)O^{ghost}_{\{l_i\}}\rangle\langle\prod_{g}^{2g-2}\beta(y_i)O^{ghost}_{\{l'_i\}}\rangle}{[det\beta_n(y_m)]}$$

In all, the integrand of the string partition function contains the following ingredients (again for the right movers in 10-dimensions):

$$\sum_\nu\left(det'_\nu\nabla_{3/2}\right)^{-1}\left(det_\nu\nabla_{1/2}\right)^5\sim\left[\prod(a_i-a_j)\right]^{-1}\sum_{\{l_i\}}\epsilon_{\{l_i\}}$$
$$\times\prod_{i>j}(a_{l_i}-a_{l_j})\prod_{i>j}(a_{l'_i}-a_{l'_j}) \tag{4.22}$$

The $\epsilon_{\{l_i\}}$are phases necessary to insure modular; that is, permutation invariance among the a_i. We now observe that the right hand side of eq. (4.22) can be written in terms of the Pfaffian of $A_{ij}=(a_i-a_j)$. Indeed,

$$\sum_{\{l_i\}}\epsilon_{\{l_i\}}\prod_{i>j}(a_{l_i}-a_{l_j})\prod_{i>j}(a_{l'_i}-a_{l'_j})\sim[Pf\ A_{ij}]^g \tag{4.23}$$

For genus one this has already been shown explicitly. For genus two it is also obvious because the left hand side of eq. (4.23) is an alternating 6-linear functional of A_{ij} and a well-known theorem of linear algebra tells us that it must be proportional to the determinant of A_{ij}. It is not difficult to see that for arbitrary g both sides of eq. (4.23) contain the same number of powers of A_{ij} and form 1-dimensional representations of Z_{2g+2} (which acts on the indices of A_{ij}) . It is then clear that they must thus be proportional. The Pfaffian of A_{ij} vanishes trivially as discussed earlier and consequently so does the partition function.

In the genus one case the vanishing of the Pfaffian was related to a theta function identity, and indeed this extends to higher genus. Thomae's formula relates the Weierstrass points to theta constants :[7]

$$\Theta[\eta_{\{l_i\}}](0)^4 = c\prod_{i<j}(a_{l_i} - a_{l_j})\prod_{i<j}(a_{l'_i} - a_{l'_j}) \tag{4.24}$$

where c is independent of the spin structure. Making use of Thomae's formula in eq. (4.23) gives:

$$\sum_{\{l_i\}} \epsilon_{\{l_i\}}\Theta[\eta_{\{l_i\}}](0)^4 = 0 \tag{4.25}$$

Remarkably enough this turns out to be a well-known identity. This is a special case of Frobenius' theta formula.[7]

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