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String Field Theory on the Conformal Plane
II. Generalized Gluing

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ABSTRACT

In our previous paper, we defined multistring vertices by conformally mapping string states onto the complex plane. In this paper, we prove the combination rules for these vertices: The contraction of two vertices by the natural inner product on the string Hilbert space yields just the composite vertex which would have been obtained by gluing together the two world-sheets in an appropriate fashion.

1. Introduction

The theory of strings contains many facets in which opposite analytical viewpoints—geometric or algebraic arguments, abstract or concretely physical reasoning—are knit together elegantly by the underlying formalism. In this series of papers, we explore the foundations of string dynamics, the formulation of a field theory of strings, in an attempt to unify various viewpoints on this subject into a coherent whole. Just as with string theory more generally, string field theory has been discussed from an essentially geometrical point of view and and from a viewpoint closely rooted in the Hilbert space picture of strings. On the one hand, string field theory can be viewed as a way of breaking down the analytic world-sheets of Polyakov's space-time approach to string dynamics[1] into more elementary string interaction vertices. On the other hand, the theory can be considered as summarizing the dynamics of the local quantum fields associated with the various modes of string excitation. Both aspects have been exemplified by the development of light cone string field theory.[2]

The relation between these two pictures is easy to imagine intuitively but not at all straightforward to derive precisely. We have found it useful to connect each viewpoint to conformally-invariant two-dimensional quantum field theory, using the beautiful formalism of Belavin, Polyakov, and Zamolodchikov[3] (BPZ) and its application to string theory by Friedan[4] and Friedan, Martinec and Shenker.[5] In the first paper of this series (referred to henceforth as I), we showed how to write the kinetic energy terms and couplings of string modes as expectation values in the BPZ two-dimensional Hilbert space. We also sketched, in the simplest special case, how the natural inner product in the space of string modes, carried into the two-dimensional formalism following BPZ, takes on a geometrical interpretation as a gluing of pieces of world-sheet, and we claimed that this connection was, in fact, completely general. In Witten's formulation of the open string theory,[6] which he presented entirely in terms of world-sheet geometry, the proof of gauge-invariance depends on the form of the surface which results from such a gluing procedure. Our method thus allowed us to prove the gauge-invariance of Witten's open-string action explicitly, in terms of transformations on the string field components, by connecting such transformations to Witten's geometrical operations.

In this paper, we will complete the analysis begun in I by deriving the connection between the string mode expansion and world-sheet geometry in its full detail, at least for the case of the bosonic string. In I, we constructed off-shell multi-string vertices explicitly in terms of mappings of conformal field theory operators onto the complex plane. Here, we will study the new vertex produced by the contraction of two of these vertices with the conformal field theory innner product of BPZ. We will show that the resulting vertex is precisely the one which results from mapping the original operators onto a new plane formed by gluing together the planes used to define the original vertices. This structure was motivated in I by the analysis of a special case of the relation. Here we will present the complete proof of this gluing identity.

Our analysis will procede as follows: We will assume that the reader is familiar with the notation and formalism that we have presented in I. To this, we must add some further elements of the formalism of 2-dimensional conformal field theory; these will be reviewed in Section 2. In Section 3, we will review the precise statement of the gluing identity and then begin the proof. This first part of the discussion will concern the terms in the vertex containing coordinate mode operators. This discussion should already make clear how we interpret mode operator contractions as building up analytic functions on the glued world-surface. The remaining sections of the paper will extend this analysis to the other elements of the vertex. Section 4 will present some further analysis of the coordinate mode osciallators which extends to the coordinate zero modes. Section 5 will generalize this analysis to general bosonized ghost system. Section 6 will present an alternative argument for the reparametrization ghosts in their fermionic formulation. Overall factors of determinants arising from the contraction are discussed in sections 5.3 and 6.3. There we will demonstrate that these factors cancel as expected when the total central charge of the string conformal field theory equals zero.

Throughout this paper we will not distinguish "Green functions" from "Neumann functions" when we refer to them in the text. Also, we will frequently call "function" what is really a conformal tensor, for example the Green "function" of the fermi ghosts $\langle b_{zz}(z)c^{w}(w)\rangle$. In all cases of interest we define these objects explicitly.

2. More Conformal Field Theory

The analysis of this paper relies on the formulation of string field theory in terms of conformal field theory vacuum expectation values which was presented in I. This treatment made essential use of the formalism for conformal field theory developed by Belavin, Polyakov, and Zamolodchikov (BPZ); the essential elements of this formalism were reviewed in Section 2 of I. For the analysis that we will present here, we will need to make use of two further sets of results in conformal field theory—the bosonization of fermions and the treatment of finite, as opposed to infinitesimal, conformal transformations. This chapter will review these new elements of the formalism.

2.1. The bosonized ghost system

Since this section is devoted mostly to fixing conventions, let us start by listing the Euclidean actions for the conformal fields that will appear later. For the open string we write the action for coordinate and free bosonized fields as

$$S_{x,open} = \frac{1}{2\pi} \int d^2z \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \tag{2.1}$$

and

$$S_{\phi,open} = -\frac{\epsilon}{2\pi} \int d^2z \sqrt{g} g^{ab} \partial_a \Phi \partial_b \Phi - \frac{Q}{2\pi} \int ds \; k\Phi - \frac{Q}{4\pi} \int d^2z \sqrt{g} R^{(2)} \Phi \ . \tag{2.2}$$

In (2.2), ds is the line element of the boundary, k is the extrinsic curvature of the boundary, and $R^{(2)}$ is the curvature of the world sheet. Since we are working in the conformal gauge, the world sheet metric g_{ab} is just δ_{ab}, up to some conformal factor. The general tree-level world sheet of the open string may be mapped conformally to the half-sphere, which we represent as the upper half complex plane, and then $g_{ab} = \delta_{ab}$. We may locate the curvature at infinity, so that $R^{(2)} = 4\pi\delta(\infty)$. This curvature then acts as a background charge, which we will have to saturate in correlation functions. The action for the field Φ describes, in the conformal gauge, a ghost system equivalent to tensor fields $b_{z\cdots z}$ and $c^{z\cdots z}$ of type λ and $1-\lambda$. The curvature terms are weighted by $Q=\epsilon(1-2\lambda)$. We also allowed different statistics of the ghost systems, namely fermionic b, c for $\epsilon=+1$ and bosonic character for $\epsilon=-1$.

For the closed string we choose

$$S_{x,closed}=\frac{1}{8\pi}\int d^2z\sqrt{g}g^{ab}\partial_a X^\mu\partial_b X^\nu\eta_{\mu\nu} \tag{2.3}$$

and

$$S_{\phi,closed}=-\frac{\epsilon}{8\pi}\int d^2z\sqrt{g}g^{ab}\partial_a\Phi\partial_b\Phi-\frac{Q}{4\pi}\int dsk\Phi-\frac{Q}{8\pi}\int d^2z\sqrt{g}R^{(2)}\Phi\ . \tag{2.4}$$

Here the tree level world sheet is taken to be the complex plane, with $R^{(2)}=8\pi\delta(\infty)$.

If we keep the ghosts in their original fermionic form, the ghost action looks the same for the open and closed string:

$$S_{bc}=\frac{1}{2\pi}\int d^2z\sqrt{g}g^{ab}b_{ca}\partial_b c^c\ . \tag{2.5}$$

In the conformal gauge this action also describes ghost systems with arbitrary conformal dimension λ. However, the zero mode structure will of course be different for each case. For the usual reparametrization ghosts of the bosonic string $\lambda=2$, and this action is equivalent to (2.2) or (2.4) with $Q=-3$ and $\epsilon=+1$.

We now display the two point functions derived from the above actions. For the open string

$$\begin{aligned}\langle X^\mu(z,\bar z)X^\nu(w,\bar w)\rangle&=-\frac{1}{4}\eta^{\mu\nu}\ln|z-w|^2|z-\bar w|^2\\ \langle\Phi(z,\bar z)\Phi(w,\bar w)\rangle&=\frac{\epsilon}{4}\ln|z-w|^2|z-\bar w|^2\end{aligned} \tag{2.6}$$

and if we decompose $X^\mu(z,\overline{z}) = \frac{1}{2}(x^\mu(z) + x^\mu(\overline{z}))$ and similarly Φ, then

$$\begin{aligned} \langle x^\mu(z) x^\nu(w)\rangle &= -\eta^{\mu\nu}\ln(z-w) \\ \langle \phi(z)\phi(w)\rangle &= \epsilon\ln(z-w) \ . \end{aligned} \tag{2.7}$$

For the closed string

$$\begin{aligned} \langle X^\mu(z,\overline{z}) X^\nu(w,\overline{w})\rangle &= -\eta^{\mu\nu}\ln|z-w|^2 \\ \langle \Phi(z,\overline{z})\Phi(w,\overline{w})\rangle &= \epsilon\ln|z-w|^2 \end{aligned} \tag{2.8}$$

and we get, with $X^\mu(z,\overline{z}) = x(z) + \tilde{x}(\overline{z})$ and similarly for Φ :

$$\langle x^\mu(z) x^\nu(w)\rangle = \langle \tilde{x}^\mu(z)\tilde{x}^\nu(w)\rangle = -\eta^{\mu\nu}\ln(z-w)$$

$$\langle \phi(z)\phi(w)\rangle = \left\langle \tilde{\phi}(z)\tilde{\phi}(w)\right\rangle = \epsilon\ln(z-w) \ . \tag{2.9}$$

In this paper, we will study correlators on world-sheets with the topology of a plane or sphere; then it will always suffice to study the analytic correlators, since the two point functions satisfying the appropriate reality conditions are generated by the decomposition of $X(z,\overline{z})$ in a canonical way. Let us introduce a shorthand notation for this process and define the analytic averages

$$\ln(z-w)_{\text{a.a.}} \ \equiv \ \frac{1}{4}\Big(\ln(z-w) + \ln(\overline{z}-w) + \ln(z-\overline{w}) + \ln(\overline{z}-\overline{w})\Big) \tag{2.10}$$

for the open string and

$$\ln(z-w)_{\text{a.a.}} \ \equiv \ \ln(z-w) + \ln(\overline{z}-\overline{w}) \tag{2.11}$$

for the closed string.

The fermionic ghosts have, for both the open and closed case:

$$\langle c(z)b(w)\rangle = \frac{1}{z-w} \tag{2.12}$$

This implies the correspondence $c(z) = e^{\phi(z)}$, $b(z) = e^{-\phi(z)}$ for the reparametrization ghosts. The correlators for conformal fields can be translated into commutators of their Fourier modes. Here we write down only the ones for the field ϕ, since all the others were already given in I. If we define

$$\epsilon\partial_z\phi(z) = j(z) = \sum_{n=-\infty}^{\infty} j_n z^{-n-1} , \tag{2.13}$$

one finds

$$[j_n, j_m] = \epsilon n \delta_{n,-m} . \tag{2.14}$$

The field $j(z)$ is the ghost current. It provides the link between the fermionic and the bosonized formalism:

$$j(z) = - : b(z)c(z) := \lim_{z\to w}[-b(z)c(w) + \frac{1}{z-w}] = \epsilon\partial_z\phi . \tag{2.15}$$

2.2. Finite conformal transformations

Even though classically $j(z)$ is a primary conformal field of weight 1, normal ordering induces an anomalous affine term in the transformation law:

$$f[j(z)] = f'(z)j(f(z)) + \frac{Q}{2}\partial_z \ln(f'(z)) \tag{2.16}$$

In the ϕ-vertex this will be reflected in extra terms with coefficients Q or Q^2 when we compare it to the vertex for the X-coordinates. The transformation law (2.16) is of course precisely what is needed to leave the actions (2.2) and (2.4) invariant, if we transform the metric (and the curvatures) at the same time.

An affine transformation rule also appears for the energy momentum tensor. The coefficient in front of the inhomogeneous term is the well known conformal anomaly. For example, if we define

$$
\begin{aligned}
T^{x}_{zz}(z) &= \frac{1}{2} : \partial_z x^{\mu}(z) \partial_z x_{\mu}(z) : \\
T^{bc}_{zz}(z) &= -\lambda : b(z)\partial_z c(z) : + (1-\lambda) : \partial_z b(z) c(z) := \\
T^{\phi}_{zz}(z) &= \frac{\epsilon}{2}[: j(z)j(z) : - Q\partial_z j(z)] \ ,
\end{aligned}
\tag{2.17}
$$

then

$$
f[T^{A}_{zz}(z)] = [f'(z)]^2 T^{A}_{zz}(f(z)) + \frac{c^A}{12}\{f,z\}
\tag{2.18}
$$

where

$$
\{f,z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}(\frac{f''(z)}{f'(z)})^2
\tag{2.19}
$$

is the Schwartzian derivative and the coefficients of the conformal anomaly are, in D-dimensional spacetime,

$$
\begin{aligned}
c^{x} &= D \\
c^{bc} = c^{\phi} &= 1 - 3\epsilon Q^2
\end{aligned}
\tag{2.20}
$$

Note that for $Q = -3$ and $D = 26$ the total conformal anomaly $c^X + c^{bc}$ vanishes and $T_{zz} = T^{X}_{zz} + T^{bc}_{zz}$ transforms as a true primary conformal field of weight 2.

The Fourier components of the energy momentum tensor L_n generate infinitesimal conformal transformations. A finite transformation then has the form

$$
f[\Psi(z)] = [f'(z)]^d \Psi(z) = U_f \Psi(z) U_f^{-1}
\tag{2.21}
$$

where

$$
U_f = \exp(v_{-n} L_n) \quad .
\tag{2.22}
$$

For a scalar field Ψ, i.e. for $d = 0$, U_f has the representation $U_f = \exp(v(z)\partial_z)$, with $v(z) = v_n z^{-n-1}$. The vector field $v(z)$ generates the transformation f, and explicitly the two are related by the differential equation

$$v(z)\partial_z f(z) = v(f(z)) \quad . \tag{2.23}$$

Sometimes it is simpler to replace $v(z)$ by $tv(z)$, and derive a differential equation with respect to the parameter t. If we set

$$f_t(z) = e^{tv(z)\partial_z} z \quad , \tag{2.24}$$

then it is easy to show that $f_t(z)$ satisfies

$$\partial_t f_t(z) = v(f_t(z)) \quad ; f_0(z) = z \quad . \tag{2.25}$$

Let us integrate this equation for $v(z) = z^{k+1}, v_{-k} = 1$. We obtain the solutions

$$\begin{aligned} k = 0: & \qquad f_t(z) = e^t z \\ k \neq 0: & \qquad f_t(z) = \frac{z}{(1 - tkz^k)^{1/k}} \quad . \end{aligned} \tag{2.26}$$

Alternatively, we may develop $f_t(z)$ in a series around $t = 0$. Then we obtain

$$f_t(z) = z + tv(z) + \frac{t^2}{2} v(z)v'(z) + \frac{t^3}{3!} \left(v(z)(v'(z))^2 + v^2(z)v''(z) \right) + O(t^4) \ . \tag{2.27}$$

These results are useful because they allow us to check all our general formulae about gluing explicitly using a perturbation expansion in small t.

A conformal transformation has a simple form when it acts on primary fields, and since we defined operators as contour intergrals of such fields, the transformed modes have the form

$$f[\Psi_n] = \oint \frac{dz}{2\pi i} z^{n+d-1} f[\Psi(z)] \quad . \tag{2.28}$$

The transform of the state $\langle 0|$ is of course $\langle 0| U_f$, but this formula is somewhat awkward, since the exponent of U_f contains creation as well as annihilation operators. It turns out that one can explicitly write down a normal ordered expression.

We will discuss here only the simplest case in detail, namely the nonzero modes of the field X. The arguments easily generalize to all the fields we consider. Let us now show that

$$\langle f| \equiv \langle 0| U_f = \langle 0| \exp\Big[-\frac{1}{2} a_n N^f_{nm} a_m\Big] \tag{2.29}$$

where the coefficients N^f_{nm} are precisely the moments of the mapped Green function $\langle \partial_z x \partial_w x\rangle$—

$$N^f_{nm} = \frac{1}{n} \oint \frac{dz}{2\pi i} z^{-n} \left(f'(z)\right) \frac{1}{m} \oint \frac{dw}{2\pi i} w^{-m} \left(f'(w)\right) \frac{1}{(f(z) - f(w))^2} \ , \tag{2.30}$$

—which appeared in the mode representation of the vertex in I. We will assume that $f(z)$ is analytic in a neighbourhood of $z = 0$. First notice that, for all n, m , even at negative values,

$$\langle f|\, a_{-n} a_{-m} \cdots |0\rangle = \langle f[a_{-n}] f[a_{-m}] \cdots \rangle \ . \tag{2.31}$$

Therefore, $\langle f|\, f^{-1}[a_{-n}] = 0$ for all $n > 0$. But by (2.21) this means $\langle f|\, U_f^{-1} a_{-n} = 0$, and since the $SL(2)$ invariant vacuum $\langle 0|$ is uniquely determined by $\langle 0|\, a_{-n} = 0$ for all $n > 0$ and $\langle 0|0\rangle = 1$, we conclude $\langle f| = \langle 0|\, U_f$. Note that $\langle f| = \langle 0|$ if and only if $f(z)$ is a $SL(2, C)$ mapping. (2.31) is then derived as follows: by $SL(2)$ invariance we may choose $f(0) = 0$ and hence

$$U_f = \exp\left(\sum_{n\geq 2} v_{-n} L_n\right) \ . \tag{2.32}$$

Then of course $U_f^{-1}\, |0\rangle = 0$ and

$$\langle 0|\, U_f a_{-n} a_{-m} \cdots |0\rangle = \langle 0|\, U_f a_{-n} U_f^{-1} U_f a_{-m} U_f^{-1} \cdots |0\rangle = \langle f[a_{-n}] f[a_{-m}] \cdots \rangle \tag{2.33}$$

by (2.21) and (2.28). Using similar methods one can establish

$$|I \circ f \circ I\rangle_A \equiv U^{-1}_{I\circ f\circ I}\, |0\rangle_A = \exp\left(-\frac{1}{2} a_{-n} (-)^n N^f_{nm} (-)^m a_{-m}\right) |0\rangle_A = \langle f|_B\, |I_{AB}\rangle \ , \tag{2.34}$$

where I is the inversion $Iz = -1/z$ and $|I_{AB}\rangle$ is the BPZ inner product, discussed in I. With these definitions,

$$U^{-1}_{I\circ f\circ I} = \exp\left(\sum_{n\geq 2} v_{-n}(-)^n L_{-n}\right) \quad . \tag{2.35}$$

In the course of gluing we will encounter expressions of the form

$$\langle f_{1,t}|\, \Psi(z)\, |I\circ f_{2,s}\circ I\rangle \ , \tag{2.36}$$

where $\Psi(z)$ is some primary conformal field located at z or a product of the latter at different positions on the complex plane. Let us manipulate this expression into the form

$$\langle g\left[\Psi(z)\right]\rangle \qquad , \tag{2.37}$$

for some function $g(z)$. First we write (2.36) as

$$\langle 0| \quad f_{1,t}\left[\Psi(z)\right] \quad U_1 U_2^{-1}\, |0\rangle \ . \tag{2.38}$$

We then note that U_1 contains only L_n with $n \geq 2$, whereas U_2^{-1} is an exponential of L_{-k} with $k \geq 2$. There exists a theorem[7] in the theory of Lie groups that assures us that for sufficiently small s and t the product of the two may be written as

$$\begin{aligned}\exp\left(t\sum_{n\geq 2} v^{(1)}_{-n} L_n\right) \exp\left(s\sum_{m\geq 2} v^{(2)}_{-m}(-)^m L_{-m}\right) = \\ \exp\left(\sum_{n\geq 2} v^{(3)}_{-n}(-)^n L_{-n}\right) \exp\left(\sum_{n\geq -1} v^{(4)}_{-m} L_m\right)\end{aligned} \tag{2.39}$$

for some vector fields $v^{(3)}(z)$ and $v^{(4)}(z)$. In general, the right hand side of this equation is multiplied by a constant, namely $\langle 0|\, U_1 U_2^{-1}\, |0\rangle$. However, if the conformal anomaly is zero, $\langle 0| L_n |0\rangle = 0$ for all n, and therefore this constant is in fact 1. If we insert (2.39) into (2.38), the exponential containing $v^{(4)}(z)$ reduces to 1, since all the Virasoro operators in the exponent annihilate the vacuum $|0\rangle$. Applying the same reasoning once more we now write (2.38) as

$$\langle 0| U_{I\circ f_3 \circ I} \quad f[\Psi(z)] \quad U^{-1}_{I\circ f_3\circ I} |0\rangle \tag{2.40}$$

and identify f_3 with $I \circ f^{-1}_{1,t} \circ g \circ I$ to obtain (2.37). We may expand (2.36) to obtain $g(z)$ as a series in s and t:

$$\begin{aligned} g(z) = z + tv^{(1)}(z) + sIv^{(2)}(z) + \frac{1}{2}t^2 v^{(1)}(z)\partial_z v^{(1)}(z) + \frac{1}{2}s^2 Iv^{(2)}(z)\partial_z Iv^{(2)}(z) \\ + stv^{(1)}(z)\partial_z Iv^{(2)}(z)\bigg|_{\substack{z^{p+1} \\ p\geq 2}} + Iv^{(2)}(z)\partial_z v^{(1)}(z)\bigg|_{\substack{z^{p+1} \\ p\leq -2}} + O(s^3, s^2 t, st^2, t^3) \ . \end{aligned} \tag{2.41}$$

$g(z)$ is not analytic around either 0 or ∞. Rather, it has a convergent series expansion in powers of z only inside an annulus with finite radii. The outer radius is the radius of convergence of $f_1(z)$, while the inner one is the inverse of the radius of convergence of $f_2(z)$.

3. Gluing Coordinate Operators

We now begin the proper subject matter of this paper, the proof that operator contractions of string field theory vertices act precisely to sew together regions of Riemann surface to form the conformal field theory on the world-sheet. Specifically, we will prove the following result, which we refer to as the Generalized Gluing and Resmoothing Theorem (GGRT): Let $\langle V_{\{Ai\}C}|$ be a BRST-invariant multi-string vertex of the form that we have presented in I. The Ai denote strings and, for each, a state in the Hilbert space of that string, defined by a boundary condition on a unit circle. C denotes one additional string and an additional state; we will act on this state with the Hilbert space contraction. The vertex function is defined as the functional integral over the complex plane subject to boundary conditions corresponding to the images of the boundary conditions which define each string state for Ai, C under a corresponding conformal transformation h_{Ai}, h_C. This vertex can be written in terms of a conformal field theory vacuum expectation value as

$$\langle V_{\{Ai\}C}| \prod_i |A_i\rangle \otimes |C\rangle \;=\; \left\langle \prod_i \left(h_{A_i}\left[\mathcal{O}_{A_i}\right]\right) h_C\left[\mathcal{O}_C\right] \right\rangle \,. \tag{3.1}$$

The physical interpretation of this expression, and its connection to more familiar formulae for the multi-string vertex, has been presented in I. Let $\langle V_{\{Bj\}D}|$ be defined similarly. Let $|\mathcal{I}_{CD}\rangle$ be the BPZ inner product for the Hilbert space of the string conformal field theory; this object is written explicitly in I. Our aim is to compute the fused vertex

$$\langle V_{\{Ai\}\{Bj\}}| \;=\; \langle V_{\{Ai\}C}|\,\langle V_{\{Bj\}D}|\;|\mathcal{I}_{CD}\rangle \tag{3.2}$$

Intuitively, this object should be given as a conformal field theory matrix element on the Riemann surface formed by cutting out the images of the unit circles defining C and D and then gluing the resulting pieces together, as suggested in Fig. 1.

A more precise statement of the construction is shown in Fig. 2. We may define the gluing procedure more carefully by mapping both holes to unit circles, inverting the Bj plane in its unit circle by the mapping $Iz = -1/z$, and then gluing the pieces together. If h_C and h_D are not transformations in $SL(2, C)$ neither they nor their inverses will map the exterior of the hole to the exterior of a unit circle in a 1-to-1 manner. Thus, the glued surface will possess branch-cut singularities. The Riemann surface, however, still has the topology of a plane. Thus, there exists a mapping g which carries this covering surface into a plane, smoothing out the branch cuts. The GGRT states that the contraction (3.2) is given by mapping all of the string states Ai, Bj to this smoothed surface in the natural way:

$$\langle V_{\{Ai\}\{Bj\}}| \prod_i |Ai\rangle \otimes \prod_j |Bj\rangle = \left\langle \prod_i \left(\hat{h}_{A_i}\left[\mathcal{O}_{Ai}\right]\right) \prod_j \left(\hat{h}_{Bj}\left[\mathcal{O}_{Bj}\right]\right) \right\rangle , \quad (3.3)$$

where

$$\begin{aligned} \hat{h}_{A_i} &= g \circ h_C^{-1} \circ h_{A_i} \\ \hat{h}_{B_j} &= g \circ I \circ h_D^{-1} \circ h_{B_j} . \end{aligned} \quad (3.4)$$

To prove this identity, we must compute the Hilbert space inner product indicated in (3.2). In Section 7 of I, we carried out this computation for the special case in which h_C and h_D were both $SL(2, C)$ transformations. That assumption allowed us some powerful simplifications which enabled us to perform the entire analysis explicitly. We now confront the general case. Here we must deal with two new complications. First, since the mappings h_C and h_D will, in general, have singularities outside the unit circle, the property that the diagonal Neumann coefficients vanish (eqs. (7.8), (7.22) of I) no longer apply. In addition, we lose the special simplifications that we used in the off-diagonal terms (eqs. (7.7), (7.20), (7.23) of I). To carry through our analysis without these simplifications, we will need to work at a higher level of abstraction, while still keeping the results of the previous calculation as a guide.

As in our previous discussion, it is easiest to begin by analyzing the Neumann coefficients of the glued vertex multiplying pairs of coordinate mode operators a_n. To compute these coefficients, we need to work out the following matrix element:

$$\langle 0|_C \otimes \langle 0|_D \, \exp\left(-\frac{1}{2} a_n^C N^{CC}_{n\,m} a_m^C \; - \; a_n^{A_i} N^{A_i C}_{n\;\;m} a_m^C\right)$$

$$\cdot \exp\left(-\frac{1}{2} a_n^D N^{DD}_{n\,m} a_m^D \; - \; a_n^{B_j} N^{B_j D}_{n\;\;m} a_m^D\right)$$

$$\cdot \exp\left(a_{-n}^C \frac{(-1)^n}{n} a_{-n}^D \right) \; |0\rangle_C \otimes |0\rangle_D \; . \tag{3.5}$$

(In this equation, summations over i and j and mode numbers n and m should be understood.) Using the commutation relations of the mode operators, eq. (2.14) of I, we can interpret the last exponential factor in (3.5) as yielding a contraction whose value is

$$\left\langle a_n^C \; a_m^D \right\rangle \; \equiv \; E_{nm} \; = \; (-1)^n n \delta_{nm} \; . \tag{3.6}$$

This contraction can be applied to the two exponentials arising from the vertices, to reduce the matrix element (3.5) to the form

$$\begin{aligned}
\exp\Bigg(& a_n^{A_i} \left[N^{A_i C} E (1 - N^{DD} E N^{CC} E)^{-1} N^{D B_j} \right]_{nm} a_m^{B_j} \\
& - \frac{1}{2} a_n^{A_i} \left[N^{A_i C} E (1 - N^{DD} E N^{CC} E)^{-1} N^{DD} E N^{C A_j} \right]_{nm} a_m^{A_j} \\
& - \frac{1}{2} a_n^{B_j} \left[N^{B_j D} E N^{CC} E (1 - N^{DD} E N^{CC} E)^{-1} N^{D B_k} \right]_{nm} a_m^{B_k} \Bigg) \\
& \cdot \mathbf{D_a} \; .
\end{aligned} \tag{3.7}$$

The final factor $\mathbf{D_a}$ is a c-number expressing the sum of all contractions which do not link to external operators $a_n^{A_i}$ or $a_n^{B_j}$:

$$
\begin{aligned}
\mathbf{D_a} &= \exp\Big(-\frac{d}{2}\mathrm{tr}\,\log(1-N^{DD}EN^{CC}E)\Big) \\
&= \Big[\det\big(1-N^{DD}EN^{CC}E\big)\Big]^{-d/2} .
\end{aligned}
\tag{3.8}
$$

$d = 26$ is the dimension of space-time. Let us store this factor away for the moment and return to deal with it in Section 5.3.

Matrix products of Neumann coefficients of the sort displayed in (3.7) are a familiar feature of detailed analyses of string field theory vertices, such as Cremmer and Gervais[8] and Green and Schwarz[9] have given for the light-cone gauge. In this literature, however, their appearance, rearrangement, and disappearance always seems a bit mysterious. Since we need to work with these expressions in their full generality, it will be important for us to understand them physically. In fact, the main idea of our proof of the GGRT is to endow these objects with a clear physical interpretation—as the Fourier coefficients of Green's functions on the glued world-sheet.

Of the various terms in (3.7), the most accessible is the new quadratic term which joins $a_n^{A_i}$ to $a_m^{B_j}$. By extracting from the Neumann coefficients the Fourier transform defined in eq. (7.11) of I, we can write this term in the form

$$
a_n^{A_i}\ a_m^{B_j}\ \mathcal{F}^{A_i}_{n,z}\,\mathcal{F}^{B_j}_{m,w}\Big[\mathcal{G}_{CD}(z,w)\Big] , \tag{3.9}
$$

where

$$
\mathcal{F}^{I}_{n,z}\omega_z(z) = \oint \frac{dz}{2\pi i}\, z^{-k} h_I'(z)\omega_z(h_I(z)) \tag{3.10}
$$

for any one-form $\omega_z(z)$ and

$$
\begin{aligned}
\mathcal{G}_{CD}(z,w) &= \mathcal{F}^{C}_{k,u}\Big[\frac{1}{(z-u)^2}\Big]\ \Big(E(1-N^{DD}EN^{CC}E)^{-1}\Big)_{k\ell} \\
&\qquad \cdot\ \mathcal{F}^{D}_{\ell,v}\Big[\frac{1}{(v-w)^2}\Big] .
\end{aligned}
\tag{3.11}
$$

The complete term quadratic in a^{A_i} operators is obtained by combining the second line of (3.7) with the quadratic term present from the beginning in the vertex. This gives as the full expression

$$a_n^{A_i}\ a_m^{A_j}\ \mathcal{F}_{n,z}^{A_i}\, \mathcal{F}_{m,y}^{A_j} \left[\mathcal{G}_{CC}(z,y)\right] , \tag{3.12}$$

where

$$\begin{aligned} \mathcal{G}_{CC}(z,y) \ &= \ \frac{1}{(z-y)^2} + \ \mathcal{F}_{k,u}^{C}\left[\frac{1}{(z-u)^2}\right] \\ &\cdot \ \left(E(1-N^{DD}EN^{CC}E)^{-1}N^{DD}E\right)_{k\ell}\ \mathcal{F}_{\ell,v}^{C}\left[\frac{1}{(v-y)^2}\right] . \end{aligned} \tag{3.13}$$

The term quadratic in a^{Bj} operators has a similar form.

The functions $\mathcal{G}_{CD}(z,w)$ and $\mathcal{G}_{CC}(z,y)$ have the form of Green's functions on the glued surface. A few properties of these functions are obvious from the definitions. Each function has a domain of analyticity on its respective planes. These domains are apparent in Fig. 2. $\mathcal{G}_{CD}(z,w)$ is analytic in z for z outside the image of the state C on the A plane and analytic in w for w outside the image of D on the B plane. $\mathcal{G}_{CC}(z,y)$ is analytic in each of its arguments outside the image of C on the A plane, except for the double pole at $z=y$.

Now we can make a crucial observation: In our construction, the image of the boundary of C on the A plane is mapped back to the unit circle in the F plane and glued to the image in that plane of the boundary of D. The functions $\mathcal{G}_{CC}$ and $\mathcal{G}_{CD}$, carried by conformal mapping onto the F plane, define functions which are analytic and singled-valued on the Riemann surface which covers this plane, respectively, outside and inside of the unit circle, except for the double pole in $\mathcal{G}_{CC}$. Since these two functions are not dissimilar, it is plausible that $\mathcal{G}_{CD}(z,w)$ could in fact be the analytic continuation of the function $\mathcal{G}_{CC}(z,y)$ from the exterior into the interior of the unit circle in the F plane. If this were true, the composite function would be a single-valued function on the covering surface of

F. The image of this function on the G plane of Fig. 2 would be a single-valued, meromorphic function on this whole plane which falls off at infinity; further, this function would be analytic except for one double pole. There is only one such function,

$$\mathcal{G}_G(p,q) = \frac{1}{(p-q)^2}, \tag{3.14}$$

where p is the image of z and q is the image of y and w on the G plane. Thus, if we can show that $\mathcal{G}_{CD}(z,w)$ is indeed the continuation of $\mathcal{G}_{CC}(z,y)$, we will have characterized both objects completely in terms of the smoothing transformation $g(z)$.

To prove that $\mathcal{G}_{CC}$ and $\mathcal{G}_{CD}$ are related by analytic continuation, we need only prove that they agree at all points on the unit circle of the F plane. To make this requirement precise, we must explain how the $\mathcal{G}$s transform as we move from one plane to another. Eq. (3.12), which gives the new Neumann coefficients between Ai operators, expresses these coefficients as the Fourier transform of $\mathcal{G}_{CC}$ using the specific transform $\mathcal{F}^{Ai}_{n,z}$ appropriate to a field of conformal dimension 1. More concretely, we expect that, if $\mathcal{G}_{CC}(z,w)$ can be identified with a Green's function, it will be precisely $\langle \partial_z x \, \partial_w x \rangle$ on the glued Riemann surface. Therefore, $\mathcal{G}_{CC}$ should transform on each variable as a field of dimension 1. Specifically, when we transform the point w on the A plane to $r = h_C^{-1}(w)$ on the F plane, the image of $\mathcal{G}_{CC}$ should be

$$\mathcal{G}_1(z,r) = h'_C(r)\, \mathcal{G}_{CC}(z, h_C(r)) \tag{3.15}$$

To transform $\mathcal{G}_{CD}(z,y)$, we move in two steps from y to $s = h_D^{-1}(y)$, and then to $t = Is = -1/h_D^{-1}(y)$. The image of $\mathcal{G}_{CD}$ is

$$\mathcal{G}_2(z,t) = -\Big(h'_D(s)/h_D^2(s)\Big)\, \mathcal{G}_{CD}(z, h_D(s)) \,. \tag{3.16}$$

By the statement that $\mathcal{G}_{CD}$ is the analytic continuation of $\mathcal{G}_{CC}$, we mean, more precisely, that

$$\mathcal{G}_1(z,r) \;=\; \mathcal{G}_2(z,r)\;. \tag{3.17}$$

on the covering surface of the F plane. This equation is equivalent to the indentification of all of the moments

$$\oint \frac{dr}{2\pi i}\, r^n\, \mathcal{G}_1(z,r) \;=\; \oint \frac{dt}{2\pi i}\, t^n\, \mathcal{G}_2(z,t)\;, \tag{3.18}$$

for $-\infty < n < \infty$.

Let us check (3.18) by inserting the expressions (3.11), (3.13), (3.15), (3.16). A first simplification is found by converting the t plane integral on the right-hand side of (3.18) to the s plane. We find that we must verify

$$\oint \frac{dr}{2\pi i}\, r^n\, h'_C(r)\mathcal{G}_{CC}(z,h_C(r)) \;=\; \oint \frac{ds}{2\pi i}\,(-s)^{-n}\, h'_D(s)\mathcal{G}_{CD}(z,h_D(s))\;; \tag{3.19}$$

that is,

$$(-n)\mathcal{F}^C_{(-n),u}\Big[\mathcal{G}_{CC}(z,u)\Big] \;=\; (-1)^n n\mathcal{F}^D_{n,u}\Big[\mathcal{G}_{CD}(z,u)\Big]\;. \tag{3.20}$$

Note that positive Fourier components of $\mathcal{G}_{CD}$ must be compared with negative Fourier components of $\mathcal{G}_{CC}$, and vice versa. This turns out to be just what we need. For $n > 0$, the Fourier components of $\mathcal{G}_{CD}$ can be simplified using the relation

$$(-1)^n n\mathcal{F}^D_{n,v}\mathcal{F}^D_{k,u}\Big[\frac{1}{(v-u)^2}\Big] \;=\; N^{DD}_{k\,n}\cdot(-1)^n \;=\; \big(N^{DD}E\big)_{kn}\;. \tag{3.21}$$

A similar simplification applies to the left-hand side of (3.20) for $n < 0$. To simplify the negative Fourier components, we need to evaluate

$$\begin{aligned}(-n)\mathcal{F}^C_{(-n),v}\mathcal{F}^C_{k,u}&\Big[\frac{1}{(v-u)^2}\Big]\\ &= \frac{1}{k}\oint \frac{dr}{2\pi i} r^n h'_C(r)\oint \frac{dw}{2\pi i} w^{-k} h'_C(w)\,\frac{1}{(h_C(w)-h_C(r))^2}\;.\end{aligned} \tag{3.22}$$

The ordering of the contours matters: The w contour is located inside the unit circle, but the r contour integral, generated by (3.18), is taken just on the unit circle. Since $h_C(r)$ is analytic inside the unit circle, we may contract the r contour onto the double pole. All dependence on h_C cancels in the residue, and (3.22) becomes

$$(-n)\mathcal{F}^C_{(-n),v}\mathcal{F}^C_{k,u}\left[\frac{1}{(v-u)^2}\right] = \frac{1}{k}\oint \frac{dw}{2\pi i}\, nw^{n-1-k} = \delta(n,k) \; . \tag{3.23}$$

This argument also applies for $n = 0$ and gives 0 in that case. To Fourier transform $\mathcal{G}_{\bar{C}C}$, we also need to compute

$$\mathcal{F}^C_{(-n),u}\left[\frac{1}{(z-u)^2}\right] , \tag{3.24}$$

with z located outside the image of the unit circle under h_C. In this case, we repeat the first half of the argument just given; since there is now no pole inside the r contour, we find 0.

The identities (3.21), (3.23) allow us to compare the two sides of (3.20). For $n > 0$, we find:

$$\begin{aligned} &\mathcal{F}^C_{k,u}\left[\frac{1}{(z-u)^2}\right] \cdot \left(E(1-N^{DD}EN^{CC}E)^{-1}N^{DD}E\right)_{k\ell} \cdot \delta(\ell,n) \\ &\quad = \mathcal{F}^C_{k,u}\left[\frac{1}{(z-u)^2}\right] \left(E(1-N^{DD}EN^{CC}E)^{-1}\right)_{k\ell} \cdot \left(N^{DD}E\right)_{\ell n} , \end{aligned} \tag{3.25}$$

which is true identically. For $n = -m < 0$, we find

$$\begin{aligned} &\mathcal{F}^C_{k,u}\frac{1}{(z-u)^2} \cdot \Big\{\delta(k,m) \\ &\qquad + \left(E(1-N^{DD}EN^{CC}E)^{-1}N^{DD}E\right)_{k\ell} \cdot \left(N^{CC}E\right)_{\ell m}\Big\} \\ &= \mathcal{F}^C_{k,u}\left[\frac{1}{(z-u)^2}\right] \left(E(1-N^{DD}EN^{CC}E)^{-1}\right)_{k\ell} \cdot \delta(\ell,m) , \end{aligned} \tag{3.26}$$

which is also an identity. By a remark in the previous paragraph, both sides give 0 for $n = 0$. We have now completely verified that $\mathcal{G}_{CC}$ and $\mathcal{G}_{CD}$ form a single meromorphic Green's function on the glued surface. The series of matrix products of Neumann coefficients then defines a multiple-reflection expansion of this Green's function.

We can now identify the image of the two $\mathcal{G}$s on the smoothed plane as being given by $\mathcal{G}_G(p,q)$ of eq. (3.14). Note that the transformation law for dimension 1 fields preserves the coefficient of the double pole: The singularity $(z-y)^{-2}$ on the A plane implies that we must also find $(p-q)^{-2}$ on the G plane. (3.14) is the only function with this singularity. Mapping back to the A and B planes, we find

$$\begin{aligned} \mathcal{G}_{CC}(z,y) &= \hat{h}'_C(z)\hat{h}'_C(y)\frac{1}{(\hat{h}_C(z)-\hat{h}_C(y))^2} , \\ \mathcal{G}_{CD}(z,w) &= \hat{h}'_C(z)\hat{h}'_D(y)\frac{1}{(\hat{h}_C(z)-\hat{h}_D(y))^2} , \end{aligned} \tag{3.27}$$

where

$$\hat{h}_C = g\circ h_C^{-1} , \qquad \hat{h}_D = g\circ I\circ h_D^{-1} . \tag{3.28}$$

Inserting these relations into (3.9) and (3.12) gives precisely the form required by the GGRT for the coordinate mode operator Neumann coefficients.

4. Resummation

In the previous chapter the main argument for the proof of the GGRT was laid out. We showed that the glued Green function has analytic properties that identify it uniquely. This will be a recurrent theme in the next sections. However, in order to prove in general that the objects we obtain by gluing vertices actually correspond to different representations of the same Green function we need sharpen our tools a little and develop some additional technology. We will explain the method for the nonzero modes of X, and then apply it to all the other sectors of the vertex.

4.1. Nonzero modes of X

Recall that the terms in the exponent of the glued vertex which are bilinear in a_n^I are given by

$$\begin{aligned}
&-\frac{1}{2}a_n^{A_i}\Big[N^{A_iA_j}+N^{A_iC}EN^{DD}E\big(1-N^{CC}EN^{DD}E\big)^{-1}N^{CA_j}\Big]_{nm}a_m^{A_j}\\
&-\frac{1}{2}a_n^{B_i}\Big[N^{B_iB_j}+N^{B_iD}EN^{CC}E\big(1-N^{DD}EN^{CC}E\big)^{-1}N^{DB_j}\Big]_{nm}a_m^{B_j}\\
&+a_n^{A_i}\Big[N^{A_iC}E\big(1-N^{DD}EN^{CC}E\big)^{-1}N^{DB_j}\Big]_{nm}a_m^{B_j}\ .
\end{aligned}\tag{4.1}$$

We would like to prove that this is nothing but

$$-\frac{1}{2}a_n^{A_i}\hat{N}_{nm}^{A_iA_j}a_m^{A_j}-\frac{1}{2}a_n^{B_i}\hat{N}_{nm}^{B_iB_j}a_m^{B_j}-a_n^{A_i}\hat{N}_{nm}^{A_iB_j}a_m^{B_j}\ ,\tag{4.2}$$

where

$$\hat{N}_{nm}^{IJ}=\frac{1}{n}\oint\frac{du}{2\pi i}u^{-n}\frac{1}{m}\oint\frac{dv}{2\pi i}v^{-m}\,\partial_u\partial_v\ln[\hat{h}_I(u)-\hat{h}_J(v)]\tag{4.3}$$

and

$$\begin{aligned}
\hat{h}_{A_i}&=g\circ h_C^{-1}\circ h_{A_i}\\
\hat{h}_{B_j}&=g\circ I\circ h_D^{-1}\circ h_{B_j}\ .
\end{aligned}\tag{4.4}$$

By a change of variables of the type

$$h_C^{-1} \circ h_{A_i}(u) = z \tag{4.5}$$

or

$$I \circ h_D^{-1} \circ h_{B_j}(v) = w \tag{4.6}$$

the Green functions appearing in (4.3) are all brought to the form

$$\partial_z \partial_w \ln(g(z) - g(w)) \tag{4.7}$$

Since the mappings $h_{C,D}(u)$ generally are branched, the above change of variables is not well defined for all z, w. However, the mappings are analytic and 1-1 in a neighbourhood of $u = 0$, and therefore the formulas (4.5) and (4.6) do make sense as long as $h_C(z)$ and $h_D \circ I(w)$ converge as power series. If we introduce the radii of convergence $R_{C,D}$ of $h_{C,D}$, we may write these conditions compactly as $|z| < R_C$; $|w| > R_D^{-1}$.

We now compare the Green functions obtained from the gluing procedure. After coordinate transformations like (4.5) ,(4.6) we obtain the following expressions:

$$\begin{aligned}
&\partial_z \partial_w \ln(h_C(z) - h_C(w)) + \\
&\frac{1}{n} \oint \frac{du}{2\pi i} u^{-n} \partial_z \partial_u \ln(h_C(z) - h_C(u)) \left[E N^{DD} E (1 - N^{CC} E N^{DD} E)^{-1} \right]_{nm} \\
&\qquad\qquad \frac{1}{m} \oint \frac{dv}{2\pi i} v^{-m} \partial_v \partial_w \ln(h_C(v) - h_C(w))
\end{aligned} \tag{4.8}$$

for $|z|,|w| < R_C$,

$$\partial_z\partial_w \ln(h_D\circ I(z) - h_D\circ I(w)) +$$

$$\frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_z\partial_u \ln(h_D\circ I(z) - h_D(u))\ \Big[EN^{CC}E\big(1-N^{DD}EN^{CC}E\big)^{-1}\Big]_{nm}$$

$$\frac{1}{m}\oint \frac{dv}{2\pi i}\, v^{-m}\partial_v\partial_w \ln(h_D(v) - h_D\circ I(w))\ , \tag{4.9}$$

for $|z|,|w| > R_D^{-1}$, and

$$-\frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_z\partial_u \ln(h_C(z) - h_C(u))\ \Big[E\big(1-N^{DD}EN^{CC}E\big)^{-1}\Big]_{nm}$$

$$\frac{1}{m}\oint \frac{dv}{2\pi i}\, v^{-m}\partial_v\partial_w \ln(h_D(v) - h_D\circ I(w))\ , \tag{4.10}$$

for $|z| < R_C$ and $|w| > R_D^{-1}$. Of course, we wish to prove that these functions are simply analytic continuations of one another, so that we can identify them with $\partial_z\partial_w \ln(g(z) - g(w))$ by the uniqueness of the Green function. For that purpose we will no show that, if $R_D^{-1} < |z|,|w| < R_C$, all three expressions are equal to

$$\partial_z\partial_w\Big[\ln(h_C(z) - h_C(w)) + \ln(h_D\circ R(z) - h_D\circ R(w)) - \ln(z-w)\Big]$$

$$+\frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_z\partial_u \ln\left(\frac{h_C(z)-h_C(u)}{z-u}\right)\ \Big[EN^{DD}E\big(1-N^{CC}EN^{DD}E\big)^{-1}\Big]_{nm}$$

$$\frac{1}{m}\oint \frac{dv}{2\pi i}\, v^{-m}\partial_v\partial_w \ln\left(\frac{h_C(v)-h_C(w)}{v-w}\right)$$

$$+\frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_z\partial_u \ln\left(\frac{h_D\circ R(z)-h_D(u)}{I(z)-u}\right)\Big[EN^{CC}E\Big]_{nk}$$

$$\Big[\big(1-N^{DD}EN^{CC}E\big)^{-1}\Big]_{km}\ \frac{1}{m}\oint \frac{dv}{2\pi i}\, v^{-m}\partial_v\partial_w \ln\left(\frac{h_D(v)-h_D\circ R(w)}{v-I(w)}\right)$$

$$
-\frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_z\partial_u \ln\left(\frac{h_D{\circ}R(z)-h_D(u)}{I(z)-u}\right)\ \left[E\left(1-N^{CC}EN^{DD}E\right)^{-1}\right]_{nm}
$$
$$
\frac{1}{m}\oint \frac{dv}{2\pi i}\, v^{-m}\partial_v\partial_w \ln\left(\frac{h_C(v)-h_C(w)}{v-w}\right)
$$

$$
-\frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_z\partial_u \ln\left(\frac{h_C(z)-h_C(u)}{z-u}\right)\ \left[E\left(1-N^{DD}EN^{CC}E\right)^{-1}\right]_{nm}
$$
$$
\frac{1}{m}\oint \frac{dv}{2\pi i}\, v^{-m}\partial_v\partial_w \ln\left(\frac{h_D(v)-h_D{\circ}R(w)}{v-I(w)}\right)\ . \tag{4.11}
$$

The calculation is straightforward. First, note that the function

$$
\oint \frac{du}{2\pi i}\, u^{-n}\partial_z\partial_u \ln(h_D(z)-h_D(u)) \tag{4.12}
$$

is not analytic across the integration contour, due to the singularity of the integrand in $z-u$. By adding and subtracting a term $\ln(z-u)$ this singularity may be isolated:

$$
\oint \frac{du}{2\pi i}\, u^{-n}\partial_z\partial_u \ln\left(\frac{h_D(z)-h_D(u)}{z-u}\right) + nz^{-n-1}\Theta(|z|>|u|)\ , \tag{4.13}
$$

where the Θ function in the last term is the Heaviside step function for the region of the complex plane outside the integration contour. Now we observe that

$$
N^{DD}_{n\,m} \ = \ \frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\, \frac{1}{m}\oint \frac{dw}{2\pi i}\, w^{-m}\, \partial_u\partial_w \ln\left(\frac{h_D(u)-h_D(w)}{u-w}\right)\ , \tag{4.14}
$$

i.e. that the integrand of the diagonal Neumann function coefficients may be written as an analytic function in $z-w$. When we replace (4.12) by (4.13) in

(4.9) , the extra terms can be summed. For example:

$$\sum_{n\geq 1} nz^{-n-1}\, n(-)^n\, N^{DD}_{n\,m} =$$

$$\sum_{n\geq 1} - \left(-\frac{1}{z}\right)^{n+1} \frac{\partial_u^{n-1}}{(n-1)!}\frac{1}{m}\oint \frac{dw}{2\pi i}\, w^{-m}\, \partial_u \partial_w \ln\left(\frac{h_D(u)-h_D(w)}{u-w}\right)\Bigg|_{u=0}$$

$$-\frac{1}{m}\oint \frac{dw}{2\pi i}\, w^{-m}\, \partial_z \partial_w \ln\left(\frac{h_D \circ I(z)-h_D(w)}{I(z)-w}\right)\ . \tag{4.15}$$

Of course, the conversion of contour integrals into derivatives is only permitted if the integrand is analytic. That is the reason why one has to rewrite N^{DD} in the form (4.14) . Applying this resummation formula and the identity

$$\left(1-N^{CC}EN^{DD}E\right)^{-1} \ = \ 1+N^{CC}EN^{DD}E\left(1-N^{CC}EN^{DD}E\right)^{-1} \tag{4.16}$$

repeatedly one finally arrives at (4.11). The conditions on the arguments of the Green function that we stated for this formula are precisely those that imply that the required extra terms in (4.13) are present and that the series they give rise to can be summed in the way shown.

Let us discuss briefly the relation between this calculation and that of the previous section. The method we employed here to demonstrate the equivalence of (4.8) , (4.9) and (4.10) , namely deriving a common form of those functions in the annular region $R_D^{-1} < |z|, |w| < R_C$, had as its counterpart the calculation of contour integrals. We chose $R_D^{-1} < 1 < R_C$ and took moments of the Green function on the unit circle $|z| = 1$ or $|w| = 1$. For the a_n^I oscillators the two methods are obviously equivalent. However, while in the nonzero mode sector taking contour integrals is a very natural thing to do, the zero modes are more conveniently dealt with by the resummation technique we just described. Let us now apply the new method and study the the terms in the exponent of the glued vertex which are bilinear in a_0^I.

4.2. ZERO MODES OF X

We will now discuss the following terms in the exponent of the glued vertex:

$$\frac{1}{2}\sum_{i,j} p_{A_i}\, p_{A_j} \left[\ln\left(h_{A_i}(0) - h_{A_j}(0)\right) + N_{0\ n}^{A_i C} M_{nm}^{DD} N_{m0}^{C\, A_j}\right]_{\text{a.a.}}$$

$$\frac{1}{2}\sum_{i,j} p_{B_i}\, p_{B_j} \left[\ln\left(h_{B_i}(0) - h_{B_j}(0)\right) + N_{0\ n}^{B_i D} M_{nm}^{CC} N_{m0}^{D\, B_j}\right]_{\text{a.a.}}$$

$$+\epsilon\sum_{i,j} p_{A_i}\, p_{B_j} \left[N_{0\ n}^{A_i C} M_{nm}^{DC} N_{m0}^{D\, B_j}\right]_{\text{a.a.}}\,, \tag{4.17}$$

where we have defined

$$\begin{aligned} M_{nm}^{CC} &= \left[E N^{CC} E\left(1 - N^{DD} E N^{CC} E\right)^{-1}\right]_{nm} \\ M_{nm}^{DD} &= \left[E N^{DD} E\left(1 - N^{CC} E N^{DD} E\right)^{-1}\right]_{nm} \\ M_{nm}^{CD} &= \left[E\left(1 - N^{CC} E N^{DD} E\right)^{-1}\right]_{nm} \\ M_{nm}^{DC} &= \left[E\left(1 - N^{DD} E N^{CC} E\right)^{-1}\right]_{nm} \end{aligned} \tag{4.18}$$

and used momentum conservation $\sum p_{A_i} + \sum p_{B_i} = 0$ to simplify the expressions. The absolute values appear in (4.17) by virtue of the decompositions that link (2.6) with (2.7) and (2.8) with (2.9). The Green functions that appear in (4.17) take, after the change of variables (4.5) and (4.6) , the form

$$\begin{aligned} \ln(h_C(z) - h_C(w)) \ + \ & \frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_u \ln(h_C(z) - h_C(u)) \\ & M_{nm}^{DD}\frac{1}{m}\oint \frac{dv}{2\pi i}\, v^{-m}\partial_v \ln(h_C(v) - h_C(w)) \end{aligned} \tag{4.19}$$

for $|z|, |w| < R_C$,

$$\ln(h_D \circ I(z) - h_D \circ I(w)) +$$

$$\frac{1}{n} \oint \frac{du}{2\pi i} u^{-n} \partial_u \ln(h_D \circ I(z) - h_D(u))$$

$$M^{CC}_{nm} \frac{1}{m} \oint \frac{dv}{2\pi i} v^{-m} \partial_v \ln(h_D(v) - h_D \circ I(w)) \tag{4.20}$$

for $|z|, |w| > R_D^{-1}$, and

$$-\frac{1}{n} \oint \frac{du}{2\pi i} u^{-n} \partial_u \ln(h_C(z) - h_C(u)) M^{DC}_{nm} \frac{1}{m} \oint \frac{dv}{2\pi i} v^{-m} \partial_v \ln(h_D(v) - h_D \circ I(w)) \ , \tag{4.21}$$

for $|z| < R_C$ and $|w| > R_D^{-1}$. These formulas differ from (4.8) , (4.9) and (4.10) only by the derivatives ∂_z, ∂_w. The reason is quite obvious: momentum conservation lets any possible integration constants disappear. It is now clear how we proceed. Resummation yields just (4.11) without ∂_z, ∂_w . We have therefore shown that the glued Neumann coefficients are given in terms of a function which in both arguments is analytic on the glued surface, except for a logarithmic singularity as the two arguments approach each other. We therefore identify it as the scalar Green function, which has the form $\ln(g(z) - g(w))$, up to functions depending on one of the arguments only. These do not appear in the vertex by virtue of momentum conservation, and hence (4.17) can be written as

$$\begin{aligned} &\sum_{i \neq j} \frac{1}{2} p_{A_i} \hat{N}^{A_i A_j}_{00} p_{A_j} + \sum_{i \neq j} \frac{1}{2} p_{B_i} \hat{N}^{B_i B_j}_{00} p_{B_j} \\ &+ \sum_{i,j} \frac{1}{2} p_{A_i} \hat{N}^{A_i B_j}_{00} p_{B_j} + \sum_{i,j} \frac{1}{2} p_{B_i} \hat{N}^{B_i A_j}_{00} p_{A_j} \ , \end{aligned} \tag{4.22}$$

where

$$\hat{N}^{IJ}_{00} \ = \ \ln \left(\hat{h}_I(0) - \hat{h}_J(0) \right)_{\text{a.a.}} \tag{4.23}$$

It is now clear that the above analysis may be taken almost verbatim and applied to the terms in the vertex linear in the zero and nonzero modes of X. The only term in the vertex fused vertex we have not yet addressed is $\mathbf{D_a}$. It will be discussed in Section 5.3, together with the corresponding factors from the ghost sector, to which we now turn.

5. Gluing Bosonized Ghosts

For the boson field Φ representing the ghost system, there is an anomalous momentum conservation law on the sphere due to the δ-function of curvature on the world sheet. This fact introduces various complications into the gluing procedure. Some of those are purely technical in nature and mainly lengthen the formulas the reader should now be familiar with from the previous sections. These results are described in section 5.2. More fundamentally important is the effect on the factors in the glued vertex which do not contain mode annihilation operators. In Section 5.3, we will show that the whole set of these factors—arising from the X and the Φ dynamics—combine and cancel.

5.1. VERTICES

Let us begin by describing the changes which occur in the definition of the vertex. We construct the vertex for the general bosonized ghost system in a way almost identical to the procedure for the X field. However, the bosonization rules imply the affine transformation law (2.16) for the ghost current, which is taken into account by a term in the exponent that is linear in the operators j_n. Also, the momentum eigenvalues are no longer continuous, but form a discrete set. The vertex therefore reads (for arbitrary Q):

$$\begin{aligned}\langle V_{1\cdots k}| &= \sum_{q_1,\cdots,q_k} \delta(q_1+\cdots+q_k+Q)\prod_{I=1}^{k}\langle -q_I - Q|_I \\ &\exp\Big(\sum_I \frac{1}{2}\epsilon\, q_I(Q+q_I)N_{00}^{II} + \sum_{I\neq J}\frac{1}{2}\epsilon\, q_I\, N_{00}^{IJ} q_J + \sum_I\sum_{n\geq 1}\frac{1}{2}\epsilon Q\, j_n^I K_n^I \\ &+ \sum_{I,J}\sum_{n\geq 1}\epsilon\, q_I\, N_{0n}^{IJ}\, j_n^J + \sum_{I,J}\sum_{n,m\geq 1}\frac{1}{2}\epsilon\, j_n^I\, N_{nm}^{IJ}\, j_m^J\Big)\ ,\end{aligned} \tag{5.1}$$

where the Neumann function coefficients N_{nm}^{IJ} are the same we defined previously for the X-part of the vertex and

$$K_n^I = \frac{1}{n}\oint \frac{dw}{2\pi i}\, w^{-n}\partial_w \ln(h_I'(w)) \tag{5.2}$$

The terms in the exponent containing the coefficient Q are a direct consequence of the transformation law (2.16) and the definition of the vertex (3.1). Using momentum conservation and $\langle -q-Q|\, j_0 = \langle -q-Q|\, q$ the exponential can be simplified to:

$$\exp\Bigl(\sum_{I,J}\sum_{n,m\geq 0}\frac{1}{2}\epsilon\, j_n^I\, \mathcal{N}_{nm}^{IJ}\, j_m^J\Bigr)\ , \tag{5.3}$$

with

$$\begin{aligned}
\mathcal{N}_{00}^{IJ} &= \begin{cases} 0 & I=J\ ,\\ -\frac{1}{2}\ln\Bigl(\partial_z\partial_w \ln[h_I(z)-h_J(w)]\Big|_{\substack{z=0\\ w=0}}\Bigr)_{\text{a.a.}} & I\neq J\ ,\end{cases}\\
\mathcal{N}_{0m}^{IJ} &= -\frac{1}{2m}\oint\frac{dw}{2\pi i}\, w^{-m}\,\partial_w \ln\Bigl(\partial_w\partial_z \ln[h_I(w)-h_J(z)]\Big|_{z=0}\Bigr)_{\text{a.a.}}\ ,\\
\mathcal{N}_{nm}^{IJ} &= \frac{1}{n}\oint\frac{dz}{2\pi i}\, z^{-n}\,\frac{1}{m}\oint\frac{dw}{2\pi i}\, w^{-m}\,\partial_z\partial_w \ln[h_I(z)-h_J(w)]\ .
\end{aligned} \tag{5.4}$$

The new Green function coefficients are related to the old ones by

$$\begin{aligned}
\mathcal{N}_{00}^{IJ} &= N_{00}^{IJ}-\frac{1}{2}N_{00}^{II}-\frac{1}{2}N_{00}^{JJ}\\
\mathcal{N}_{0m}^{IJ} &= N_{0m}^{IJ}-\frac{1}{2}K_m^J\\
\mathcal{N}_{nm}^{IJ} &= N_{nm}^{IJ}\quad .
\end{aligned} \tag{5.5}$$

They correspond to different contractions of operators which are equivalent to the usual ones inside correlation functions. For example, by momentum conservation the following equations hold:

$$\Bigl\langle f[j(z)]e^{q_1\phi(z_1)}\ldots e^{q_n\phi(z_n)}\Bigr\rangle$$

$$=\Bigl[\sum_i q_i\partial_z \ln(f(z)-z_i)\ +\ \frac{1}{2}Q\partial_z \ln f'(z)\Bigr]\Bigl\langle e^{q_1\phi(z_1)}\ldots e^{q_n\phi(z_n)}\Bigr\rangle$$

$$= \sum_i q_i \left[\partial_z \ln(f(z) - z_i) - \frac{1}{2}\partial_z \ln f'(z)\right] \left\langle e^{q_1\phi(z_1)} \ldots e^{q_n\phi(z_n)}\right\rangle$$

$$= \sum_i q_i(-)\frac{1}{2}\partial_z \ln\left(\partial_z\partial_{z_i} \ln[f(z) - z_i]\right) \left\langle e^{q_1\phi(z_1)} \ldots e^{q_n\phi(z_n)}\right\rangle \ . \tag{5.6}$$

In the last line we find the contractions whose moments are the coefficients $\mathcal{N}^{IJ}_{n0}$ in equation (5.5).

It is important to realize that the $\mathcal{N}$'s are based on a different Green function than the N's, since the transformation properties are affected. Under coordinate transformations

$$z \to f(z), \quad w \to g(w) \tag{5.7}$$

the functions change in the following way:

$$\ln\left(z - w\right) \to \ln\left(f(z) - g(w)\right) \tag{5.8}$$

$$-\frac{1}{2}\ln\left(\partial_z\partial_w \ln(z - w)\right) \to -\frac{1}{2}\ln\left(\partial_z\partial_w \ln(f(z) - g(w))\right) \ . \tag{5.9}$$

Let us rewrite the last line so that the affine transformation rule becomes explicit:

$$\ln\left(z - w\right) \to \ln\left(f(z) - g(w)\right) - \frac{1}{2}\ln f'(z) - \frac{1}{2}\ln g'(w) \ . \tag{5.10}$$

Whereas in section 4 we patched together the Green function according to (5.8), now we will have to use (5.10) when we compare different representations.

Writing the vertex in the fashion (5.3) eliminates all the explicit dependence on the background charge Q from the exponent. By this trick the X-exponent has been put on the same footing as the Φ-exponent. Upon replacement of ϵ with $-\eta^{\mu\nu}$ the expressions derived below will reduce to the formulas obtained in the previous sections for the case $Q = 0$. Of course, various terms will drop out as a consequence of proper (instead of anomalous) momentum conservation. Also note that $SL(2)$-invariance is now manifest, since the function $\partial_z\partial_w \ln[h_I(z) - h_J(w)]$

obviously does not change under $SL(2,C)$ transformations, *i.e.* under $h_{I,J} \to T \circ h_{I,J}$ with $T(z) \in SL(2,C)$.

As a simple example of such a vertex we construct the two point function

$$\langle \mathcal{I}_{AB}| = \sum_p \langle p|_A \otimes \langle -p-Q|_B \exp\Bigg(\sum_{n\geq 1} \frac{\epsilon}{n}(-)^{n+1} j_n^A j_n^B \Bigg) . \tag{5.11}$$

Its inverse is the inner product:

$$|\mathcal{I}_{BC}\rangle = \sum_q \exp\Bigg(\sum_{n\geq 1} \frac{\epsilon}{n}(-)^{n+1} j_{-n}^B j_{-n}^C \Bigg) \; |q\rangle_B \otimes |-q-Q\rangle_C . \tag{5.12}$$

For later convenience we also define the one point function

$$\langle f, -q-Q| \equiv \langle -q-Q| \exp \left[\frac{\epsilon}{2} \sum_{n\geq 0} j_n \mathcal{N}^{ff}_{nm} j_m \right] \tag{5.13}$$

and its "inversion"

$$\begin{aligned} |f,-q-Q\rangle &\equiv \langle f,-q-Q|_A \;\; |\mathcal{I}_{A.}\rangle \\ &= \exp \left[\frac{\epsilon}{2} \sum_{n\geq 0} j_n (-)^{n+1} \mathcal{N}^{ff}_{nm} (-)^{m+1} j_m \right] |-q-Q\rangle . \end{aligned} \tag{5.14}$$

We have now displayed all the ingredients needed to evaluate the fused vertex (3.2). Using coherent state techniques, in particular the identity

$$\mathbf{1} = \sum_p \prod_{n\geq 1} \left[\int \frac{d^2\alpha_n}{\pi} \exp(-|\alpha_n|^2) \exp(\frac{\epsilon}{\sqrt{n}} \alpha_n j_{-n}) \, |p\rangle \, \langle -p-Q| \exp(\frac{1}{\sqrt{n}} \alpha_n^\dagger j_n) \right] , \tag{5.15}$$

one performs the contractions and obtains

$$\begin{aligned} \langle V_{\{A_i\}\{B_j\}}| = \sum_{q_{A_i}, q_{B_j}} & \delta(\sum_i q_{A_i} + \sum_j q_{B_j} + Q) \prod_i \langle -q_{A_i} - Q|_{A_i} \prod_j \langle -q_{B_j} - Q|_{B_j} \\ & \mathbf{D_j} \cdot \exp(\text{ determinant term }) \cdot \exp(\text{ q-q term }) \\ & \cdot \exp(\text{ j-q term }) \cdot \exp(\text{ j-j term }) , \end{aligned} \tag{5.16}$$

where the various terms will be explained below, one at a time. We will show that

$\mathbf{D_j}$ and the exponential of the determinant term, multplied by the correponding terms arising in the gluing of X, give just the trivial factor 1. The cancellations that take place are somewhat delicate, and are treated in detail in section 5.3. The remaining terms then define a new vertex, based on the Green functions of the glued surface.

5.2. AFFINE GLUING

We will first describe the j-j term, i.e. the nonzero modes of ϕ. It was already noted in (5.5) that the Neumann coefficients coincide with those of the nonzero modes of X. That happens simply because the pair of derivatives $\partial_z\partial_w$ annihilates the affine terms that appeared in (5.10). The result of gluing is therefore almost identical to the analogous formula (4.1) for X, namely

$$\begin{aligned}
\text{j-j term} = & \frac{\epsilon}{2} j_n^{A_i} \Big[N^{A_iA_j} + N^{A_iC} E N^{DD} E \big(1 - N^{CC} E N^{DD} E\big)^{-1} N^{CA_j}\Big]_{nm} j_m^{A_j} \\
& + \frac{\epsilon}{2} j_n^{B_i} \Big[N^{B_iB_j} + N^{B_iD} E N^{CC} E \big(1 - N^{DD} E N^{CC} E\big)^{-1} N^{DB_j}\Big]_{nm} j_m^{B_j} \\
& - \epsilon j_n^{A_i} \Big[N^{A_iC} E \big(1 - N^{DD} E N^{CC} E\big)^{-1} N^{DB_j}\Big]_{nm} j_m^{B_j} \\
= & \ \frac{\epsilon}{2} j_n^{A_i} \hat{N}_{nm}^{A_iA_j} j_m^{A_j} + \frac{\epsilon}{2} j_n^{B_i} \hat{N}_{nm}^{B_iB_j} j_m^{B_j} + \epsilon j_n^{A_i} \hat{N}_{nm}^{A_iB_j} j_m^{B_j} \ .
\end{aligned} \tag{5.17}$$

The last equality was shown in sections 3 and 4, and the hatted Neumann coefficients are defined in (4.3) and (4.4).

The real challenge in the study of the bosonized ghosts lies in the zero mode sector. Two subtleties arise, one associated with the identification of the correct Green function, and the other with the cancellation of determinants. Both are of course closely related, but we can understand the solution of the first without addressing the second by studying the j-q term. Recall that now the Green function defining the coefficients $\mathcal{N}_{0n}^{IJ}$ transforms with an affine term. It it given by

$$-\frac{1}{2}\partial_w \ln\Big(\partial_w\partial_z \ln[h_I(w) - h_J(z)]\Big) \quad . \tag{5.18}$$

We may identify it uniquely by its affine transformation properties, *i.e.* its analytic structure for well separated arguments, and its short distance expansion, which we study in coordinates where $I = J$. Then (5.18) takes the form

$$\frac{1}{w-z} - \frac{1}{6}(w-z)\{h_I, w\} + O\left((w-z)^2\right) \ , \tag{5.19}$$

where $\{h_I, w\}$ is the Schwartzian derivative which already made its appearance in section 2.2. The important observations are that there is a pole in $w - z$ with residue 1 and *no* term of order $(w-z)^0$.

Now that we have characterized the affine Green function, let us examine its mutiple reflection expansion that we obtain from the gluing procedure. The contractions yield

$$\begin{aligned}
&\text{j-q terms} = \\
&\sum_{i,j} \epsilon\, j_n^{A_i} q_{A_j} \left(\mathcal{N}_{n\,0}^{A_i A_j} + \left[N^{A_i C} M^{DD}\right]_{nm} \mathcal{N}_{m0}^{C\,A_j} - \left[N^{A_i C} M^{DC}\right]_{nm} \mathcal{N}_{m0}^{D\,D} \right)_{\text{a.a.}} \\
&\sum_{i,j} \epsilon\, j_n^{A_i} q_{B_j} \left(\mathcal{N}_{n\,0}^{A_i C} + \left[N^{A_i C} M^{DD}\right]_{nm} \mathcal{N}_{m0}^{C\,C} - \left[N^{A_i C} M^{DC}\right]_{nm} \mathcal{N}_{m0}^{D\,B_j} \right)_{\text{a.a.}} \\
&\sum_{i,j} \epsilon\, j_n^{B_i} q_{A_j} \left(\mathcal{N}_{n\,0}^{B_i D} + \left[N^{B_i C} M^{CC}\right]_{nm} \mathcal{N}_{m0}^{D\,D} - \left[N^{B_i D} M^{CD}\right]_{nm} \mathcal{N}_{m0}^{C\,A_j} \right)_{\text{a.a.}} \\
&\sum_{i,j} \epsilon\, j_n^{B_i} q_{B_j} \left(\mathcal{N}_{n\,0}^{B_i B_j} + \left[N^{B_i D} M^{CC}\right]_{nm} \mathcal{N}_{m0}^{D\,B_j} - \left[N^{B_i D} M^{CD}\right]_{nm} \mathcal{N}_{m0}^{C\,C} \right)_{\text{a.a.}}
\end{aligned} \tag{5.20}$$

In the usual way we extract by a change of coordinates the Green functions corresponding to (4.8), (4.9), (4.10) and derive the function analogous to (4.11), thus establishing the analyticity of the affine Green function. We will not write down all of these formulas. The interested reader can easily derive them by taking a z-derivative of the formulas (5.32) to (5.35) below. We will, however, display one set, namely for $|z|, |w| < R_C$. The Green function then has the form

$$-\frac{1}{2}\partial_z \ln\left(\partial_z\partial_w \ln\left[h_C(z) - h_C(w)\right]\right)$$

$$-\ \frac{1}{2}\ H_n^C(z) M_{nm}^{DD} G_m^C(w) +\ \frac{1}{2}\ H_n^C(z) M_{nm}^{DC} G_m^D(\infty)\ , \tag{5.21}$$

where

$$H_n^C(z) = \frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_z\partial_u \ln[h_C(z) - h_C(u)] \tag{5.22}$$

$$H_n^D(z) = \frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_z\partial_u \ln[h_D \circ I(z) - h_D(u)] \tag{5.23}$$

and

$$G_n^C(z) = \frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_u \ln\left[\partial_z\partial_u \ln(h_C(z) - h_C(u))\right] \tag{5.24}$$

$$G_n^D(z) = \frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_u \ln\left[\partial_z\partial_u \ln(h_D \circ I(z) - h_D(u))\right]\ . \tag{5.25}$$

Let us examine this function for small $(z - w)$. The first term we already know to have a simple pole in $(z - w)$ with residue 1, and no term of order 1. The characterization of the affine Green function is complete if for $z = w$ the last two terms cancel, i.e. if

$$H_n^C(z) M_{nm}^{DD} G_m^C(z) = H_n^C(z) M_{nm}^{DC} G_m^D(\infty)\ . \tag{5.26}$$

We have checked this identity for mappings $h_{C,D}(z)$ close to the identity (or $SL(2)$ transforms of it). More precisely, we introduced parameters s and t , constructed the expansion (2.27) and verified (5.26) perturbatively for small s, t. The general argument follows from equation (2.37). The short distance expansion

$$-\frac{1}{2}\partial_z \ln\left(\partial_z\partial_w \ln[f(z)-f(w)]\right) = \frac{1}{z-w}-\frac{1}{6}(z-w)\{f,w\}+O(|z-w|^2) \quad (5.27)$$

holds for any function f which is analytic around z. We now observe that the contraction (5.21) is generated in the correlation function of conformal fields. For example, it appears in

$$\langle h_C, 0|\, j(z)\mathrm{e}^{q_1\phi(w_1)}\ldots\mathrm{e}^{q_n\phi(w_n)}\,|I\circ h_D\circ I, 0\rangle \quad (5.28)$$

with $\sum_i q_i = -Q$, which we know from the argument following (2.37) to be given by

$$\left\langle f[j(z)]f\left[\mathrm{e}^{q_1\phi(w_1)}\right]\cdots f\left[\mathrm{e}^{q_n\phi(w_n)}\right]\right\rangle \quad (5.29)$$

for some function $f(z)$. Hence (5.27) holds and we have characterized the affine Green function. We can now claim that in fact f is identical with the smoothing function g. Therefore (5.20) may be put into the form

$$\epsilon j_n^{A_i}\hat{\mathcal{N}}_{n0}^{A_iA_j}q_{A_j}+\epsilon j_n^{B_i}\hat{\mathcal{N}}_{n0}^{B_iB_j}q_{B_j}+\epsilon j_n^{B_i}\hat{\mathcal{N}}_{n0}^{B_iA_j}q_{A_j}+\epsilon j_n^{A_i}\hat{\mathcal{N}}_{n0}^{A_iB_j}q_{B_j}\,, \quad (5.30)$$

where

$$\hat{\mathcal{N}}_{n0}^{IJ} = \frac{1}{n}\oint\frac{dz}{2\pi i}\,z^{-n}\,\partial_z\ln\left(\partial_z\partial_w\ln[\hat{h}_I(z)-\hat{h}_J(w)]\Big|_{w=0}\right)_{\text{a.a.}} \quad (5.31)$$

is the Neumann function coefficient generated by the glued surface just as we expected. This concludes our dicussion of the j-q terms.

Now let us extend the analysis to the terms in the exponent of the fused vertex that are bilinear in zero modes. First we display the result of performing the contractions in (3.3). One obtains

$$\text{determinant term} + \text{q-q term} =$$

$$-\frac{\epsilon}{4}\sum_{i\neq j} q_{A_i}\; q_{A_j}\;\ln\left(\partial_z\partial_w \ln\left(h_{A_i}(z) - h_{A_j}(w)\right)\Big|_{\substack{z=0\\w=0}}\right)_{\text{a.a.}}$$

$$+\frac{\epsilon}{2}\sum_{i,j} q_{A_i}\; q_{A_j}\;\Big[\;\mathcal{N}^{A_i C}_{0\;n} M^{DD}_{nm}\mathcal{N}^{C\,A_j}_{m0} - \mathcal{N}^{A_i C}_{0\;n} M^{DC}_{nm}\mathcal{N}^{D\,D}_{m0}$$

$$-\mathcal{N}^{DD}_{0\;n} M^{CD}_{nm}\mathcal{N}^{C\,A_j}_{m0} + \mathcal{N}^{DD}_{0\;n} M^{CC}_{nm}\mathcal{N}^{D\,D}_{m0}\Big]_{\text{a.a.}}$$

$$-\frac{\epsilon}{4}\sum_{i\neq j} q_{B_i}\; q_{B_j}\;\ln\left(\partial_z\partial_w \ln\left(h_{B_i}(z) - h_{B_j}(w)\right)\Big|_{\substack{z=0\\w=0}}\right)_{\text{a.a.}}$$

$$+\frac{\epsilon}{2}\sum_{i,j} q_{B_i}\; q_{B_j}\;\Big[\;\mathcal{N}^{B_i D}_{0\;n} M^{CC}_{nm}\mathcal{N}^{D\,B_j}_{m0} - \mathcal{N}^{B_i D}_{0\;n} M^{CC}_{nm}\mathcal{N}^{C\,D}_{m0}$$

$$-\mathcal{N}^{DD}_{0\;n} M^{DC}_{nm}\mathcal{N}^{D\,B_j}_{m0} + \mathcal{N}^{DD}_{0\;n} M^{DD}_{nm}\mathcal{N}^{C\,D}_{m0}\Big]_{\text{a.a.}}$$

$$-\frac{\epsilon}{2}\sum_{i,j} q_{A_i}\; q_{B_j}\;\ln\left(\partial_z\partial_w \ln\left(h_{A_i}(z) - h_{C}(w)\right)\Big|_{\substack{z=0\\w=0}}\right)_{\text{a.a.}}$$

$$-\frac{\epsilon}{2}\sum_{i,j} q_{A_i}\; q_{B_j}\;\ln\left(\partial_z\partial_w \ln\left(h_{D}(z) - h_{B_j}(w)\right)\Big|_{\substack{z=0\\w=0}}\right)_{\text{a.a.}}$$

$$+\epsilon\sum_{i,j} q_{A_i}\; q_{B_j}\;\Big[-\mathcal{N}^{A_i C}_{0\;n} M^{DC}_{nm}\mathcal{N}^{D\,B_j}_{m0} + \mathcal{N}^{A_i C}_{0\;n} M^{DD}_{nm}\mathcal{N}^{C\,C}_{m0}$$

$$-\mathcal{N}^{CC}_{0\;n} M^{DC}_{nm}\mathcal{N}^{D\,D}_{m0} + \mathcal{N}^{DD}_{0\;n} M^{CC}_{nm}\mathcal{N}^{D\,B_j}_{m0}\Big]_{\text{a.a.}}\;. \tag{5.32}$$

These expressions bear a close resemblance to (5.20) . After all, the Green functions defining the coefficients differ only by a derivative and integration constants. Therefore we proceed in analogy to the discussion of the j-q terms. The affine Green function

$$-\frac{1}{2}\ln\Big(\partial_z\partial_w \ln[h_I(z) - h_J(w)]\Big) \quad . \tag{5.33}$$

is determined uniquely by its transformation properties under analytic coordinate transformations, and its short distance expansion. For $I = J$ and small $(z - w)$ we expand it as follows:

$$\ln(z - w) - \frac{1}{12}(z - w)^2\{h_I, w\} + O\left((z - w)^3\right) \ . \tag{5.34}$$

Again, the crucial observation is that terms of order $(z-w)^0$ are absent. Clearly, then, we will need relations of the type (5.26) to identify the affine Green function properly. This might not be entirely unexpected. The careful reader may have noticed that, while the vertex (5.3) does not contain any term diagonal in the zero modes, *i.e.* with coefficients $q_{A_i}^2$ or $q_{B_j}^2$, the fused vertex (5.32) certainly does. Note that we cannot use momentum conservation to solve that problem, because we would introduce new terms linear in Q and q_{A_i} or q_{B_j}, and the point of (5.3) was to eliminate all expressions of that type. If life were simple, the terms multiplying $q_{A_i}^2$ and $q_{B_j}^2$ in (5.32) would cancel, an expectation that gains credibility by an inspection of the j-q sector, and we would be finished. This is not what happens. A few paragraphs ago we noted that there is a second subtlety in the gluing of ghosts, which was not touched upon by our discussion of the j-q terms. We will explain below that by converting (5.32) in the usual way into the four different representations of the affine Green function, one misses the expected function (5.33) by a constant, which may depend on h_C and h_D, but not on z or w. In (5.32) this constant, let us call it $\mathbf{k}$, gives rise to a contribution of the form

$$-\frac{\epsilon}{2}\left(\sum_{i,j} q_{A_i}q_{A_j} + \sum_{i,j} q_{B_i}q_{B_j} + 2\sum_{i,j} q_{A_i}q_{B_j}\right)\ \mathbf{k} = -\frac{\epsilon}{2}\ Q^2\ \mathbf{k}\ , \tag{5.35}$$

and this is something we should have been waiting for all along. The determinants $\mathbf{D_a}$ and $\mathbf{D_j}$ need to be cancelled; we will show later that they are just the determinants of the scalar Laplacian, and therefore we expect a mechanism that works by adding up (2.20) to zero:

$$c^x + c^\phi = D + 1 - 3\epsilon Q^2 = 0 \tag{5.36}$$

The Q^2 part of that equality just appeared in (5.35).

Now that we have explained how the various pieces in (5.32) fit together, let us present the calculations. First, the constant $\mathbf{k}$ is given by

$$\mathbf{k} = K^C_n M^{DC}_{nm} \mathcal{N}^{DD}_{m0} + \mathcal{N}^{DD}_{00} M^{CC}_{nm} \mathcal{N}^{DD}_{m0} + \frac{1}{4} K^C_n M^{DD}_{nm} K^C_m \ . \tag{5.37}$$

We will motivate its form in section 5.3. The technique of resummation, applied to (5.32) plus (5.35) yields the equality of the following four representations of the affine Green function: for $|z|, |w| < R_C$,

$$\begin{aligned} &-\frac{1}{2} \ \ln \left(\partial_z \partial_w \ln \left(h_C(z) - h_C(w) \right) \right) \ - \mathbf{k} \\ &+ \ \frac{1}{4} \ G^C_n(z) M^{DD}_{nm} G^C_m(w) - \ \frac{1}{4} \ G^C_n(z) M^{DC}_{nm} G^D_m(\infty) \\ &- \ \frac{1}{4} \ G^D_n(\infty) M^{CD}_{nm} G^C_m(w) + \ \frac{1}{4} \ G^D_n(\infty) M^{CC}_{nm} G^D_m(\infty) \ ; \end{aligned} \tag{5.38}$$

for $\left|\frac{1}{z}\right|, \left|\frac{1}{w}\right| < R_D$,

$$\begin{aligned} &-\frac{1}{2} \ \ln \left(\partial_z \partial_w \ln \left(h_D \circ I(z) - h_D \circ I(w) \right) \right) \ - \mathbf{k} \\ &+ \ \frac{1}{4} \ G^C_n(0) M^{DD}_{nm} G^C_m(0) - \ \frac{1}{4} \ G^C_n(0) M^{DC}_{nm} G^D_m(w) \\ &- \ \frac{1}{4} \ G^D_n(z) M^{CD}_{nm} G^C_m(0) + \ \frac{1}{4} \ G^D_n(z) M^{CC}_{nm} G^D_m(w) \ ; \end{aligned} \tag{5.39}$$

for $|z| < R_C$, $\left|\frac{1}{w}\right| < R_D$,

$$
\begin{aligned}
&-\frac{1}{2}\ \ln\left(\partial_z\partial_v \ln\left(h_C(z) - h_C(v)\right)\Big|_{v=0}\right) \\
&-\frac{1}{2}\ \ln\left(\partial_u\partial_w \ln\left(h_D(u) - h_D\circ I(w)\right)\Big|_{u=0}\right)\ -\mathbf{k} \\
&+\ \frac{1}{4}\ G^C_n(z)M^{DD}_{nm}G^C_m(0) -\ \frac{1}{4}\ G^C_n(z)M^{DC}_{nm}G^D_m(w) \\
&-\ \frac{1}{4}\ G^D_n(\infty)M^{CD}_{nm}G^C_m(0) +\ \frac{1}{4}\ G^D_n(\infty)M^{CC}_{nm}G^D_m(w)\ ;
\end{aligned}
\tag{5.40}
$$

and finally, for $|z|, |w| < R_C$ and $\left|\frac{1}{z}\right|, \left|\frac{1}{w}\right| < R_D$,

$$
\begin{aligned}
&-\frac{1}{2}\ \ln\left(\partial_z\partial_w \ln\left(h_C(z) - h_C(w)\right)\right) \\
&-\frac{1}{2}\ \ln\left(\partial_z\partial_w \ln\left(h_D\circ I(z) - h_D\circ I(w)\right)\right) \\
&-\ln(z-w) - \mathbf{k} \\
&+\ \frac{1}{4}\ F^C_n(z)M^{DD}_{nm}F^C_m(w) -\ \frac{1}{4}\ F^C_n(z)M^{DC}_{nm}F^D_m(w) \\
&-\ \frac{1}{4}\ F^D_n(z)M^{CD}_{nm}F^C_m(w) +\ \frac{1}{4}\ F^D_n(z)M^{CC}_{nm}F^D_m(w)\ ,
\end{aligned}
\tag{5.41}
$$

where

$$
F^C_n(z) = \frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_u \ln\left[\partial_z\partial_u \ln\left(\frac{h_C(z) - h_C(u)}{z-u}\right)\right]
\tag{5.42}
$$

$$
F^D_n(z) = \frac{1}{n}\oint \frac{du}{2\pi i}\, u^{-n}\partial_u \ln\left[\partial_z\partial_u \ln\left(\frac{h_D\circ I(z) - h_D(u)}{I(z)-u}\right)\right]\ .
\tag{5.43}
$$

The identification of the Neumann function requires the correct short distance expansion. The identities that guarantee the absence of terms of order $(z-w)^0$

are

$$\mathbf{k} = \frac{1}{4} F_n^C(z) M_{nm}^{DD} F_m^C(z) - \frac{1}{4} F_n^C(z) M_{nm}^{DC} F_m^D(z)$$
$$- \frac{1}{4} F_n^D(z) M_{nm}^{CD} F_m^C(z) + \frac{1}{4} F_n^D(z) M_{nm}^{CC} F_m^D(z) \ , \tag{5.44}$$

and coordinate transforms of this equation. Again, we have checked them in a perturbation expansion of $h_{C,D}$ around the identity, which we performed to third order in small parameters. The general argument proceeds as before in the paragraph after (5.27). The Green functions (5.38) - (5.41) appear in

$$\langle h_C, 0 | \, e^{q_1\phi(w_1)} \ldots e^{q_n\phi(w_n)} \, | I \circ h_D \circ I, 0 \rangle = \left\langle f\left[e^{q_1\phi(w_1)}\right] \cdots f\left[e^{q_n\phi(w_n)}\right] \right\rangle \tag{5.45}$$

with some $f(z)$. This implies the proper short distance behaviour, and hence we can write (5.38) through (5.41) as $\ln\left(\partial_z \partial_w \ln[g(z) - g(w)]\right)$ and bring the exponent (5.32) into the form

$$-\sum_{i\neq j} \frac{\epsilon}{4} q_{A_i} \hat{\mathcal{N}}_{00}^{A_i A_j} q_{A_j} - \sum_{i\neq j} \frac{\epsilon}{4} q_{B_i} \hat{\mathcal{N}}_{00}^{B_i B_j} q_{B_j}$$
$$- \frac{\epsilon}{4} \sum_{i,j} q_{A_i} \hat{\mathcal{N}}_{00}^{A_i B_j} q_{B_j} - \frac{\epsilon}{4} \sum_{i,j} q_{B_i} \hat{\mathcal{N}}_{00}^{B_i A_j} q_{A_j} \tag{5.46}$$
$$+ \frac{\epsilon}{2} \, Q^2 \, \mathbf{k}_{\text{a.a.}} \ ,$$

where

$$\hat{\mathcal{N}}_{00}^{IJ} = \ln\left(\partial_z \partial_w \ln\left(\hat{h}_I(z) - \hat{h}_J(w)\right)\Big|_{\substack{z=0\\w=0}}\right)_{\text{a.a.}} \tag{5.47}$$

and the functions $\hat{h}_{A_i}$, $\hat{h}_{B_i}$ are defined by equation (4.4).

5.3. The Determinant

The only pieces of our gluing formula (5.16) that we have not explained yet are

$$\mathbf{D_j} \;=\; \left|\det\left(1 - N^{DD} E N^{CC} E\right)\right|^{-1/2} , \tag{5.48}$$

and its counterpart for the X- oscillators,

$$\mathbf{D_a} \;=\; \left|\det\left(1 - N^{DD} E N^{CC} E\right)\right|^{-D/2} . \tag{5.49}$$

The task is now to make sense out of these rather unwieldy expressions. First, we will motivate our final result by demonstrating that in fact

$$\mathbf{D_j} = \left|\frac{\det'\Delta_{g(z)}}{\det'\Delta_z}\right|^{-1/2} , \tag{5.50}$$

i.e. that we are dealing with determinants of the Laplacian. Then it is quite obvious why these determinants should vanish, namely by an argument á la Polyakov,[1] and the fact that we are in the critical dimension and therefore the conformal anomaly is zero. However, the actual proof must be somewhat involved, since it is well known since the days of light cone string field theory that in general, the above determinants are not trivial at all, but provide crucial measure factors in the calculation of scattering amplitudes.

Let us first take a look back at section 4.1 and display some moments of the Green function $\partial_z \partial_w \ln(z-w)$. Define

$$\begin{aligned}
D_{-n,-m} \;&=\; \frac{1}{n}\oint \frac{dz}{2\pi i} z^{-n} \frac{1}{m}\oint \frac{dw}{2\pi i} w^{-m} \partial_z \partial_w \ln(g(z) - g(w)) \\
D_{n,m} \;&=\; \frac{1}{n}\oint \frac{dz}{2\pi i} z^{n} \frac{1}{m}\oint \frac{dw}{2\pi i} w^{m} \partial_z \partial_w \ln(g(z) - g(w)) \\
D_{n,-m} \;&=\; \frac{1}{n}\oint \frac{dz}{2\pi i} z^{n} \frac{1}{m}\oint \frac{dw}{2\pi i} w^{-m} \partial_z \partial_w \ln(g(z) - g(w)) \\
D_{-n,m} \;&=\; \frac{1}{m}\oint \frac{dw}{2\pi i} w^{m} \frac{1}{n}\oint \frac{dz}{2\pi i} z^{-n} \partial_z \partial_w \ln(g(z) - g(w)) ,
\end{aligned} \tag{5.51}$$

where for the first two expressions the order in which the contour intetgrals are performed is irrelevant, whereas in the last two formulae the order is crucial. We derive then from (4.11)

$$\begin{aligned}
D_{-n,-m} &= \left[(1 - N^{CC}EN^{DD}E)^{-1}N^{CC}\right]_{nm} \\
D_{n,m} &= \frac{1}{n}\left[EN^{DD}E(1 - N^{CC}EN^{DD}E)^{-1}\right]_{nm}\frac{1}{m} \\
D_{n,-m} &= \frac{1}{n}(1 - N^{DD}EN^{CC}E)^{-1}{}_{nm} \\
D_{-n,m} &= (1 - N^{CC}EN^{DD}E)^{-1}{}_{nm}\frac{1}{m} ,
\end{aligned} \tag{5.52}$$

and observe that

$$\det\begin{pmatrix} D_{-n,-m} & D_{n,-m} \\ D_{-n,m} & D_{n,m} \end{pmatrix} = \det\left(n\left[1 - N^{DD}EN^{CC}E\right]_{nm} m\right)^{-1} . \tag{5.53}$$

The determinant we are evaluating is a determinant of moments of the Green function. The moments correspond to the nonzero modes of the field X. Since the Green function is just the inverse of the Laplacian, we may write the above equation as:

$$\det{}'\Delta_{g(z)} = \det\left(n\left[1 - N^{DD}EN^{CC}E\right]_{nm} m\right) . \tag{5.54}$$

Dj is then essentially the ratio of the determinant associated with the glued surface to the determinant we obtain for $N^{DD} = N^{CC} = 0$, i.e. the expression we get if the map g can be taken to be the identity. It is now clear that this factor is just the change of the determinant of the Laplacian under the map g, and the work of Polyakov[1] and Alvarez[10] has made it clear that this factor must vanish between the matter and the ghost fields, if we are in the critical dimension. This cancellation must occur for any g.

However, from the calculation of the 4-point scattering amplitude in light-cone string field theory we know that this vanishing happens in a nontrivial manner. There one obtains

$$\det(1 - N^{CC}EN^{DD}E)^{12} \;=\; \frac{dT}{dx}\exp\left[\sum_{i=1,2} N^{A_iA_i}_{0\ 0} + N^{B_iB_i}_{0\ 0} + T - f(\alpha_i, \beta_j)\right] . \tag{5.55}$$

In this formalism the determinant provides the jacobian for the change of variables from the propagator length T in the ρ-plane to the positions $0, 1, x, \infty$ of the vertex operators on the z-plane and some normalization factors involving the string lengths α_i, β_j.[8] Obviously these determinants are nontrivial and can combine to 1 only with other terms in the vertex.

We will now show how that happens. In the same way we proved (2.29) in section 2 one can prove that

$$\langle f, -q-Q| \equiv \langle -q-Q|\exp\Big[\frac{\epsilon}{2}\sum_{n,m\geq 0} j_n\mathcal{N}^{ff}_{nm}j_m\Big] = \langle -q-Q|\, U_f \ . \tag{5.56}$$

Note that $\langle -q-Q|\, j_0 = \langle -q-Q|\, q$, and hence

$$\begin{aligned}|I\circ f\circ I, 0\rangle_C &\equiv \langle f, 0|_D\, |I_{DC}\rangle \\ &= \exp\Big[\frac{\epsilon}{2}\sum_{n,m\geq 1} j_{-n}(-)^n\mathcal{N}^{ff}_{nm}(-)^m j_{-m} + \epsilon Q\sum_{n\geq 1} j_{-n}(-)^n\mathcal{N}^{ff}_{n0}\Big]\, |0\rangle_C \\ &= U^{-1}_{I\circ f\circ I}\, |0\rangle_C \ .\end{aligned} \tag{5.57}$$

In the two pieces (5.56) and (5.57), the vacua are transformed by unitary transformations generated by moments of the total energy momentum tensor $T_{zz} = T^x_{zz} + T^\phi_{zz}$. But this energy momentum tensor has zero central charge and therefore, on the sphere, all of its correlations in the vacuum vanish:[3]

$$\langle T_{zz}(z)\rangle = \langle -Q|\, T_{zz}(z)\, |0\rangle = 0 \ , \quad \langle T_{zz}T_{ww}\rangle = \langle -Q|\, T_{zz}T_{ww}\, |0\rangle = 0 \ , \quad etc. \tag{5.58}$$

We thus obtain

$$\langle -Q|\, U_{h_C} U_{h_D}^{-1}\, |0\rangle = 1 \ . \tag{5.59}$$

This leads us finally to the formula

$$1 = \langle h_C, -Q| I \circ h_D \circ I, 0\rangle \ = \exp\left[-\frac{D+1}{2}\left(1 - N^{CC} E N^{DD} E\right) + \frac{\epsilon}{2} Q^2 \mathbf{k} \right]_{\text{a.a.}}$$

$$= \det\left(1 - N^{CC} E N^{DD} E\right)_{a.a.}^{-\frac{D+1}{2}} \cdot$$

$$\cdot \exp\left[\frac{\epsilon}{2}\, Q^2\, K^C_n M^{DC}_{nm} \left(N^{DD}_{m0} - \frac{1}{2} K^D_m \right) \right.$$

$$+ \frac{\epsilon}{2}\, Q^2 \left(N^{DD}_{0\,n} - \frac{1}{2} K^D_n \right) M^{CC}_{nm} \left(N^{DD}_{m0} - \frac{1}{2} K^D_m \right) \tag{5.60}$$

$$\left. + \frac{\epsilon}{8}\, Q^2\, K^C_n M^{DD}_{nm} K^C_m \right]_{\text{a.a.}} \ .$$

We derived (5.60) using general arguments, but we also checked it perturbatively, in the same way we demonstrated the equations (5.26) and (5.44). This is of course the way to find $\mathbf{k}$. We now combine (5.60) with (5.46) to establish the GGRT. The fused vertex contains precisely the terms which the vertex definition for the glued surface requires, as stated in equation (3.3).

6. Gluing the bc System

The basic strategy for the proof of the GGRT in the context of fermionic ghosts is the same as in the previous sections. However, now we have to account for the different tensor character of the conformal fields we are considering and the associated zero modes. Here we will concentrate on the reparametrization ghosts of the bosonic string, but the generalization of our results to general ghost systems will be straightforward.

The two point function was defined in (2.12), and since the conformal weights of b and c are 2 and -1, the coordinate transformations (5.7) have the effect

$$\langle c(z)b(w)\rangle = \frac{1}{z-w} \rightarrow \frac{[g'(w)]^2}{f'(z)} \frac{1}{f(z)-g(w)} \; . \tag{6.1}$$

The field $c(z)$ has three zero modes, namely $Z_i(z) = z^{i+1}$ for $i = -1, 0, 1$. Their transformation behaviour is given by

$$z^{i+1} \rightarrow \frac{1}{f'(z)}[f(z)]^{i+1} \; . \tag{6.2}$$

In I the vertex was constructed out of the Fourier modes of these quantities. We obtained

$$\langle V_{1\cdots k}| \;=\; \prod_{I=1}^{k} \langle 3|_I \int_{\varsigma_1,\varsigma_0,\varsigma_{-1}} \exp\Big(- \sum_{m\geq -1} \varsigma_i M_{i\,m}^{\,J} b_m^J \;+\; \sum_{\substack{n\geq 2\\ m\geq -1}} c_n^I \mathsf{N}_{n\,m}^{I\,J} b_m^J \Big) \; , \tag{6.3}$$

with

$$\mathsf{N}_{nm}^{I\,J} \;=\; \oint \frac{dz}{2\pi i} z^{-n+1} \oint \frac{dw}{2\pi i} w^{-m-2} \frac{[h_I'(z)]^2}{h_J'(w)} \frac{-1}{(h_I(z)-h_J(w))} \tag{6.4}$$

and

$$M_{i\,m}^{\,J} \;=\; \oint \frac{dw}{2\pi i} w^{-m-2} \frac{1}{h_J'(w)} Z_i(h_J(w)) \; . \tag{6.5}$$

A simple example is the 2-vertex $\langle I_{AB}|$, based on $h_A(z) = z$ and $h_B(z) = -1/z$, *i.e.*

$$\langle \mathcal{I}_{AB}| = \langle 3|_A \otimes \langle 3|_B \int_{\varsigma_1,\varsigma_0,\varsigma_{-1}} \exp\Bigg(\varsigma_i((-)^i b^A_{-i} - b^B_i) \\ + \sum_{n\geq 2}((-)^n b^A_n c^B_n + (-)^{n+1} c^A_n b^B_n)\Bigg) . \tag{6.6}$$

Then, with

$$\mathbf{1}_{AC} = \langle 3|_A \exp\left(\sum_{n\geq -1} c^C_{-n} b^A_n \; + \; \sum_{n\geq 2} b^C_{-n} c^A_n \right) |0\rangle_C \tag{6.7}$$

and the definition

$$\langle \mathcal{I}_{AB}|\mathcal{I}_{BC}\rangle = \mathbf{1}_{AC} \tag{6.8}$$

we obtain the inner product

$$|\mathcal{I}_{AB}\rangle = \int_{\rho_1,\rho_0,\rho_{-1}} \exp\Bigg(\rho_i((-)^i c^A_{-i} + c^B_i) \\ + \sum_{n\geq 2}((-)^{n+1} b^A_{-n} c^B_{-n} + (-)^n c^A_{-n} b^B_{-n})\Bigg) |0\rangle_A \otimes |0\rangle_B . \tag{6.9}$$

In the vertices (6.3) and (6.6) one has to pay close attention to the range of the index sums $\sum_n$. It will prove to be convenient if we introduce the following matrix notation:

$$\begin{aligned} \left(\mathsf{N}^{IJ}\right)_{nm} &= \mathsf{N}^{IJ}_{nm} & \left(\mathsf{N}^{IJ}_0\right)_{nj} &= \mathsf{N}^{IJ}_{nj} \\ \left(\mathsf{N}^{I}\right)_{nm} &= \mathsf{N}^{IJ}_{nm} & \left(\mathsf{N}^{I}_0\right)_{nj} &= \mathsf{N}^{IJ}_{nj} \\ (M^I)_{in} &= M^I_{in} & (M^I_0)_{ij} &= M^I_{ij} \\ (E)_{nm} &= (-)^n \delta_{n,m} & (E_0)_{ij} &= (-)^i \delta_{i,j} \\ & & (P)_{ij} &= \delta_{i,-j} . \end{aligned} \tag{6.10}$$

The indices i, j run over the set $\{-1, 0, 1\}$, while $n, m \geq 2$. A lower index 0 on a matrix indicates that we are summing on the right side of the matrix only over indices j that come from ghost zero modes. For example, the vertex (6.3) now takes the form

$$\langle V_{1\cdots k}| = \prod_{I=1}^{k} \langle 3|_I \int_{\varsigma_1, \varsigma_0, \varsigma_{-1}} \exp\Big(c^I \mathsf{N}^{IJ} b^J + c^I \mathsf{N}^{IJ}_{\ \ 0} b^J - \varsigma M^J b^J - \varsigma M_0^J b^J \Big) , \quad (6.11)$$

It is most convenient to begin the gluing procedure by performing the nonzero mode contractions. This gives the result:

$$\Big\langle V_{\{A_i\}\{B_j\}}\Big| \equiv \Big\langle V_{\{A_i\}C}\Big| \Big\langle V_{\{B_j\}D}\Big| \, |I_{CD}\rangle$$

$$= \prod_i \langle 3|_{A_i} \prod_j \langle 3|_{B_j} \int_{\eta_1, \eta_0, \eta_{-1}} \int_{\theta_1, \theta_0, \theta_{-1}} \int_{\rho_1, \rho_0, \rho_{-1}} \mathbf{D_{bc}}$$

$$\begin{aligned}
\exp\Bigg(c^{A_i}\Big[& \Big(\mathsf{N}^{A_i A_j} b^{A_j} + \mathsf{N}^{A_i A_j}_{\ \ 0} b^{A_j}\Big) \\
& + \mathsf{N}^{A_i C} E \mathsf{N}^D E \big(1 - \mathsf{N}^C E \mathsf{N}^D E\big)^{-1} \Big(\mathsf{N}^{C A_j} b^{A_j} + \mathsf{N}^{C A_j}_{\ \ 0} b^{A_j}\Big)\Big] \\
+ c^{B_i}\Big[& \Big(\mathsf{N}^{B_i B_j} b^{B_j} + \mathsf{N}^{B_i B_j}_{\ \ 0} b^{B_j}\Big) \\
& + \mathsf{N}^{B_i C} E \mathsf{N}^D E \big(1 - \mathsf{N}^C E \mathsf{N}^D E\big)^{-1} \Big(\mathsf{N}^{C B_j} b^{B_j} + \mathsf{N}^{C B_j}_{\ \ 0} b^{B_j}\Big)\Big] \qquad (6.12) \\
- c^{A_i} & \mathsf{N}^{A_i C} E \big(1 - \mathsf{N}^D E \mathsf{N}^C E\big)^{-1} \Big(\mathsf{N}^{D B_j} b^{B_j} + \mathsf{N}^{D B_j}_{\ \ 0} b^{B_j}\Big) \\
& - c^{B_i} \mathsf{N}^{B_i D} E \big(1 - \mathsf{N}^C E \mathsf{N}^D E\big)^{-1} \Big(\mathsf{N}^{C A_j} b^{A_j} + \mathsf{N}^{C A_j}_{\ \ 0} b^{A_j}\Big) \\
& + \theta \mathcal{C} + \eta \mathcal{B} + \mathcal{A} \rho + \theta \mathcal{D} \rho + \eta \mathcal{E} \rho \Bigg)
\end{aligned}$$

where

$$\mathbf{D_{bc}} = \det\Big(1 - E \mathsf{N}^D E \mathsf{N}^C\Big) \qquad (6.13)$$

and where we have defined the row vector

$$\begin{aligned}\mathcal{A} =& - c^{A_i} \mathsf{N}^{A_i C}_{0} P \\ &+ c^{A_i} \mathsf{N}^{A_i C} E \big(1 - \mathsf{N}^{D} E \mathsf{N}^{C} E\big)^{-1} \left(\mathsf{N}^{D}_{0} E_0 - \mathsf{N}^{D} E \mathsf{N}^{C}_{0} P\right) \\ &- c^{B_i} \mathsf{N}^{B_i D}_{0} E_0 \\ &+ c^{B_i} \mathsf{N}^{B_i D} E \big(1 - \mathsf{N}^{C} E \mathsf{N}^{D} E\big)^{-1} \left(\mathsf{N}^{C}_{0} P - \mathsf{N}^{C} E \mathsf{N}^{D}_{0} E_0\right) \ ,\end{aligned} \tag{6.14}$$

the column vectors

$$\begin{aligned}\mathcal{B} =& - M^{B_j} b^{B_j} - M^{B_j}_{0} b^{B_j} + M^{D} E \big(1 - \mathsf{N}^{C} E \mathsf{N}^{D} E\big)^{-1} \cdot \\ & \left[\mathsf{N}^{C A_i} b^{A_i} + \mathsf{N}^{C A_i}_{0} b^{A_i} - \mathsf{N}^{C} E \left(\mathsf{N}^{D B_j} b^{B_j} + \mathsf{N}^{D B_j}_{0} b^{B_j}\right)\right] \ ,\end{aligned} \tag{6.15}$$

$$\begin{aligned}\mathcal{C} =& - M^{A_j} b^{A_j} - M^{A_j}_{0} b^{A_j} + M^{C} E \big(1 - \mathsf{N}^{D} E \mathsf{N}^{C} E\big)^{-1} \cdot \\ & \left[\mathsf{N}^{D B_i} b^{B_i} + \mathsf{N}^{D B_i}_{0} b^{B_i} - \mathsf{N}^{D} E \left(\mathsf{N}^{C A_j} b^{A_j} + \mathsf{N}^{C A_j}_{0} b^{A_j}\right)\right] \ ,\end{aligned} \tag{6.16}$$

and the (3 by 3) matrices

$$\mathcal{D} = M^{C}_{0} P - M^{C} E \big(1 - \mathsf{N}^{D} E \mathsf{N}^{C} E\big)^{-1} \left(\mathsf{N}^{D}_{0} E_0 - \mathsf{N}^{D} E \mathsf{N}^{C}_{0} P\right) \tag{6.17}$$

and

$$\mathcal{E} = M^{D}_{0} E_0 - M^{D} E \big(1 - \mathsf{N}^{C} E \mathsf{N}^{D} E\big)^{-1} \left(\mathsf{N}^{C}_{0} P - \mathsf{N}^{C} E \mathsf{N}^{D}_{0} E_0\right) \ . \tag{6.18}$$

In the exponent matrices are multiplied according to the following rules: each upper index C is connected to an upper index D through a matrix E; only zero mode matrices allow the connection of upper indices of the same type, via the link P. (6.12) is of course not the final form of the glued vertex, since the expression is still written in terms of three fermionic integrals, of which we will have to perform two. The expressions simplify if we do the η and the ρ integrals. The above formula is then multiplied by the zero-mode determinant $\mathbf{d_{bc}} = \det \mathcal{E}$, so that the exponential contining θ, ρ or η gets replaced by

$$\det \mathcal{E} \cdot \exp\left[\theta\left(\mathcal{C} - \mathcal{D}\mathcal{E}^{-1}\mathcal{B}\right) - \mathcal{A}\mathcal{E}^{-1}\mathcal{B}\right] \ . \tag{6.19}$$

We will show that this expression is precisely the vertex expected from the geometrical gluing picture, multiplied by a factor that cancels the X-determinant $\mathbf{D_a}$. More explicitly, in the critical dimension,

$$\langle V_{\{A_i\}\{B_j\}}| = \mathbf{D}_a^{-1} \prod_{I,J} \langle 3|_I \int_{\theta_1,\theta_0,\theta_{-1}} \exp\left(c^I \hat{\mathsf{N}}^{IJ} b^J + c^I \hat{\mathsf{N}}^{IJ}_0 b^J - \theta \hat{M}^I b^I - \theta \hat{M}^I_0 b^I \right) , \tag{6.20}$$

where the indices I, J take the values A_i, B_j for all i, j and the hatted Neumann coefficients are given by the hatted versions of (6.4) and (6.5). The argument relies again on the analyticity and singularity structure of the glued Green functions, which we will now prove.

6.1. Nonzero modes

The Neumann coefficients are just moments of the Green function. The gluing procedure provides us with four different forms of this function, and in this section we will outline how one proves their equality. The conversion of Fourier modes into Green functions and the change of coordinates (5.7) is accomplished by the following replacements:

$$
\begin{aligned}
\mathsf{N}^{A_iA_j}_{\cdot\ \cdot} &\rightarrow \frac{[h'_C(z)]^2}{h'_C(w)} \frac{-1}{h_C(z) - h_C(w)} \\
\mathsf{N}^{A_iC}_{\cdot\ n} &\rightarrow \mathcal{G}^C_n(z) = \oint \frac{du}{2\pi i} u^{-n-2} \frac{[h'_C(z)]^2}{h'_C(u)} \frac{-1}{h_C(z) - h_C(u)} \\
\mathsf{N}^{CA_j}_{m.} &\rightarrow \mathcal{H}^C_m(w) = \oint \frac{dv}{2\pi i} v^{-m+1} \frac{[h'_C(v)]^2}{h'_C(w)} \frac{-1}{h_C(v) - h_C(w)} \\
M_{i.}^{\ A_j} &\rightarrow \frac{1}{h'_C(z)} [h_C(z)]^{i+1}
\end{aligned} \tag{6.21}
$$

for $|z|, |w| < R_C$ and

$$
\begin{aligned}
\mathsf{N}^{B_iB_j}_{\cdot\ \cdot} &\rightarrow \frac{[(h_D \circ I)'(z)]^2}{h'_D(w)} \frac{-1}{h_D(z) - h_D \circ I(w)} \\
\mathsf{N}^{B_iD}_{\cdot\ n} &\rightarrow \mathcal{G}^D_n(z) = \oint \frac{du}{2\pi i} u^{-n-2} \frac{[(h_D \circ I)'(z)]^2}{h'_D(u)} \frac{-1}{h_D \circ I(z) - h_D(u)} \\
\mathsf{N}^{DB_j}_{m.} &\rightarrow \mathcal{H}^D_m(w) = \oint \frac{dv}{2\pi i} v^{-m+1} \frac{[h'_D(v)]^2}{(h_D \circ I)'(w)} \frac{-1}{h_D(v) - h_D \circ I(w)} \\
M_{i.}^{\ B_j} &\rightarrow \frac{1}{(h_D \circ I)'(w)} [h_D \circ I(w)]^{i+1}
\end{aligned} \tag{6.22}
$$

for $|z|, |w| > R_D^{-1}$. Using these substitutions we derive the representations

$$
\begin{aligned}
&\frac{[h'_C(z)]^2}{h'_C(w)} \frac{-1}{h_C(z) - h_C(w)} \\
&+ \mathcal{G}^C_n(z) \left[E \left(1 - \mathsf{N}^D E \mathsf{N}^C E\right)^{-1} \mathsf{N}^D E \right]_{nm} \mathcal{H}^C_m(w) \\
&+ \mathcal{G}^C_i(z) \left[P \mathcal{E}^{-1} M^D E \left(1 - \mathsf{N}^C E \mathsf{N}^D E\right)^{-1} \right]_{im} \mathcal{H}^C_m(w)
\end{aligned}
$$

$$
\begin{aligned}
&- \mathcal{G}_n^C(z)\Bigg[E\big(1-\mathsf{N}^D E \mathsf{N}^C E\big)^{-1}\left(\mathsf{N}_0^D E_0 - \mathsf{N}^D E \mathsf{N}_0^C P\right) \\
&\qquad\qquad \mathcal{E}^{-1} M^D E\big(1-\mathsf{N}^C E \mathsf{N}^D E\big)^{-1}\Bigg]_{nm} \mathcal{H}_m^C(w)
\end{aligned}
\tag{6.23}
$$

for $|z|, |w| < R_C$, and

$$
\begin{aligned}
&- \mathcal{G}_n^C(z)\left[E\big(1-\mathsf{N}^D E \mathsf{N}^C E\big)^{-1}\right]_{nm} \mathcal{H}_m^D(w) \\
&- \mathcal{G}_i^C(z)[P\mathcal{E}^{-1}]_{ij}\frac{1}{(h_D\circ I)'(w)}[(h_D\circ I)(w)]^{j+1} \\
&- \mathcal{G}_i^C(z)\left[P\mathcal{E}^{-1} M^D E\big(1-\mathsf{N}^C E \mathsf{N}^D E\big)^{-1}\mathsf{N}^C E\right]_{im} \mathcal{H}_m^D(w) \\
&+ \mathcal{G}_n^C(z)\left[E\big(1-\mathsf{N}^D E \mathsf{N}^C E\big)^{-1}\left(\mathsf{N}_0^D E_0 - \mathsf{N}^D E \mathsf{N}_0^C P\right)\mathcal{E}^{-1}\right]_{nj} \cdot \\
&\qquad \left[\frac{[(h_D\circ I)(w)]^{j+1}}{(h_D\circ I)'(w)} + \left[M^D E\big(1-\mathsf{N}^C E \mathsf{N}^D E\big)^{-1}\mathsf{N}^C E\right]_{jm} \mathcal{H}_m^D(w)\right]
\end{aligned}
\tag{6.24}
$$

for $|z| < R_C$ and $|w| > R_D^{-1}$. The most convenient way of showing the equivalence of these two expressions consists of applying the contour integral method. Since in this paper we wish to be explicit in the proof of the GGRT, let us do this exercise in some detail as a model for all the other cases, *i.e.* the other forms of the Green function and the different representations of the Killing vectors. The contour integral in w is taken in the strip $R_C > |w| > R_D^{-1}$, or $R_D > |w^{-1}| > R_C^{-1}$. We now list a set of identities which are well defined and in which the ordering of contours is determined by virtue of the bounds on $|w|$ and $|w^{-1}|$.

$$\oint \frac{dw}{2\pi i} w^{-k-2} \frac{1}{h'_C(w)} [h_C(w)]^{i+1} = \begin{cases} M_{i\,k}^{\,C} & \text{for } k \geq -1 \\ 0 & \text{for } k \leq -2 \end{cases}$$

$$\oint \frac{dw}{2\pi i} w^{-k-2} \frac{1}{(h_C \circ I)'(w)} [h_C \circ I(w)]^{i+1} = \begin{cases} 0 & \text{for } k \geq 2 \\ -\left[M^D E\right]_{ik} & \text{for } k \leq 1 \end{cases}$$

$$\oint \frac{dw}{2\pi i} w^{-k-2} \mathcal{H}_m^C(w) = \oint \frac{dw}{2\pi i} w^{-k-2} \cdot$$

$$\cdot \oint \frac{dv}{2\pi i} v^{-m+1} \frac{[h'_C(v)]^2}{h'_C(w)} \frac{-1}{h_C(v) - h_C(w)} = \begin{cases} \delta_{m,k} & \text{for } k \geq -1 \\ \mathsf{N}_{m\,k}^{C\,C} & \text{for } k \leq -2 \end{cases}$$

$$\oint \frac{dw}{2\pi i} w^{-k-2} \mathcal{H}_m^D(w) = \oint \frac{dw}{2\pi i} w^{-k-2} \cdot$$

$$\cdot \oint \frac{dv}{2\pi i} v^{-m+1} \frac{[h'_D(v)]^2}{(h_D \circ I)'(w)} \frac{-1}{h_D(v) - h_D \circ I(w)} = \begin{cases} -E_{mk} & \text{for } k \geq 2 \\ -\left[\mathsf{N}^D E\right]_{mk} & \text{for } k \leq 1 \end{cases} \tag{6.25}$$

Upon substitution in (6.23) and (6.24) one sees after a little algebra that the two expressions indeed represent the same function. Notice that the precise mechanism is different for the zero modes, *i.e.* for $k = -1, 0, 1$ and for the nonzero modes, *i.e.* for $|n| \geq 2$.

It is now clear how to proceed for the other two representations of the Green function. Alternatively, we could have employed the resummation technique and rewritten (6.23) and (6.24) in terms of the analytic function

$$\frac{[h'_C(w)]^2}{h'_C(z)} \frac{1}{h_C(z) - h_C(w)} - \frac{1}{z - w} \tag{6.26}$$

and its coordinate transforms. Again, after some algebra the equality of the two representations becomes apparent. Therefore, we may identify the expressions given above with the function

$$\frac{[g'(w)]^2}{g'(z)} \frac{1}{g(z)-g(w)} + \sum_{i=\pm 1,0} a_i \frac{1}{g'(z)} [g(z)]^{i+1} \quad , \tag{6.27}$$

where a_i are arbitrary constant coefficients. This ambiguity arises because we can determine the Green function from its analytic structure only up to the addition of zero modes. However, just as in the X-system, where momentum conservation made the zero modes irrelevant, these extra terms do not contribute to correlation functions. This follows from the Fermi statistics of the ghosts and is made obvious in the formula (A.2) in the appendix of I, where we represented the correlator as a determinant of a matrix of zero modes and two point functions. Of course, this assumes that we can prove the correct zero mode structure of the ghost sector.

6.2. Zero modes

The coefficients $M_{i\,n}^{\,J}$ are the n-th Fourier modes of the analytic vector fields labelled by i. At tree level we have to show that the corresponding quantities, i.e. the coefficients of the fermionic variables ς, are also just moments of linearly independent analytic vector fields. Thus we determine the correlation function of three $c(z)$-fields, or, equivalently the Vandermonde determinant of the zero modes (eq. (2.26) of I), up to a constant. The correct normalization of this correlator is proved by studying its short distance properties. At this point we have to take recourse to a perturbative expansion of the functions $h_{C,D}$. This situation sounds worse than it actually is, because we will prove the proper normalization to all orders in the parameters we expand in.

First, however, let us prove analyticity. As usual, we show that the two expressions we obtain for each vector field actually represent the same object. The two functions we then have to compare are

$$\frac{(h_C(z))^{i+1}}{h'_C(z)}$$

$$+\left[\left(M^{C}E\mathsf{N}^{D}+\mathcal{D}\mathcal{E}^{-1}M^{D}\right)E\left(1-\mathsf{N}^{C}E\mathsf{N}^{D}E\right)^{-1}s\right]_{im}\mathcal{H}_{m}^{C}(z)$$

for $|z| < R_C$ and

$$-\left[\mathcal{D}\mathcal{E}^{-1}\right]_{ij}\frac{[(h_D\circ I)(z)]^{j+1}}{(h_D\circ I)'(z)}$$

$$-\left[\left(M^{C}+\mathcal{D}\mathcal{E}^{-1}M^{D}E\mathsf{N}^{C}\right)E\left(1-\mathsf{N}^{D}E\mathsf{N}^{C}E\right)^{-1}\right]_{im}\mathcal{H}_{m}^{D}(z)$$

for $|z| > R_D^{-1}$. The methods shown above are again applicable to prove that these are but two equivalent representations of the same vector field. Notice that both functions are linear in M^C or M_0^C. Hence linear independence of the glued vector fields follows from the linear independence of the vector fields $1, z, z^2$ for the original vertices.

We now examine the ghost correlation function

$$\langle g[c(z)]g[c(w)]g[c(u)]\rangle = \frac{[g(z)-g(w)][g(z)-g(u)][g(w)-g(u)]}{g'(z)g'(w)g'(u)} \ . \tag{6.28}$$

For $|z-w|, |z-u|, |w-u| < \epsilon$ this function is equal to

$$(z-w)(z-u)(w-u)+O(\epsilon^2) \tag{6.29}$$

if ϵ is sufficiently small. This is the result we seek to prove for the glued zero modes. We start by asserting that the glued correlation function is given by

$$\langle h_C|\, c(z)c(w)c(u)\, |I\circ h_D\circ I\rangle \ , \tag{6.30}$$

where $\langle h_C|$ is the part of the vertex $\langle V_{\{A_i\}C}|$ that depends only on h_C, and $|I\circ h_D\circ I\rangle \equiv \langle h_D|I_{DC}\rangle$. Now we make use of a result that will be proved in the next section, namely

$$\langle h_C| = \langle 0| U_{h_C}; \qquad |I \circ h_D \circ I\rangle = U^{-1}_{I\circ h_D \circ I} |0\rangle \ . \tag{6.31}$$

$SL(2)$ invariance guarantees that both $h_C(z)$ and $h_D(z)$ can be chosen to be analytic around $z = 0$, with $h_{C,D}(0) = 0$. Therefore

$$U_{h_C} = \exp\left(\sum_{k\geq 2} v^C_{-k} L_k\right) \tag{6.32}$$

and

$$U^{-1}_{I\circ h_D \circ I} = \exp\left(\sum_{k\geq 2} v^D_{-k} (-)^k L_{-k}\right) \tag{6.33}$$

for some vector fields v^C and v^D. We replace v^C by tv^C and v^D by sv^D, with s, t small so that a perturbation expansion in these parameters makes sense. According to equation (2.39) we may rewrite (6.30) as

$$\langle 0| h_C[c(z)c(w)c(u)] U_A |0\rangle \ , \tag{6.34}$$

where

$$U_A |0\rangle = \exp\left(\sum_{k\geq 2} v^A_k L_k\right) |0\rangle \tag{6.35}$$

and the vector field v^A is given by some complicated formula in terms of tv^C, sv^D. At this point it is obvious that (6.30) is equal to $\langle f[c(z)c(w)c(u)]\rangle$ for some function $f(z)$ that we may write as $f(z) = z + d(z)$, where $d(z)$ is of order s or t. From the form of $f(z)$ one immediately derives (6.29), and concludes the proof that the ghost zero modes are correctly normalized, to any order in s, t. Note that $f(z)$ is generally not analytic around $z = 0$ or $z = \infty$. This is the reason why $\langle h_C| c_{-1}c_0c_1 |0\rangle \neq 1$, contrary to the expectation one might have had from $\langle h_C| c_{-1}c_0c_1 |0\rangle = \det M^C_{in} = 1$.

6.3. DETERMINANTS

As the final step in the proof of gluing for the bc system let us derive the cancellation of determinants. Again we start by deriving $\langle f| = \langle 0| U_f$. However, due to the existence of ghost zero modes the expression

$$\langle 0| U_f U_{g\circ I}^{-1} |0\rangle = 0 \tag{6.36}$$

does not yield the desired information. We therefore define

$$\begin{aligned} \langle f,0| \equiv \langle f| &= \langle 3| \int_{\eta_1,\eta_0,\eta_{-1}} \exp\left(\sum_{\substack{n\geq -1\\ m\geq 2}} c_m N^{f\,f}_{m\,n} b_n - \sum_{n\geq -1} \eta_i M_{i\,n}^{\,f} b_n \right) \\ \langle f,3| &\equiv \langle 3| \exp\left(\sum_{\substack{n\geq -1\\ m\geq 2}} c_m N^{f\,f}_{m\,n} b_n \right) . \end{aligned} \tag{6.37}$$

The 1-point functions $\langle f, 0, 3|$ are uniquely characterized by

$$\begin{aligned} &\langle f,0| c_{-n_1} \cdots b_{-n_k} |0\rangle = \langle f[c_{-n_1}] \cdots f[b_{-n_k}]\rangle \ ; \qquad \langle f,0|3\rangle = 1 \\ &\langle f,3| c_{-n_1} \cdots b_{-n_k} |0\rangle = \langle c_{-1} c_0 c_1 f[c_{-n_1}] \cdots f[b_{-n_k}]\rangle \ ; \qquad \langle f,3|0\rangle = 1 \ . \end{aligned} \tag{6.38}$$

Consequently

$$\begin{aligned} \langle f,0| f^{-1}[c_n] &= 0 \qquad \text{for} \qquad n \geq 2 \\ \langle f,0| f^{-1}[b_n] &= 0 \qquad \text{for} \qquad n \geq -1 \end{aligned} \tag{6.39}$$

and

$$\begin{aligned} \langle f,3| f^{-1}[c_n] &= 0 \qquad \text{for} \qquad n \geq -1 \\ \langle f,3| f^{-1}[b_n] &= 0 \qquad \text{for} \qquad n \geq 2 \ . \end{aligned} \tag{6.40}$$

Together with $f[\mathcal{O}] = U_f \mathcal{O} U_f^{-1}$ these equations imply

$$\langle f,0| = \langle 0|\, U_f \qquad ; \qquad \langle f,3| = \langle 3|\, U_f \ . \tag{6.41}$$

Then

$$\langle f|_A \, |\mathcal{I}_{AB}\rangle = U^{-1}_{I\circ f\circ I}\, |0\rangle \tag{6.42}$$

and now the product $\langle h_C, 3| I \circ h_D \circ I, 0\rangle$ is nontrivial. But then we can again apply the logic used at the end of Section 5.3. By combining the ghost factor the the factor $\langle h_C| I \circ h_D \circ I\rangle$ from the X dynamics, we find a structure $\langle 3|\, U_{h_C} U_{h_D}^{-1}\, |0\rangle$, with transformations U_h generated by an energy-momentum tensor with central charge $c = 0$. As before, we find that inclusion of the terms from X gives

$$\langle h_C, 3| I \circ h_D \circ I, 0\rangle = 1 \ . \tag{6.43}$$

All that remains is to translate this result into the specific form needed for the proof of (6.20). If there are no external states, all the terms with indices A_i, B_j vanish; in addition, use of $\langle h_C, 3|$ eliminates the variable θ. The remaining fermionic integrations give the zero mode determinant $\det \mathcal{E}$ we encountered before. Then, finally,

$$\begin{aligned} 1 &= \det\left(1 - N^{CC} E N^{DD} E\right)^{-13} \det\left(1 - \mathsf{N}^C E \mathsf{N}^D\right) \ \det \mathcal{E} \\ &= \mathbf{D_a D_{bc} d_{bc}} \ . \end{aligned} \tag{6.44}$$

This completes our proof of the gluing identity.

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FIGURE CAPTIONS

1) Simple representation of gluing.

2) Precise representation of gluing.

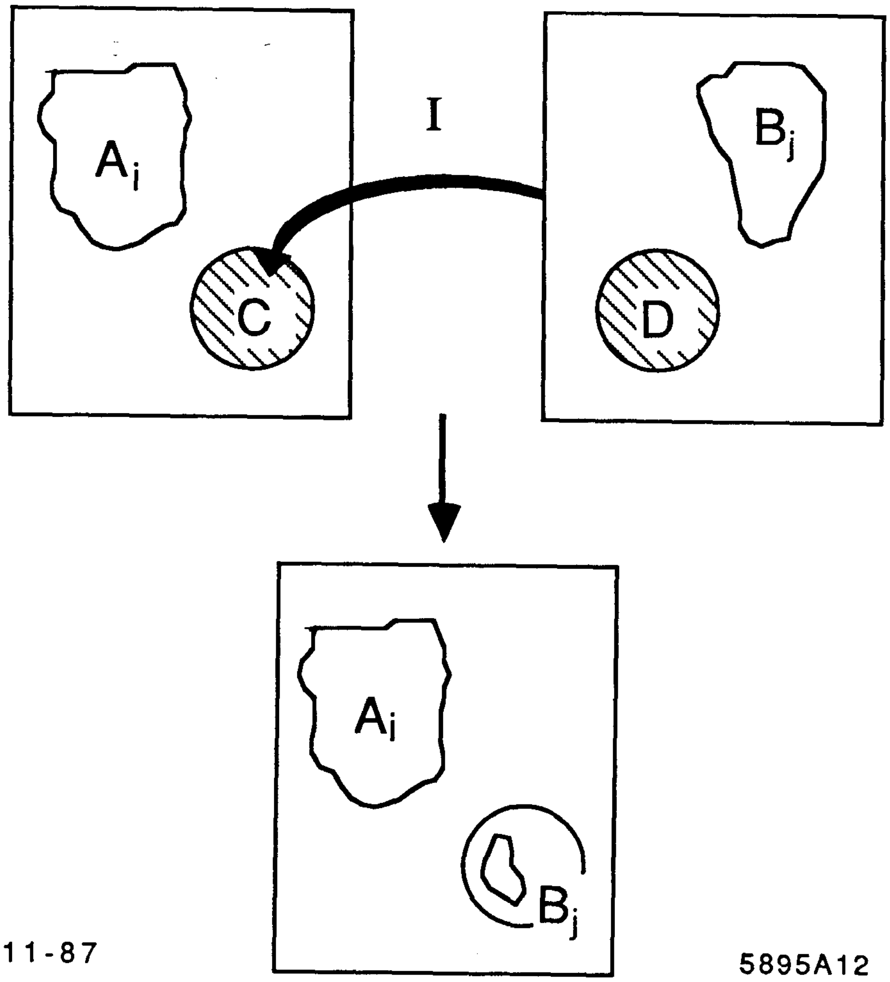

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Fig. 1

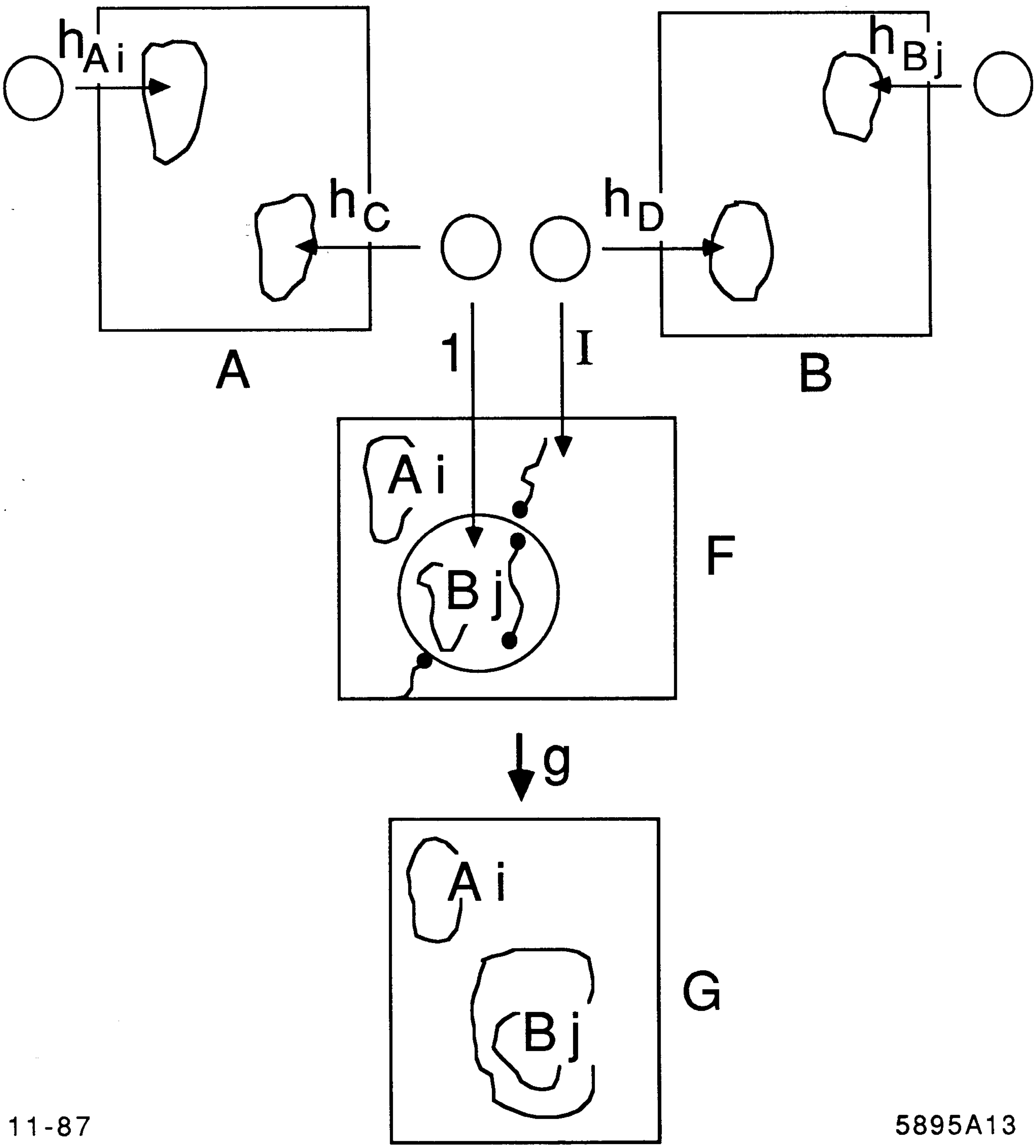

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Fig. 2