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THE $\eta - \eta'$ MIXING ANGLE*

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ABSTRACT

The current experimental evidence on the value of the $\eta - \eta'$ mixing angle is summarized in the light of our present theoretical understanding. A value of $\theta_p \simeq -20°$ is consistent with all present evidence.

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1. Introduction

The value of the $\eta - \eta'$ mixing angle has been the subject of discussion almost from the time that SU(3) flavor symmetry was proposed. In the simplest possible situation where one assumes the presence of only an octet and a singlet, the quadratic Gell-Mann - Okubo mass formula yields a pseudoscalar mixing angle of $\theta_p \simeq -10°$. With the same assumption, a Gell-Mann - Okubo mass formula which is linear in the masses gives $\theta_p \simeq -23°$. For reasons that have to do with both theory and experiment at a given time, over the years most authors[1] have taken $\theta_p \simeq -10°$.

However, in the past few years new data,[2,3] particularly on $\psi \rightarrow \eta(\eta')\gamma$ and $\eta \rightarrow \gamma\gamma$, have accumulated which favor a mixing angle of $\theta_p \simeq -20°$. Some of this evidence has already been pointed to as favoring such a mixing angle.[4,5]

In this paper we make an up-to-date summary of all the different experimental data and the theoretical arguments from which the pseudoscalar mixing angle can be determined. We show that a value of $\theta_p \simeq -20°$is consistent with all present evidence if we do not admix other quark model, gluonium, or exotic states into the ground state pseudoscalar system. No single piece of evidence is ironclad; aside from experimental errors, one can argue with the theoretical analysis of any particular experiment. It is the weight of the combination of all the data that leads to our conclusion. Moreover, the analysis gives a consistent result within the assumptions; it does not rule out small admixtures of gluonium or other states, particularly to the η'.

The paper is organized as follows: in Section 2 we define the notation and interrelate quark content and mixing angles. In Section 3 we discuss the $\gamma\gamma$ widths of the η and η'. Sections 4 and 5 cover $\psi \rightarrow \gamma\eta(\eta')$ and $\psi \rightarrow$ pseudoscalar + vector, respectively. Radiative decays, vector $\rightarrow$ pseudoscalar + photon and pseudoscalar $\rightarrow$ vector + photon are covered in Section 6. In Section 7 we discuss the evidence provided by $\pi^- p \rightarrow \eta(\eta')n$ scattering. Section 8 deals with decays of the tensor mesons involving the η, and in particular, $f \rightarrow \eta\eta$ and $a_2 \rightarrow \pi\eta$.

Finally, we return to the historical starting point of the subject, namely mass formulae. The predictions of mass matrix phenomenology, linear and quadratic, are reviewed in Sections 9 and 10. Our conclusions are found in Section 11. Relegated to the Appendix are two topics which are of interest in their own right and are related to the new width of the η, as derived from the branching ratio and absolute width for $\eta \to \gamma\gamma$: $\psi' \to \psi\pi^0$ and $\eta \to 3\pi$.

2. Notation

We are interested in consistency with the simplest possible situation. Thus we assume a two-state system and neglect possible mixing of the η and η' with other pseudoscalar states, whether radially excited quarkonium states, gluonium, or exotics. We also assume that the physical states are orthogonal, *i.e.*, that the mixing is independent of energy. Implicit in the analysis is the assumption that it is sensible to apply the mixing formalism to the processes of interest below.

The SU(3) basis states are then

$$|\eta_8\rangle = \frac{1}{\sqrt{6}} \left|u\bar{u} + d\bar{d} - 2s\bar{s}\right\rangle \tag{2.1}$$

and

$$|\eta_0\rangle = \frac{1}{\sqrt{3}} \left|u\bar{u} + d\bar{d} + s\bar{s}\right\rangle . \tag{2.2}$$

In terms of these states the η and η' wave functions are defined to be

$$|\eta\rangle = \cos\theta_p \, |\eta_8\rangle - \sin\theta_p \, |\eta_0\rangle \tag{2.3}$$

$$|\eta'\rangle = \sin\theta_p \, |\eta_8\rangle + \cos\theta_p \, |\eta_0\rangle . \tag{2.4}$$

For some purposes it is more convenient to use a quark basis:

$$|\eta\rangle = X_\eta \frac{1}{\sqrt{2}} \left|u\bar{u} + d\bar{d}\right\rangle + Y_\eta \, |s\bar{s}\rangle \tag{2.5}$$

$$\left|\eta'\right\rangle = X_{\eta'}\frac{1}{\sqrt{2}}\left|u\bar{u}+d\bar{d}\right\rangle + Y_{\eta'}\left|s\bar{s}\right\rangle . \tag{2.6}$$

With our assumption of no mixing with other pseudoscalar states, we require

$$X_\eta^2 + Y_\eta^2 = X_{\eta'}^2 + Y_{\eta'}^2 = 1. \tag{2.7}$$

In terms of θ_p the X's and Y's can be written

$$\begin{aligned} X_\eta = Y_{\eta'} &= \frac{1}{\sqrt{3}}\cos\theta_p - \sqrt{\frac{2}{3}}\sin\theta_p \\ Y_\eta = -X_{\eta'} &= -\sqrt{\frac{2}{3}}\cos\theta_p - \frac{1}{\sqrt{3}}\sin\theta_p, \end{aligned} \tag{2.8}$$

and conversely,

$$\tan\theta_p = -\frac{\sqrt{2}X_\eta + Y_\eta}{X_\eta - \sqrt{2}Y_\eta} = \frac{X_{\eta'} - \sqrt{2}Y_{\eta'}}{\sqrt{2}X_{\eta'} + Y_{\eta'}}. \tag{2.9}$$

A mixing angle of $-10°$ then corresponds to $X_\eta = Y_{\eta'} = 0.71$, while $-20°$ corresponds to $X_\eta = Y_{\eta'} = 0.82$.

3. $\gamma\gamma$ Widths

Current algebra predicts the ratios:

$$\frac{\Gamma(\eta\to\gamma\gamma)}{\Gamma(\pi^0\to\gamma\gamma)} = 18\left(\frac{m_\eta}{m_\pi}\right)^3 (F_\pi)^2 \left[\frac{\cos\theta_p}{F_8}\frac{(e_u^2+e_d^2-2e_s^2)}{\sqrt{6}} - \frac{\sin\theta_p}{F_0}\frac{(e_u^2+e_d^2+e_s^2)}{\sqrt{3}}\right]^2 \tag{3.1}$$

and

$$\frac{\Gamma(\eta'\to\gamma\gamma)}{\Gamma(\pi^0\to\gamma\gamma)} = 18\left(\frac{m_{\eta'}}{m_\pi}\right)^3 (F_\pi)^2 \left[\frac{\sin\theta_p}{F_8}\frac{(e_u^2+e_d^2-2e_s^2)}{\sqrt{6}} + \frac{\cos\theta_p}{F_0}\frac{(e_u^2+e_d^2+e_s^2)}{\sqrt{3}}\right]^2 \tag{3.2}$$

where F_π, F_8, and F_0 are the decay constants of the pion, eighth component of the octet, and singlet, respectively.

Let us start in the limit where SU(3) flavour symmetry is exact, and we have

$$F_8 = F_\pi. \tag{3.3}$$

Note that SU(3) symmetry alone does not imply $F_0 = F_\pi$ and we do not assume it. The latest experimental results are:[2,3]

$$\begin{aligned} \Gamma(\pi^0 \to \gamma\gamma) &= 7.3 \pm 0.2 \text{ eV} \\ \Gamma(\eta \to \gamma\gamma) &= 0.56 \pm 0.04 \text{ keV} \\ \Gamma(\eta' \to \gamma\gamma) &= 4.16 \pm 0.30 \text{ keV}. \end{aligned} \tag{3.4}$$

Using these results we can solve for F_0/F_π and θ_p:

$$\frac{F_0}{F_\pi} = 1.06 \pm 0.04 \tag{3.5}$$

$$\theta_p = -20^\circ \pm 2^\circ. \tag{3.6}$$

However, we do not expect $F_8 = F_\pi$ to be accurate to better than about 30% because of SU(3) breaking. The calculation of one-loop chiral corrections to F_8 and F_π yield the result[4]

$$\frac{F_8}{F_\pi} = 1.25. \tag{3.7}$$

To get an estimate of the error in this result we note that a similar calculation for F_K gave the result[6] $F_K/F_\pi = 1.20$, within 5% of the experimental value of $1.26 \pm .02$. We see that the calculated correction to the SU(3) symmetry relation $F_K = F_\pi$ is not very large, agrees with experiment, and is comparable to that calculated for the relation $F_8 = F_\pi$ in Eq. (3.7). We therefore assign an uncertainty to the calculation of F_8/F_π of 5%. Using Eq. (3.7) gives

$$\theta_p = -23^\circ \pm 3^\circ \pm 1^\circ \tag{3.8}$$

$$\frac{F_0}{F_\pi} = 1.04 \pm .04 \pm .05, \tag{3.9}$$

where the first error is statistical and the second accounts for the 5% uncertainty assigned to Eq. (3.7). Both with and without SU(3) symmetry we therefore have values of θ_p near $-20°$, and values of F_0/F_π within 10% of unity, although we made no assumption equivalent to nonet symmetry in either case.

There is one other argument[7] based on η and η' decays which is relevant to this subject. Current algebra can be used to predict the amplitudes for $\eta \to \pi^+\pi^-\gamma$ and $\eta' \to \pi^+\pi^-\gamma$ in the soft pion limit.[7] This result is irrelevant for the physical η' where the pions are not "soft" and there is a strong ρ resonance that is visible in the final $\pi\pi$ mass spectrum. It is less clear how relevant the soft pion result is for the η.

Nevertheless, applying the soft pion result to the physical η, and taking account of the presence of a virtual ρ through a multiplicative Breit - Wigner factor,[7] one obtains the prediction

$$\Gamma(\eta \to \pi^+\pi^-\gamma) = 1.04 \times 10^{-9}\mathrm{GeV}^7|G_\eta|^2, \tag{3.10}$$

where

$$G_\eta = -\frac{e}{4\sqrt{3}\pi^2 F_\pi^2}\left(\frac{\cos\theta_p}{F_8} - \frac{\sqrt{2}\sin\theta_p}{F_0}\right). \tag{3.11}$$

Using $F_\pi = 94$ MeV and the values of F_8 and F_0 obtained from Eqs. (3.7) and (3.9), respectively, we find

$$\Gamma(\eta \to \pi^+\pi^-\gamma) = \begin{cases} 28\ \mathrm{eV}, & \text{for } \theta_p \simeq -10°; \\ 37\ \mathrm{eV}, & \text{for } \theta_p \simeq -20°. \end{cases} \tag{3.12}$$

If we use the Crystal Ball result[3] for $\Gamma(\eta \to \gamma\gamma)$ to determine the total width of the η, then the experimental partial width is

$$\Gamma(\eta \to \pi^+\pi^-\gamma) = 71 \pm 13\ \mathrm{eV}, \tag{3.13}$$

which is significantly larger than the current algebra prediction in Eq. (3.12) for either $\theta_p \simeq -10°$ or $\theta_p \simeq -20°$. We conclude that, even though one can argue[8] that the presence of the ρ is incorporated into the current algebra predictions at threshold, the correct manner of extrapolation of the amplitude from the soft pion point to the physical region for η decay is unknown. (An identical calculation for the η' also gives a result much smaller than the experimental rate, but we do not expect the current algebra result to be relevant in this case.) The large discrepancy between the current algebra predictions and the experimental results effectively eliminates $\eta \to \pi^+\pi^-\gamma$ as a constraints on θ_p.

4. $\psi \rightarrow \gamma\eta(\eta')$

The processes $\psi \to \gamma\eta$ and $\psi \to \gamma\eta'$ occur primarily through radiation of the photon from the charmed quark or charmed antiquark in the initial state, as evidenced by the very small rate for $\psi \to \gamma\pi^0$ as compared to either of the former processes. Assuming such a mechanism and the applicability of SU(3) symmetry for the decay amplitudes, the decay proceeds through the SU(3) singlet part of the pseudoscalar and one finds[9]

$$\frac{\Gamma(\psi \to \gamma\eta')}{\Gamma(\psi \to \gamma\eta)} = \left(\frac{k_{\eta'}}{k_\eta}\right)^3 \frac{1}{\tan^2\theta_p}. \tag{4.1}$$

The current experimental value[2] of the left-hand-side is 4.8 ± 0.2. Using this we find

$$\theta_p = -22^\circ \pm 1^\circ \pm 4^\circ, \tag{4.2}$$

where the first error is from experiment and the second is an estimated theoretical uncertainty which reflects a possible 25% symmetry breaking (see particularly Eq. (3.7)). This seems like a fairly conclusive result, however it is possible to argue for an even larger breaking of the symmetry. In a physical picture where the decay proceeds through an intermediate two gluon state, the latter (nominally SU(3) singlet) may couple to the final pseudoscalar through an amplitude with a strong mass dependence. It has even been argued by Novikov *et al.*[10] that the mixing formalism can not be justified *a priori* in this case. It is not possible to simply dismiss these criticisms. One can note *a posteriori* though that the prediction of Novikov *et al.* for $\Gamma(\psi \to \gamma\eta')/\Gamma(\psi \to \gamma\eta)$ disagrees with experiment, while the application of the mixing formalism yields a value of θ_p which is consistent with that obtained from several other sources.

5. $\psi \to$ Pseudoscalar + Vector

We consider next purely hadronic decays of the ψ such as $\psi \to \omega\eta$ and $\psi \to \phi\eta$. An extensive analysis of all decays of this type was done by the Mark III collaboration[11] in which these decays were assumed to proceed through diagrams involving $c\bar{c}$ annihilation into three gluons or a (virtual) photon. Mixing of the η and η' with other exotic states was also allowed. The conclusions of this analysis were that there is very large SU(3) breaking and substantial mixing of the η' with exotic states, all within the context of a mixing angle of $\theta_p \simeq -10°$.

However, a recent re-analysis[5] comes to a very different conclusion. This analysis includes the possibility of both SU(3) breaking and doubly-OZI-suppressed amplitudes, thereby breaking nonet symmetry in a very particular way. It gives an excellent fit to the data. The breaking of SU(3) is moderate in this fit, and, assuming no mixing of either the η or η' with exotic states, it yields a value for the non-strange quark content of the of the η that corresponds to:

$$|X_\eta| = 0.79 \pm 0.02. \tag{5.1}$$

This is consistent with $|X_\eta| = 0.82$ (corresponding to $\theta_p = -20°$), and is inconsistent at the 4σ level with $\theta_p = -10°$. We conclude that the data on $\psi \to$ pseudoscalar + vector favors $\theta_p \simeq -20°$. We note that the analysis does not break SU(3) and nonet symmetry in the most general way and that our conclusion may depend somewhat on the form of breaking chosen. A further data analysis is in progress, and may well sharpen this result further.[5]

6. Radiative Decays of Light Mesons

We calculate these magnetic dipole transition amplitudes in the framework of the quark model, with SU(3) (and nonet symmetry) broken in the time-honored manner by a difference between the down and strange quark magnetic moments.[12] A summary of the results is presented in Table 1, with all decay rates normalized to the recent Novosibirsk result[13] $\Gamma(\omega \to \pi\gamma) = 764 \pm 69$ keV, which is 11% below the central value of the Particle Data Group[2] . Both this result and the associated result for the total width of the ω are significantly smaller than the previous world average, but appear to be very clean, systematics-free measurements. We have also included other current experimental data.[14–16]

Process	Ratio (theoretical)	Ratio (experimental)	Result
$\rho \to \pi\gamma$	0.105	0.093±0.015[2]	
$K^{*+} \to K^+\gamma$	0.088	0.067±0.010[2]	
$K^{*0} \to K^0\gamma$	0.19	0.153±0.021[2]	
$\rho \to \eta\gamma$	$\lvert X_\eta\rvert^2(k_\eta/k_\pi)^3$	0.072±0.019[12,14]	$\lvert X_\eta\rvert = 0.76 \pm 0.06$
$\phi \to \eta\gamma$	$\frac{4}{9}(\frac{m_u}{m_s})^2\lvert Y_\eta\rvert^2(k_\eta/k_\pi)^3$	0.066 ± 0.013[2,12]	$\lvert Y_\eta\rvert = 0.52 \pm 0.05$
$\eta' \to \rho\gamma$	$3\lvert X_{\eta'}\rvert^2(k_\rho/k_\pi)^3$	0.086 ± 0.015[2,15]	$\lvert X_{\eta'}\rvert = 0.57 \pm 0.05$
$\eta' \to \omega\gamma$	$\frac{1}{3}\lvert X_{\eta'}\rvert^2(k_\omega/k_\pi)^3$	0.0103 ± 0.0023[16]	$\lvert X_{\eta'}\rvert = 0.65 \pm 0.07$

Table 1. Radiative Decays of Light Mesons.

The theoretical and experimental widths are normalized to that for $\omega \to \pi\gamma$.

Over the years, the experimental data have evolved from gross disagreement to better and better agreement with the quark model. The latest data extend this trend. The theoretical values in the Table were calculated taking the ratio of strange to down quark magnetic moments, or equivalently m_d/m_s, equal to 0.8, chosen to best fit the data from the K^* decays. A value of $m_d/m_s = 0.7$ yields a prediction 15% higher for $K^{*+} \to K^+\gamma$ and 15% lower for $K^{*0} \to K^0\gamma$ and changes the value of $|Y_\eta|$ as derived from $\phi \to \eta\gamma$ to 0.59 ± 0.06, a result which favors $\theta_p \simeq -20°$. Reading from Table 1, we see that the data for $\phi \to \eta\gamma$ and $\eta' \to \rho\gamma$ favor $\theta_p \simeq -20°$ by several standard deviations, while the data on $\rho \to \eta\gamma$ and $\eta' \to \omega\gamma$ favor an angle midway between$-10°$ and $-20°$ but are consistent within one standard deviation with either value. The first three lines in the Table show us the level of SU(3) violation. We see that SU(3) is broken at the level of 30% in the rate. This means that our values for the X's and Y's should be considered to be uncertain to within 15%. Unfortunately, this uncertainty prevents us from discriminating decisively between $\theta_p \simeq -10°$ and $\theta_p \simeq -20°$ on this basis alone. We conclude that the radiative decays of the light mesons favor $\theta_p \simeq -20°$, but cannot rule out $\theta_p \simeq -10°$.

7. $\pi^- p$ Scattering

We consider the reactions $\pi^- p \to \eta n$ and $\pi^- p \to \eta' n$. At very high energies the difference in the phase space for the two processes becomes negligible and then SU(3) symmetry and the OZI rule predict the ratio of cross sections

$$\frac{\sigma(\pi^- p \to \eta' n)}{\sigma(\pi^- p \to \eta n)} = \left|\frac{X_{\eta'}}{X_\eta}\right|^2 . \tag{7.1}$$

There is some disagreement over the experimental value of this ratio. One group[17] finds 0.55 ± 0.06, which implies a mixing angle of $\theta_p = -18° \pm 1.4°$, while another group[18] finds 0.67 ± 0.03, yielding $\theta_p = -15° \pm 1°$. There is an ambiguity in the extraction of the left-hand side of Eq. (7.1) from experiment centered around the theoretical question of whether to use the whole cross section[17] or only the part coming from the spin-flip amplitude.[18] This adds an additional uncertainty. In any case, we see that the first result favors $\theta_p \simeq -20°$while the second result falls exactly between $\theta_p \simeq -10°$and $\theta_p \simeq -20°$, favoring neither value.

8. Tensor Meson Decays

First we consider the decay $f \to \eta\eta$. SU(3) and the OZI rule (or equivalently, nonet symmetry) lead to the prediction,

$$\frac{\Gamma(f \to \eta\eta)}{\Gamma(f \to \pi\pi)} = \frac{1}{3}|X_\eta|^4 \left(\frac{k_\eta}{k_\pi}\right)^5 , \tag{8.1}$$

where the d-wave character of the final state has been used to correct for the phase space difference between the two decays.

The present experimental data is conflicting. One group[19] finds $BR(f \to \eta\eta) = (5.2 \pm 1.7) \times 10^{-3}$ which implies $|X_\eta| = 0.83 \pm 0.07$, corresponding to $\theta_p \simeq -20°$. A second group measures[20] $BR(f \to \eta\eta) = (2.2 \pm 0.8) \times 10^{-3}$, which implies $|X_\eta| = 0.71 \pm 0.05$, and corresponds to $\theta_p \simeq -10°$.

As a check on the validity of SU(3) and the phase space correction factor, we note that the prediction,

$$\frac{\Gamma(f \to KK)}{\Gamma(f \to \pi\pi)} = 0.036, \tag{8.2}$$

is in excellent agreement with the experimental value[2] of 0.034 ± 0.003.

Lastly, we examine the decay $a_2 \to \pi\eta$. We predict that

$$\frac{\Gamma(a_2 \to \pi\eta)}{\Gamma(a_2 \to KK)} = 2|X_\eta|^2 \left(\frac{k_\pi}{k_K}\right)^5 . \tag{8.3}$$

The experimental value[2] of this ratio is 2.96 ± 0.54. Inserting this into Eq. (8.3) yields $|X_\eta| = 0.72 \pm 0.06$, consistent with $\theta_p \simeq -10°$. We conclude that the tensor meson decay data prefer $\theta_p \simeq -10°$ by a couple of standard deviations in somewhat conflicting experiments.

9. Quadratic Mass Matrix

In a basis of SU(3) octet and singlet states the most general quadratic mass matrix is

$$\mathcal{M} = \begin{bmatrix} m_{\eta_8}^2 & a^2 \\ a^2 & m_{\eta_0}^2 \end{bmatrix} . \tag{9.1}$$

First order SU(3) breaking is incorporated through the Gell-Mann-Okubo mass relation

$$m_{\eta_8}^2 = \frac{4}{3} m_K^2 - \frac{1}{3} m_\pi^2 = (0.56 \text{ GeV})^2 . \tag{9.2}$$

We leave the other elements of the matrix as free parameters, although in the quark model the octet-to-singlet (SU(3) breaking) transition mass matrix element a^2 and the mass of the SU(3) singlet state are calculable in terms of m_K^2 and m_π^2. Leaving $m_{\eta_0}^2$ free also accounts for possible contributions to the singlet mass matrix element from two-gluon intermediate states or from the QCD anomaly in the divergence of the ninth axial-vector current.[21]

Requiring that the physical η and η' be eigenvectors of this matrix with eigenvalues, m_η^2 and $m_{\eta'}^2$, respectively, yields $\theta_p \simeq -10°$. This was basically the original motivation for the use of $\theta_p = -10°$. It is interesting to note that if we take the viewpoint that a^2 is fixed by the quark model to be $a^2 = \frac{2}{3}\sqrt{2}(m_K^2 - m_\pi^2)$, keeping $m_{\eta_8}^2$ as given by the Gell-Mann - Okubo formula then we find a mixing angle of $\theta_p \simeq -18°$,[22] while if we keep a^2 fixed at its quark-model value and allow both $m_{\eta_8}^2$ and $m_{\eta_0}^2$ to be determined by the eigenvalue equation then we derive a mixing angle of $\theta_p \simeq -22°$.

In employing the form of the mass matrix, Eq. (9.1), we have assumed that all the deviation of the η mass from the prediction of the Gell-Mann - Okubo mass formula is caused by mixing with the η'. However, there are other corrections to the η mass of the same order. Donoghue *et al.*[4] have calculated one-loop chiral corrections to $m_{\eta_8}^2$:

$$m_{\eta_8}^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2 - \frac{2}{3}\frac{m_K^2}{(4\pi F_\pi)^2}\log m_K^2/\mu^2 \simeq (0.61 \text{ GeV})^2, \tag{9.3}$$

where μ is a typical hadronic mass scale, $\mu \simeq 1$ GeV. Using this value of $m_{\eta_8}^2$ in the mass-squared matrix and diagonalizing (allowing a^2 to vary) gives $\theta_p \simeq -20°$. Thus the small shift in m_{η_8} of 0.05 GeV from chiral corrections is enough to change the mixing angle from one standard choice to another. Gasser and Leutwyler[23] calculated all the $O(m_q^2)$ contributions to the η and η' mass in the context of chiral perturbation theory. Their analysis yields $\theta_p = -20° \pm 4°$.

These analyses are supported by similar calculations in the $1/N$ expansion,[24] which give $\theta_p \simeq -18°$, and by the semi-phenomenological treatment of Filippov,[25] which derives $\theta_p \simeq -19°$.

In order to retain predictive power the analysis following from Eq. (9.1) neglects mixing with other pseudoscalar states, whether from the quark model or involving gluons and assumes that a^2 and $m_{\eta_0}^2$ are independent of energy. This is aside from any uncertainty in the one-loop chiral correction given in Eq. (9.3) (For example, increasing m_{η_8} by another 0.05 GeV to 0.66 GeV causes θ_p to go from $-20°$ to $-28°$). Given this sensitivity, it is remarkable that the results of the various calculations presented in this section agree with those found in other ways in the previous Sections and, although each analysis can be argued with, taken collectively they strongly favor $\theta_p \simeq -20°$ over $\theta_p \simeq -10°$.

10. Linear Mass Matrix

The linear mass matrix which is the analogue of the quadratic mass matrix in Eq. (9.1) in the octet - singlet basis is

$$\mathcal{M} = \begin{bmatrix} m_{\eta_8} & \alpha \\ \alpha & m_{\eta_0} \end{bmatrix}, \tag{10.1}$$

where now m_{η_8} is given by the linear Gell-Mann-Okubo mass formula:

$$m_{\eta_8} = \frac{4}{3} m_K - \frac{1}{3} m_\pi. \tag{10.2}$$

Diagonalizing this matrix gives $\theta_p \simeq -24°$.

Up to this point we have neglected self-mixing and energy dependent mixing. To investigate more elaborate forms of the mass matrix we rotate it to a basis of non-strange and strange quark states:

$$\mathcal{M} = \begin{bmatrix} X_\eta^2 m_\eta + Y_\eta^2 m_{\eta'} & -X_\eta Y_\eta (m_{\eta'} - m_\eta) \\ -X_\eta Y_\eta (m_{\eta'} - m_\eta) & X_\eta^2 m_{\eta'} + Y_\eta^2 m_\eta \end{bmatrix}, \tag{10.3}$$

and parametrize the it with the form:

$$\mathcal{M} = \begin{bmatrix} m_\pi + 2a^2 & \sqrt{2}ab \\ \sqrt{2}ab & m_{s\bar{s}} + b^2 \end{bmatrix}. \tag{10.4}$$

Physically, the mixing amplitudes a and b may be interpreted in terms of a pseudoscalar quark-antiquark state passing through an intermediate two gluon state to another quark-antiquark state with allowance for a mass dependence (and therefore nonet and SU(3) symmetry breaking). If we impose a value for θ_p, the mass matrix is fully determined (given also the physical η and η' masses). We can then compare to Eq. (10.4) and read off values for a^2, b^2, and $m_{s\bar{s}}$. In particular, choosing $\theta_p = -10°$, we obtain

$$\mathcal{M} = \begin{bmatrix} 746 & 203 \\ 203 & 752 \end{bmatrix} \text{MeV}, \tag{10.5}$$

which implies a^2 =304 MeV, b^2 =68 MeV, and $m_{s\bar{s}}$ =684 MeV. If instead, we choose $\theta_p = -20°$, then we find

$$\mathcal{M} = \begin{bmatrix} 680 & 191 \\ 191 & 823 \end{bmatrix} \text{MeV}, \tag{10.6}$$

giving a^2 =271 MeV, b^2 =67 MeV, and $m_{s\bar{s}}$ =756 MeV. The values of a^2 and b^2 in the two cases are very close numerically and both values of $m_{s\bar{s}}$ are within the bounds placed on this quantity by meson hyperfine splitting[26] in potential models. Thus, depending on what assumptions are made, the linear mass matrix is consistent with both $\theta_p \simeq -10°$ and $\theta_p \simeq -20°$ and is unable to discriminate decisively between them.

11. Conclusion

We have summarized the various data pertaining to the pseudoscalar mixing angle and found that $\eta(\eta') \to \gamma\gamma$, $\psi \to \eta(\eta')\gamma$, $\psi \to$ pseudoscalar + vector, and $\pi^- p$ scattering favor the choice of $-20°$ over $-10°$. Other data weakly favor $-10°$ or are inconclusive and consistent with $-20°$, but unable to distinguish between them. We should note that our conclusion rests heavily on the simple mixing scenario we have chosen and, to a somewhat lesser degree, on the manner in which SU(3) and/or nonet symmetry breaking have used in the various arguments.

Even given our assumptions we do not claim to rule out some mixing of η or η' with exotic states or values of the mixing angle a few degrees different than $\theta_p \simeq -20°$. We have just shown that the present data is consistent with the simplest mixing scenario and with a mixing angle of $\theta_p \simeq -20°$.

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APPENDIX : $\eta \longrightarrow \pi^+\pi^-\pi^0$ and $\psi' \longrightarrow \psi\pi^0$

The isospin-violating decay $\eta \to \pi^+\pi^-\pi^0$ can be interpreted in terms of an $\eta - \pi$ transition (which violates isospin and proceeds through the up - down quark mass difference) followed by a strong interaction π to 3π transition that can calculated using current algebra techniques yielding[27]

$$\Gamma(\eta \to \pi^+\pi^-\pi^0) = |A|^2 \ 487 \text{ eV}, \tag{A1}$$

where

$$A = \frac{8m_k^2}{3\sqrt{3}F_\pi^2}\left(\frac{m_u - m_d}{2m_s}\right). \tag{A2}$$

The experimental value of $\Gamma(\eta \to \pi^+\pi^-\pi^0)$ increased when the new rate for $\eta \to \gamma\gamma$ increased the total width of the η. Inserting the new experimental value, $\Gamma(\eta \to \pi^+\pi^-\pi^0) = 250 \pm 36$ eV, into Eq. (A1) yields

$$\frac{m_d - m_u}{2m_s} = 0.017 \pm 0.0012. \tag{A3}$$

Now consider the other isospin-violating process of interest, $\psi' \to \psi\pi^0$. A pole model where the decay proceeds through an intermediate η or η' gives[28]

$$\frac{\Gamma(\psi' \to \psi\pi^0)}{\Gamma(\psi' \to \psi\eta)} = \left(\frac{\lambda_{\pi\eta}g_\eta + \lambda_{\pi\eta'}g_{\eta'}}{g_\eta}\right)^2 \left(\frac{k_\pi}{k_\eta}\right)^3, \tag{A4}$$

where $\lambda_{\pi\eta}$ and $\lambda_{\pi\eta'}$ describe the mixing of the η and η' with the π^0 and g_η and g'_η are the couplings for $\psi' \to \psi\eta$ and $\psi' \to \psi\eta'$. The mixing parameters can be expressed in terms of θ_p and $(m_d - m_u)/m_s$. Taking $\theta_p = -10°$ gives

$$\lambda_{\pi\eta} + \lambda_{\pi\eta'}\frac{g_{\eta'}}{g_\eta} = 3.1\left(\frac{m_d - m_u}{2m_s}\right), \tag{A5}$$

while $\theta_p = -20°$ implies

$$\lambda_{\pi\eta} + \lambda_{\pi\eta'} \frac{g_{\eta}}{g_{\eta}} = 2.1 \left(\frac{m_d - m_u}{2m_s} \right) . \tag{A6}$$

With the experimental value[2]

$$\frac{\Gamma(\psi' \to \psi\pi^0)}{\Gamma(\psi' \to \psi\eta)} = 0.37 \pm 0.001, \tag{A7}$$

$\theta_p = -10°$ gives

$$\frac{m_d - m_u}{2m_s} = 0.021 \pm 0.004, \tag{A8}$$

while $\theta_p = -20°$ gives

$$\frac{m_d - m_u}{2m_s} = 0.014 \pm 0.003. \tag{A9}$$

These two results bound that from the decay $\eta \to \pi^+\pi^-\pi^0$ given above. The value derived using $\theta_p = -20°$ is consistent with that derived from other sources,[29] *e.g.*, baryon masses and $\rho - \omega$ mixing, which give $(m_d - m_u)/(2m_s) \simeq 0.11$.

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