SLAC – PUB – 4298
ITP – 825
April 1987
(T)

STRING LOOP DIVERGENCES AND EFFECTIVE LAGRANGIANS

Willy Fischler*
University of Texas at Austin
Austin, Texas 78712

Igor Klebanov**
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Leonard Susskind***
Stanford University
Stanford, California 94305

ABSTRACT

We isolate logarithmic divergences from bosonic string amplitudes on a disc. These divergences are compared with "tadpole" divergences in the effective field theory with a covariant cosmological term implied by the counting of string coupling constants. We find an inconsistency between the two. This is a serious problem which could undermine the ability to remove divergences from the bosonic string.

Submitted to *Nuclear Physics B*

---

*Work supported by NSF PHY 86-05978 and Robert Welch Foundation.

**Work supported by DOE grant DE-AC03-76SF00515

***Work supported by NSF PHY 812280.

I. Introduction

The equations of motion for background fields in which strings may consistently propagate are summarized by the condition that the β function of the two dimensional world-sheet field theory vanish.[1] Furthermore these equations can be derived from a Lagrangian which resembles and generalizes the usual gravity and matter field action. The same Lagrangian may be used to generate Feynman diagrams for scattering amplitudes. The amplitudes reproduce the usual string S-matrix elements.

Recently this idea has been generalized to include the divergent effects of string loop corrections. Briefly the idea is that small "fixtures" such as handles and holes attached to an otherwise smooth world sheet renormalize the effective two dimensional action and correct the β functions. In ref. 2 it was shown how such an effect for a single handle generates a cosmological constant in the Einstein equations.

Subsequently it was emphasized by Callan et al. and by Seiberg [ref. 3] that a consistent treatment requires inclusion of the dilaton field background along with the gravitational field.

In this paper we study the consistency of the effective Lagrangian and the string amplitudes derived by standard string theory methods when a fixture, in this case a hole, is installed on the world sheet. This can be done either by computing the fixture corrected β function or by direct comparison of the effective field theory scattering amplitudes with string amplitudes. In the latter case we compare logarithmic divergences in string-theoretic amplitudes with the effective-Lagrangian tadpole and mass insertion diagrams shown in Fig. 1. These diagrams are also divergent because they contain a massless propagator at zero momentum. We find that an obvious method of calculation fails to produce consistency between string log divergences and effective field theory tadpoles. However, we shall see that in addition to log divergences, the string theory also has certain quadratic divergences. These quadratic divergences can be eliminated by a renormalization of the 2-dimensional cosmological constant. This raises the possibility that a careful evaluation may reveal an additional subleading logarithmic divergence. At present we have no evidence that this is so.

II. Effective Lagrangian

The effective Lagrangian for massless string states[1] can be obtained by the following procedure. Introduce background gravitational and dilaton fields into the world sheet action:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\,\Big(\gamma^{ab}g_{\mu\nu}(X)\,\partial_a X^\mu \partial_b X^\nu - \\ - \frac{\alpha'}{2}R^{(2)}\phi(X)\Big) \tag{1}$$

where $\gamma_{ab}(\sigma_1,\sigma_2)$ is the world sheet metric, $g_{\mu\nu}(X)$ -- the 26 dimensional metric and $\phi(X)$ is the dilaton field. The fields $g_{\mu\nu}(X)$ and $\phi(X)$ can be thought of as an infinite collection of couplings in the 2-dimensional field theory. These couplings become renormalized and satisfy renormalization group equations

$$\begin{aligned} \frac{\partial g_{\mu\nu}}{\partial \log\lambda} &= \beta_{\mu\nu}(g,\phi) \\ \frac{\partial \phi}{\partial \log\lambda} &= \beta_\phi(g,\phi) \end{aligned} \tag{2}$$

---

[1]We concentrate on the gravitational and dilaton degrees of freedom. Other massless states are ignored.

The field equations for g and ϕ are

$$\beta_{\mu\nu} = \beta_\phi = 0 \tag{3}$$

Remarkably, these equations can be derived from a generally covariant action[1]

$$I = \int d^{26}X \sqrt{g}\, e^{\phi}\left(R + (\nabla\phi)^2 + O(\alpha')\right) \tag{4}$$

The order α' corrections in eq. 4 are higher derivative terms where each additional derivative introduces a factor $\sqrt{\alpha'}$.[1,4]

The processes leading to eq. (4) are string tree graphs with spherical world sheet topology. The factor e^{ϕ} in eq. (4) is understood as follows: the path integral for the string $\int [DX] e^{-S}$ has a factor

$$\exp\left(\phi_0 \frac{\int d^2\sigma \sqrt{\gamma}\, R^{(2)}}{8\pi}\right) = e^{\phi_0 (1-g)} \tag{5}$$

where ϕ_0 is the zero momentum part of ϕ and g is the genus of the world sheet. Since for a sphere (g = 0) the effective action must be weighted by e^{ϕ_0}, locality requires this to be replaced by e^{ϕ}.

Let us now consider the correction to the action (I) due to processes with a small hole in the world sheet. Such processes occur in the theory of coupled open and closed strings. The coupling of the dilaton field $\phi(X)$ to such a world sheet has an extra boundary term given by

$$S_\phi^{bry} = -\frac{1}{4\pi} \oint ds\, K\, \phi(X) \tag{6}$$

where K is the extrinsic curvature of the boundary. For the topology in question the sum

$$\frac{1}{8\pi}\int d^2\sigma\sqrt{\gamma}\,R^{(2)}+\frac{1}{4\pi}\oint ds\,K=\frac{1}{2} \tag{7}$$

so that amplitudes all contain the factor $e^{\phi/2}$. We expect the corrected action to have this factor. The term with the least possible number of derivatives is

$$\delta I=\int d^{26}X\sqrt{g}\,e^{\phi/2}\Lambda \tag{8}$$

where Λ is constant. To compute Λ and investigate the consistency of eq. (5) with string theory we will compare scattering amplitudes derived using (4) and (8) with string scattering amplitudes.

III. String Calculation

We will consider the scattering of gravitons from a world sheet with the topology of a sphere with a hole. We can represent this situation by a plane with a hole or radius a centered at position z. For simplicity we will write the formulae for the 3-graviton amplitude but the results are general. The gravitons enter at z_1, z_2, and z_3. We may hold the radius and location of the hole and one of the graviton insertions fixed. Alternatively we can hold the 3 gravitons fixed and integrate the radius a and location z. The latter form is more convenient for our purpose. The amplitude is given by[9]

$$\begin{gathered}e^{\phi_0/2}\int\frac{da}{a^3}\int d^2z\,|z_1-z_2|^2|z_2-z_3|^2|z_1-z_3|^2\\ \langle\rho^1_{ij}\,\partial X^i\bar\partial X^j e^{ik_1\cdot X}(z_1,\bar z_1)\,\rho^2_{k\ell}\,\partial X^k\bar\partial X^\ell e^{ik_2\cdot X}(z_2,\bar z_2)\\ \rho^3_{st}\,\partial X^s\bar\partial X^t e^{ik_3\cdot X}(z_3,\bar z_3)\rangle\end{gathered} \tag{9}$$

where the expectation value is computed on the plane with a hole of radius a centered at z. The integration over z covers the whole plane excluding only those regions which would cause one of the z_i to be in the hole (Fig. 2).

The Green's function needed to compute (9) is given by

$$\langle X(z_1,\bar{z}_1)\,X(z_2,\bar{z}_2)\rangle = -\log|z_1-z_2|^2 - \log\left|1-\frac{a^2}{(z_1-z)(\bar{z}_2-\bar{z})}\right|^2 \tag{10}$$

The second term represents the effect of the hole. Expanding (10) for small a gives

$$-\log|z_1-z_2|^2 + \frac{a^2}{(z_1-z)(\bar{z}_2-\bar{z})} + \frac{a^2}{(\bar{z}_1-\bar{z})(z_2-z)} + O(a^4) \tag{11}$$

The order a^2 contribution to the propagator is identical to the effect of the operator

$$a^2 :\partial X\cdot\bar{\partial} X: \tag{12}$$

inserted at the point z. Thus to obtain the coefficient of the logarithmic divergence in the amplitude (g) we expand the integrand to order a^2 and integrate over a.

$$e^{\phi_0/2}\int_{a_0}\frac{da}{a}\int d^2z\;|z_1-z_2|^2|z_2-z_3|^2|z_1-z_3|^2$$

$$\langle V_1(z_1,\bar{z}_1)\,V_2(z_2,\bar{z}_2)\,V_3(z_3,\bar{z}_3) :\partial X\bar{\partial}X:(z,\bar{z})\rangle_{sphere} \tag{13}$$

This expression schematically summarizes the logarithmic divergence relevant for our purposes. An analogous expression for a multi-tachyon amplitude was derived in ref. 9. In section (V) the consequences of eq. (13) for the β-functions and equations of motion will be discussed.

To compare (9) with effective field theoretic amplitudes we have carried out integration over the location and size of the hole. The details of this calculation are provided in the appendix. The end result is that the insertion of eq. (13) is equivalent to operating with

$$-\pi \log a_0 \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} \tag{14}$$

Thus the log divergent part of the amplitude is given by

$$-\pi \log a_0 \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} A_{tree}(\rho_1, k_1; \rho_2, k_2; \rho_3, k_3) \tag{15}$$

where A_{tree} is the amplitude with no hole.

IV. Effective Field Theory Amplitude

The effective Lagrangian is

$$I = \int d^{26}X \sqrt{g} \left[e^{\phi} \left(R + (\nabla\phi)^2 + O(\alpha') \right) + e^{\phi/2} \Lambda \right] \tag{16}$$

The rule for the α' dependence in the first term in (16) is that a factor of $\sqrt{\alpha'}$ is included for each derivative in the higher derivative terms in the Lagrangian. Calculating with (16) is inconvenient because the term $e^{\phi}R$ introduces mixing between the graviton and dilaton. This may be eliminated by a field redefinition. Define a rescaling of the metric $g_{\mu\nu} \rightarrow g_{\mu\nu} e^{-\frac{2\phi}{d-2}}$. The action becomes[5]

$$\int\sqrt{g}\left(R-\frac{(\nabla\phi)^2}{d-2}+\dots\right)+\int\sqrt{g}\,\Lambda\, e^{-\phi\left(\frac{2}{d-2}+\frac{1}{2}\right)} \tag{17}$$

The graphs of interest are those of first order in Λ which we will compare with the contribution of the small hole in string theory. These are shown in Fig. 1. Each graph contains a zero momentum propagator of the form

$$\left.\frac{1}{k^2}\right|_{k^2=0} \tag{18}$$

which renders it divergent. This divergence is not unexpected and should be identified with the log divergence in the hole size integration in the string case.

To compute the diagrams it is useful to recall the soft dilaton and soft graviton emission theorems. The soft dilaton emission amplitude is given by[6]

$$A_\phi=-\frac{1}{d-2}\left(\sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}}-2\right)A \tag{19}$$

where A is the amplitude without the soft dilaton. For our purposes this theorem can be derived by observing that the power of e^ϕ multiplying each term in the rescaled tree level lagrangian (17) is determined by the number of derivatives in that term. Multiplying (19) by the dilaton one point function from the cosmological term and dilaton propagator

$$\Lambda\left(\frac{2}{d-2}+\frac{1}{2}\right)\left(\frac{d-2}{2k^2}\right)\Bigg|_{k^2=0} \tag{20}$$

from eq. (17) we obtain the dilaton contribution of Fig. 1a.

Similarly one can compute the soft graviton contribution of Fig. 1a. The combined result has the simple form

$$-\frac{1}{4}\Lambda\left(\sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}}+2\right)A\cdot\left(\frac{1}{k^2}\right)_{k^2=0} \tag{21}$$

Evidently eq. (15) and (21) do not agree. More should be said about the external leg and mass insertions of Fig. 1b, and c. For our purposes they can be separated from the contact terms because their contributions grow linearly with the number of external legs. It turns out that our regularization of the string amplitudes sets the external leg contributions to zero. A detailed discussion of these issues is included in the appendix.

V. The β Functions

The results of the previous section can be restated using the language of the world sheet beta functions. For our purposes the β functions are the coefficients of the logarithmically divergent counterterms in the two-dimensional string action. In a particular renormalization scheme they have the form

$$\begin{aligned} \beta_{\mu\nu} &= R_{\mu\nu} - \nabla_\mu \nabla_\nu \phi \\ \beta_\phi &= -\frac{1}{2}\nabla^2\phi - \frac{1}{2}(\nabla\phi)^2 \end{aligned} \tag{22}$$

As was pointed out recently by several authors,[8] these beta functions are actually linear combinations of the ones computed in ref. 1. β-functions can be expressed in terms of variations of effective action

$$\begin{aligned} \beta_{\mu\nu} &= \frac{\delta I}{\delta g^{\mu\nu}} + \frac{1}{2} g_{\mu\nu}\frac{\delta I}{\delta\phi} \\ \delta\beta_\phi &= -\frac{d-2}{4}\frac{\delta I}{\delta\phi} - \frac{1}{2}\frac{\delta I}{\delta g^{\mu\nu}} g^{\mu\nu} \end{aligned} \tag{23}$$

These relations allow us to infer what world sheet insertions we expect to be induced at one loop. Substituting the cosmological term into (23) we find that, for agreement with the generally covariant effective action, the small hole should induce a counterterm proportional to

$$\log a_0 \left(\partial X \cdot \bar{\partial} X + \frac{d}{8} R^{(2)} + \frac{1}{4} R^{(2)} \right)$$

According to de-Alwis,[10]

$$\partial X \cdot \bar{\partial} X + \frac{d}{8} R^{(2)} = :\partial X \cdot \bar{\partial} X:$$

On the other hand, we just saw that the hole naively induces only the insertion proportional to $:\partial X \bar{\partial} X:$[2] We can see that the discrepancy between our calculation and the result required by consistency of β functions is an extra renormalization of $R^{(2)}$. In terms of string amplitude calculations, this discrepancy translates into missing a multiplicative renormalization of the tree amplitude.

The problem can be made to disappear if an additional log divergence can be found in string amplitudes. In particular eqns. (21) and (15) suggest a missing term of the form

$$-2 \pi \log a_0 A_{tree} \tag{24}$$

The only consistent interpretation of such a term is that the small hole at z introduces an additional counterterm equal to $-\frac{1}{4}R^{(2)}(z) \log a_0$. Integrating

---

[2]This result was emphasized also in ref. 9.

over the world sheet would give (24).

The fact that the coefficient of the missing term is independent of spacetime dimension d suggests that it is hiding in the ghost determinant. Let us return to eq. (9). In addition to the log divergence, integration over a has a quadratic divergence. The coefficient of this divergence is obtained by using the leading term in eq. (11). The result is

$$e^{\phi_0/2} \int \frac{da}{a^3} \int d^2z \, A_{tree} \tag{25}$$

This divergence can be removed by a renormalization of the 2-dimensional cosmological constant.

The naive separation of quadratic and logarithmic divergences may have overlooked an additional logarithmic divergence. At the moment we do not know how to verify this. The issues raised in this paper can affect the ultimate consistency of string theory.

Note added:

After this paper was written we received a paper by Callan et al.[7] in which loop-corrected string equations of motion were derived using BRST methods. Although these equations are consistent, it appears that, as far as amplitude calculations go, the methods of ref. 7 are equivalent to our calculations and do not produce the missing piece in scattering amplitudes.

Acknowledgements

We thank M. Peskin and B. Ratra for considerable help during the early stages of this work. We also acknowledge many enlightening conversations with T. Banks.

APPENDIX

In this appendix we present an evaluation of the divergent parts of the three-graviton amplitude on a disc. Clearly, eq. (9) contains a quadratic divergence proportional to the area of the world sheet $\int d^2z\sqrt{\gamma(z)}$ and the tree-level scattering amplitude. This divergence is usually absorbed by a renormalization of the world sheet cosmological constant. Let us proceed now to the logarithmic divergences. Obviously, a generic contribution arises from the sum of terms in the expectation value of eq. (9), where each propagator is in turn replaced by its $O(a^2)$ part. This is precisely the part of the expectation proportional to an insertion of $a^2 :\partial X\bar{\partial}X:$ There are two types of contractions involved:

$$\langle \bar{\partial} X^{\mu}(z_1,\bar{z}_1)\, e^{ik\cdot X}(z_2,\bar{z}_2)\rangle \quad \text{and} \quad \langle \bar{\partial} X^{\mu}(z_1,\bar{z}_1)\, \bar{\partial} X^{\nu}(z_2,\bar{z}_2)\rangle$$

Of course, there are similar contractions involving ∂X whose contributions can be evaluated with identical methods. Let us concentrate on $\langle \bar{\partial} X^{\mu} e^{ik\cdot X}\rangle$ Its contribution to the amplitude on a sphere is $-i\dfrac{k^{\mu}}{\bar{z}_1-\bar{z}_2}$. The contribution from $O(a^2)$ part of the propagator (11), with a subsequent integration over the position of the hole is

$$-ik_{\mu}a^2\int d^2z\left[-\frac{1}{(\bar{z}_1-\bar{z})^2(z_2-z)} - 4\pi\delta^2(z_1-z)\frac{1}{\bar{z}_2-\bar{z}}\right] \qquad \text{(A1)}$$

The z-integration is restricted to the region of Fig. 2. Therefore the second term in (A1) does not contribute. The first term can be manipulated with the use of Green's theorem:

$$-\int d^2z\; \bar{\partial}\left(\frac{1}{\bar{z}-\bar{z}_1}\right)\frac{1}{z-z_2} = -\int d^2z\; \bar{\partial}\,\frac{1}{(\bar{z}_1-\bar{z})(z_2-z)} =$$

$$-\frac{1}{2i}\oint dz\,\frac{1}{(\bar{z}-\bar{z}_1)(z-z_2)} = -\frac{\pi}{\bar{z}_2-\bar{z}_1} + O(a) \tag{A2}$$

Note that only the contour around z_2 contributes to (A2). (A1) finally reduces to

$$-i\pi a^2 k^\mu \frac{1}{\bar{z}_1-\bar{z}_2}$$

The method illustrated above can be applied in turn to all contractions. We find that substituting the $O(a^2)$ piece of the propagator into $\langle\partial X\partial X\rangle$, $\langle\bar{\partial}X\bar{\partial}X\rangle$, and $\langle\partial X\bar{\partial}X\rangle$ integrates to zero, while $\langle\partial Xe^{ik\cdot X}\rangle$ and $\langle\bar{\partial}Xe^{ik\cdot X}\rangle$, upon the integration over the position of the hole contribute πa^2 times their contribution on a sphere.

Putting everything together, we find that the log divergence arising from the insertion of $:\partial X\bar{\partial}X:$ on the world sheet is given by

$$-\log a_0 \times \pi\sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}}A_{tree}(1,2,3) \tag{A3}$$

$\sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}}$ just weighs each piece of the tree amplitude proportionately to its net power in momenta. Since the three-graviton amplitude has contributions $O(k^2)$, $O(k^4)$, and $O(k^6)$, this effect can be easily distinguished from the multiplicative renormalization of the tree amplitude.

Although (A3) turns out to be the full answer, we would like to point out that there are additional log divergences that turn out to cancel exactly. These divergences are proportional to the number of particles being scattered. In the field-theoretic interpretation, these are the external leg divergences shown in Fig. (1). We find two sources of such divergences. One comes from using all propagators to $O(a^0)$. Then the integrand in the d^2z integration is independent of z.

We note, however, that the result of the d^2z integration still depends on a through the region (circles of area πa^2 are excluded). This gives rise to a contribution $\pi n \log a_0 A_{tree}$, where n is the number of external states. Another external leg contribution appears if one uses the $O(a^2)$ propagator to contract, ∂X and $\bar\partial X$ from the same graviton vertex with other vertex operators. Then the z-integration will give rise to a quadratic divergence cut off at the size of the hole. The result of the z-integration is $\sim a^4/a^2$. Therefore, after integrating over the size of the hole, the answer is logarithmic in the cut off. We find that the two contributions described above cancel each other. This indicates that the string S-matrix calculation effectively sets the external leg divergences to zero. An independent check on this is a calculation of the graviton two-point function on a disc in the path integral formalism.

We represent the disc by an upper half-plane and use the image method to determine the propagator:

$$\langle X(z,\bar z)X(w,\bar w)\rangle = -\log|z-w|^2 - \log|z-\bar w|^2 \tag{A4}$$

Fixing the location of one of the vertex operators by SL(2,R) we find

$$\begin{aligned} A(1,2) &\sim \int d^2z \langle \rho^1_{ij}\, \partial X^i \bar\partial X^j e^{ik\cdot X}(z,\bar z) \\ &\rho^2_{k\ell}\, \partial X^k \bar\partial X^\ell e^{-ik\cdot X}(w,\bar w)\rangle \sim \int d^2z \left(\frac{1}{|z-w|^4} + \frac{1}{|z-\bar w|^4}\right), \end{aligned} \tag{A5}$$

where we extended the region of integration by symmetry to the entire complex plane. The expression is $\sim \int_0^\infty \frac{dr}{r^3}$. If we regularize this expression in the obvious way by $\int_a^\infty \frac{dr}{r^3} =$, it is proportional to a "pure" quadratic divergence. The finite part underlying this divergence is zero. This agrees with the fact that the external leg log divergences cancel in the 3-h amplitude.

In conclusion of this section we would like to point to one more important implication of our calculation. Some standard lore of string physics which dates back to ref. [6] states that the logarithmic divergence of a string diagram is proportional to the zero-momentum dilaton emission amplitude. This statement cannot possibly agree with low energy field theory due to the presence in it of a tadpole for the trace of the graviton. In fact the dilaton emission amplitude is proportional to $\sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}} - 2$ which agrees with neither the naive string amplitude (A3) or the result implied by effective field theory tadpole analysis.

Figure Captions

1. Three types of diagrams to first order in the cosmological constant that give rise to divergences in the effective field theory. Wavy line corresponds to a dilaton or a trace of graviton.

2. The region of integration over z, the position of the center of the hole, for a hole of radius a.

REFERENCES

1. C. Callan, E. Martinec, M. Perry and D. Friedan, Nucl. Phys. B262 (1985) 593.

2. W. Fishler and L. Susskind, Phys. Lett. 171B (1986) 383; 173B (1986) 262.

3. C. Callan, C. Lovelace, C. Nappi and S. Yost, Princeton preprint PUPT-1027 (1986). N. Seiberg, Weizmann Inst. preprint WIS/86/61.

4. C. Callan, I.R. Klebanov and M.J. Perry, Nucl. Phys. B272 (1986) 111.

5. R. Nepomechie, Phys. Rev. 32D (1986) 413.

6. M. Ademollo, R. D'Auria, F. Gliozzi, E. Napolitano, S. Sciuto, and P. di Vecchia, Nucl. Phys. B94 (1975) 221.

7. C. Callan, C. Lovelace, C. Nappi and S. Yost, Princeton preprint PUPT-1045 (1987).

8. G. Curci and G. Paffuti, CERN preprint CERN-TH 4558/86.
A.A. Tseytlin, Phys. Lett. 178B, 34 (1986).

9. S.R. Das and S.J. Rey, CalTech preprint CALT-68-1389.

10. S. de Alwis, Phys. Lett. 168B, 59 (1986).

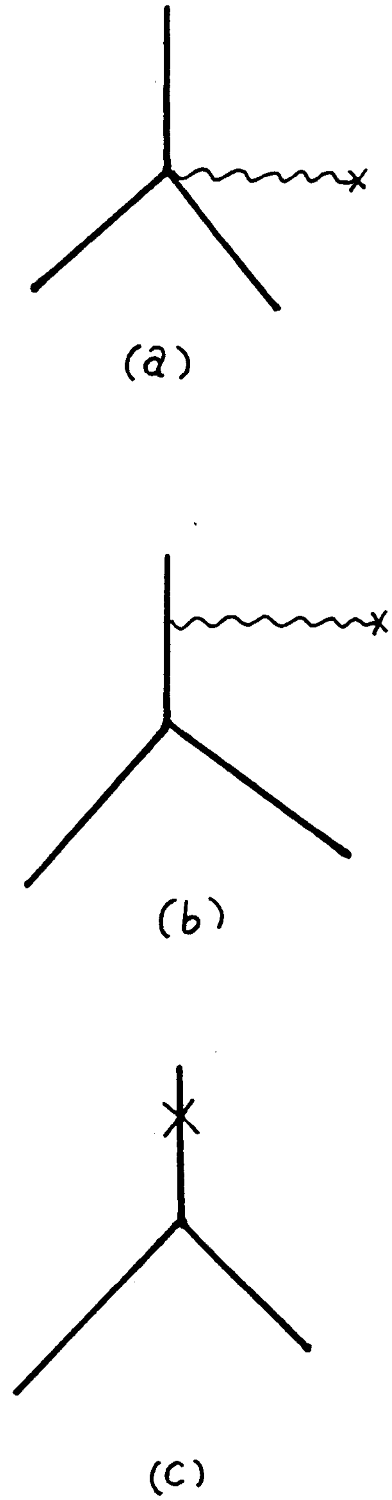

Fig. 1

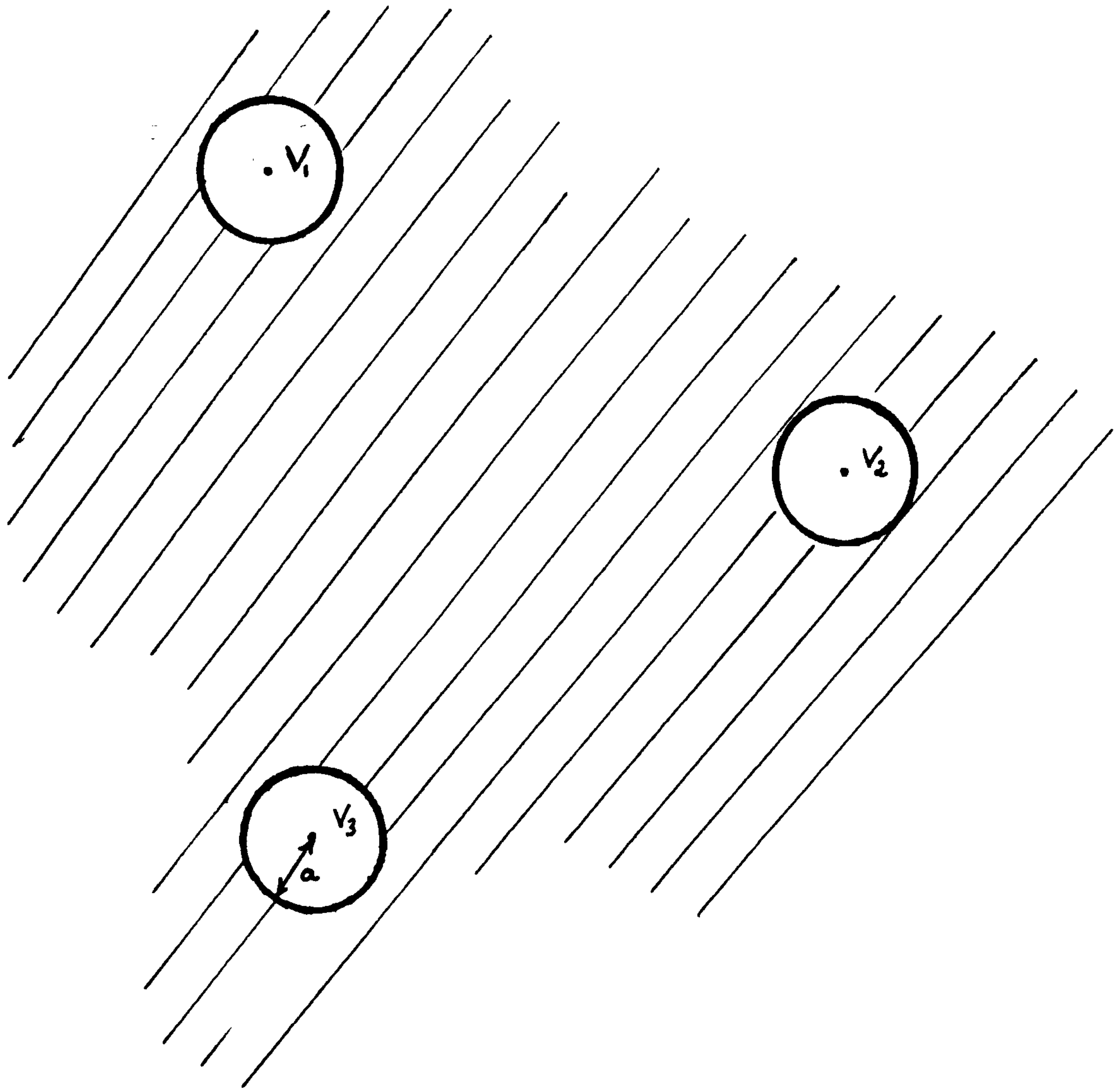

Fig. 2